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BOUNDED COHOMOLOGY AND BINATE GROUPS

FRANCESCO FOURNIER-FACIO, CLARA LÖH and MARCO MORASCHINI

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Abstract

A group is boundedly acyclic if its bounded cohomology with trivial real coefficients vanishes in all positive degrees. Amenable groups are boundedly acyclic, while the first nonamenable examples are the group of compactly supported homeomorphisms of $\mathbb{R}^n$ (Matsumoto–Morita) and mitotic groups (Löh). We prove that binate (alias pseudo-mitotic) groups are boundedly acyclic, which provides a unifying approach to the aforementioned results. Moreover, we show that binate groups are universally boundedly acyclic. We obtain several new examples of boundedly acyclic groups as well as computations of the bounded cohomology of certain groups acting on the circle. In particular, we discuss how these results suggest that the bounded cohomology of the Thompson groups F, T, and V is as simple as possible.

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1. Introduction

Bounded cohomology is defined via the topological dual of the simplicial resolution. It was introduced by Johnson and Trauber in the context of Banach algebras [35], then extended by Gromov to topological spaces [28]. Since then, it has become a fundamental tool in several fields, including the geometry of manifolds [28], rigidity theory [13], the dynamics of circle actions [25], and stable commutator length [14].

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Despite a good understanding in degree 2 and a partial understanding in degree 3, the full bounded cohomology of a group seems to be hard to compute [22, Section 7]. Therefore, it is fundamental to produce alternative resolutions that compute the bounded cohomology of a group. In this respect, amenable groups played a fundamental role in the approach of Ivanov [32]. One can also exploit a larger class of groups for computing bounded cohomology, namely the class of *boundedly acyclic groups* [33, 48].

DEFINITION 1.1. Let $n \geq 1$. A group Γ is said to be *n-boundedly acyclic* if $\mathrm{H}_b^i(\Gamma;\mathbb{R}) \cong 0$ for all $i \in \{1,\dots,n\}$. The group Γ is *boundedly acyclic* if it is n-boundedly acyclic for every $n \geq 1$.

Amenable groups are boundedly acyclic [28, 35]. The first nonamenable example, due to Matsumoto and Morita [42], is the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of compactly supported homeomorphisms of $\mathbb{R}^n$. Their proof relies on the *acyclicity* of this group, which for the purposes of this paper will always be considered with respect to the integers.

DEFINITION 1.2. A group Γ is said to be *acyclic* if $\mathrm{H}_n(\Gamma;\mathbb{Z}) \cong 0$ for all $n \geq 1$.

It was shown by Mather that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is acyclic [40], and thus the proof of bounded acyclicity reduces to the proof of injectivity of the comparison map from bounded to ordinary cohomology. The same approach was employed by Löh to prove that *mitotic* groups are boundedly acyclic [39]. This class was introduced by Baumslag, Dyer and Heller [1] to produce embedding results into finitely generated acyclic groups. Bounded acyclicity of mitotic groups, together with co-amenability of ascending HNN extensions, eventually led to finitely generated and finitely presented examples of nonamenable boundedly acyclic groups [22].

The two bounded acyclicity results mentioned above are similar in spirit but independent of one another, since $\mathrm{Homeo}_c(\mathbb{R}^n)$ is *not* mitotic [49]. However, there is a larger framework that includes both $\mathrm{Homeo}_c(\mathbb{R}^n)$ and mitotic groups: *binate groups* (see Section 3 for the definition). This class was introduced by Berrick [4] and independently by Varadarajan [55], under the name *pseudo-mitotic*. They proved that binate groups are acyclic, and thus provided a unified approach to the proofs of Mather and Baumslag–Dyer–Heller, as well as several new and interesting examples of acyclic groups, mainly among groups of homeomorphisms (see Section 3.1). We adapt this unification to bounded cohomology.

THEOREM 1.3 (Theorem 3.5). *All binate groups are boundedly acyclic.*

We remark that, in general, binate groups are nonamenable, since they typically contain free subgroups. However, there are a few exceptions (Section 3.1.4).

Binate groups reflect enough of group theory to serve as a faithful testing class for open problems such as the Bass conjecture, a modified version of the Baum–Connes conjecture, or the Kervaire conjecture [6]. By Theorem 1.3, boundedly acyclic groups also serve as conjecture testers. For instance, if the Bass conjecture holds for all boundedly acyclic groups, then it holds for all groups. This is especially interesting

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since amenable groups, which serve as the prototypical example of boundedly acyclic groups, are known to satisfy the Bass conjecture [7].

The bounded acyclicity of binate groups is a phenomenon that is not strictly linked to real coefficients. Indeed we prove the following theorem.

THEOREM 1.4 (Corollary 5.4). *Binate groups are* universally boundedly acyclic*: if* Γ *is a binate group, then for every complete valued field* $\mathbb{K}$ *and every* $n \geq 1$*, we have* $\mathrm{H}_b^n(\Gamma;\mathbb{K}) \cong 0$.

More generally, we characterize universally boundedly acyclic groups as those groups that are simultaneously acyclic and boundedly acyclic (Theorem 5.2). In this sense, Theorem 1.4 is a combination of the acyclicity result of Berrick and Varadarajan, together with Theorem 1.3, but it also contains both results in its statement.

1.1. Hereditary properties of boundedly acyclic groups. By analogy with the amenable case, it is interesting to check which group-theoretic constructions preserve bounded acyclicity. It is known that extensions, as well as quotients with boundedly acyclic kernels, do [48]. It is therefore natural to wonder whether these two results extend to a two-out-of-three property for bounded acyclicity and group extensions.

Using the fact that every group embeds 2-step subnormally into a binate group, we show that this cannot hold.

THEOREM 1.5 (Theorem 4.5). *There exists a boundedly acyclic group* Γ *with a normal subgroup* H *such that* Γ/H *is boundedly acyclic, but* $\mathrm{H}_b^n(H;\mathbb{R})$ *is continuum-dimensional for every* $n \geq 2$.

We also look at directed unions of boundedly acyclic groups, and show that these are boundedly acyclic under an additional technical requirement (Proposition 4.13), which is however not needed in degree two (Corollary 4.16).

1.2. Application to Thompson groups. The advantage of Theorem 1.3 is that the class of binate groups is flexible enough that one can construct several concrete examples of boundedly acyclic groups. We use this to study the bounded cohomology of certain analogs of the classical Thompson groups F, T, and V. The amenability question for F is one of the most influential open questions in modern group theory. It is therefore natural to wonder whether F is at least boundedly acyclic. It is known that F is 2-boundedly acyclic, but nothing seems to be known in higher degrees. Using Theorem 1.3, we show that a countably singular analog of the Thompson group F is boundedly acyclic (Proposition 6.7).

Moreover, we prove that if F is n-boundedly acyclic, then the bounded cohomology of T is generated by the real Euler class and its cup-powers, up to degree n (Corollary 6.17). In particular, we obtain the following theorem.

THEOREM 1.6 (Corollaries 6.17 and 6.18). *If the Thompson group* F *is boundedly acyclic, then* $\mathrm{H}_b^*(T;\mathbb{R})$ *(with the cup-product structure) is isomorphic to the polynomial*

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ring $\mathbb{R}[x]$ *with* $|x| = 2$ *and the bounded Euler class of* T *is a polynomial generator of* $\mathrm{H}_b^*(T;\mathbb{R})$. *Moreover, the canonical semi-norm on* $\mathrm{H}_b^*(T;\mathbb{R})$ *is a norm.*

Therefore, the bounded acyclicity of F would make T into the first group of type F_∞ that is not boundedly acyclic, and whose bounded cohomology ring can be completely and explicitly computed. Similarly, if F is boundedly acyclic, then the bounded cohomology ring of $T^{\times r}$ is a polynomial ring in r generators of degree two (Corollary 6.18).

Independently, Monod and Nariman recently established analogous results for the bounded cohomology of the group of orientation-preserving homeomorphisms of S^1 [46].

1.3. A note from the future. After this paper was finished, Monod proved that the Thompson group F is boundedly acyclic [45]. Therefore, Theorem 1.6 shows that the bounded cohomology of the Thompson group T is isomorphic to the polynomial ring $\mathbb{R}[x]$ with $|x| = 2$ and that the bounded Euler class of T is a polynomial generator of $\mathrm{H}_b^*(T;\mathbb{R})$.

1.4. Organization of this article. We recall the definition of bounded cohomology and the uniform boundary condition in Section 2. Binate groups are surveyed in Section 3. In Section 4, we study hereditary properties of boundedly acyclic groups. Section 5 is devoted to universal bounded acyclicity. The applications to Thompson groups are discussed in Section 6. Finally, Appendix A contains the proof of Theorem 1.3.

2. Bounded cohomology

We quickly recall basic notions concerning bounded cohomology.

2.1. Definition of bounded cohomology. Let Γ be a group and let $\mathbb{R} \to \ell^\infty(\Gamma^{*+1})$ be the bounded simplicial Γ-resolution of $\mathbb{R}$. More generally, if V is a normed Γ-module, we consider the complex $\ell^\infty(\Gamma^{*+1}, V)$ and set

$$\mathrm{C}_b^*(\Gamma; V) := \ell^\infty(\Gamma^{*+1}, V)^\Gamma.$$

The *bounded cohomology* of Γ with coefficients in V is defined as

$$\mathrm{H}_b^*(\Gamma; V) := \mathrm{H}^*(\mathrm{C}_b^*(\Gamma; V)).$$

The norm on $\mathrm{C}_b^*(\Gamma; V)$ induces a semi-norm on $\mathrm{H}_b^*(\Gamma; V)$, the so-called *canonical semi-norm*.

The canonical inclusion $\mathrm{C}_b^*(\Gamma; V) \hookrightarrow \mathrm{C}^*(\Gamma; V)$ induces a natural transformation between bounded cohomology and ordinary cohomology, the *comparison map*

$$\mathrm{comp}^*_{\Gamma,V} : \mathrm{H}_b^*(\Gamma; V) \to \mathrm{H}^*(\Gamma; V).$$

Further information on the bounded cohomology of groups (and spaces) can be found in the literature [24, 28, 32].

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In Section 5, we also deal with bounded cohomology over different valued fields. Recall that an *absolute value* on a field $\mathbb{K}$ is a multiplicative map $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that $|x| = 0$ if and only if $x = 0$; and the triangle inequality holds: $|x + y| \leq |x| + |y|$. We say that $\mathbb{K}$ is a *complete valued field* if the metric induced by the absolute value is complete. One can then define bounded cohomology over $\mathbb{K}$ in exactly the same way.

If the strong triangle inequality $|x + y| \leq \max\{|x|, |y|\}$ holds for all $x, y \in \mathbb{K}$, then $\mathbb{K}$ is said to be *non-Archimedean*. Concerning the bounded cohomology over non-Archimedean fields [20], we only use the following result.

LEMMA 2.1 [20, Corollary 9.38]. *Let Γ be a group, let $n \geq 1$, and suppose that $\mathrm{H}_{n-1}(\Gamma; \mathbb{Z})$ is finitely generated. Then the comparison map*

$$\mathrm{comp}^n_{\Gamma,\mathbb{K}} : \mathrm{H}^n_b(\Gamma; \mathbb{K}) \to \mathrm{H}^n(\Gamma; \mathbb{K})$$

is injective.

2.2. The uniform boundary condition. We recall the uniform boundary condition, originally due to Matsumoto and Morita [42], and some of its variations [39].

DEFINITION 2.2 (Uniform boundary condition). Let $n \in \mathbb{N}$ and let $\kappa \in \mathbb{R}_{>0}$. A group Γ satisfies the (n, κ)-*uniform boundary condition*, or simply (n, κ)-UBC, if for every $z \in \operatorname{im} \partial_{n+1} \subset \mathrm{C}_n(\Gamma; \mathbb{R})$, there exists a chain $c \in \mathrm{C}_{n+1}(\Gamma; \mathbb{R})$ with

$$\partial_{n+1} c = z \quad \text{and} \quad \|c\|_1 \leq \kappa \cdot \|z\|_1.$$

A group Γ satisfies n-UBC if it satisfies (n, κ)-UBC for some $\kappa \in \mathbb{R}_{>0}$.

The uniform boundary condition can lead to bounded acyclicity.

THEOREM 2.3 [42, Theorem 2.8]. *Let Γ be a group and let $n \in \mathbb{N}$. Then, the following are equivalent:*

(1) *the group Γ satisfies n-UBC;*
(2) *the comparison map $\mathrm{comp}^{n+1}_{\Gamma,\mathbb{R}} : \mathrm{H}^{n+1}_b(\Gamma; \mathbb{R}) \to \mathrm{H}^{n+1}(\Gamma; \mathbb{R})$ is injective.*

In particular: every acyclic group that satisfies UBC *in all positive degrees is boundedly acyclic.*

In the proof of Theorem 1.3, it is useful to extend the definition of UBC from groups to group homomorphisms [39, Definition 4.5].

DEFINITION 2.4 (UBC for homomorphisms). Let $n \in \mathbb{N}$ and let $\kappa \in \mathbb{R}_{>0}$. A group homomorphism $\varphi : H \to \Gamma$ satisfies the (n, κ)-uniform boundary condition, or simply (n, κ)-UBC, if there exits a linear map

$$S : \partial_{n+1}(\mathrm{C}_{n+1}(H; \mathbb{R})) \to \mathrm{C}_{n+1}(\Gamma; \mathbb{R})$$

with

$$\partial_{n+1} \circ S = \mathrm{C}_n(\varphi; \mathbb{R}) \quad \text{and} \quad \|S\| \leq \kappa.$$

Here $\|S\|$ is the operator norm of S with respect to the ℓ^1-norms.

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The uniform boundary condition will be more systematically reviewed in a forthcoming paper [38].

3. Binate groups (alias pseudo-mitotic groups)

We recall basic notions, properties, and examples of binate (alias pseudo-mitotic) groups. We begin with the original definition given by Berrick [4].

DEFINITION 3.1 (Binate). Let Γ be a group. We say that Γ is *binate* if for every finitely generated subgroup $H \leq \Gamma$, there exists a homomorphism $\varphi : H \to \Gamma$ and an element $g \in \Gamma$ such that for every $h \in H$, we have

$$h = [g, \varphi(h)] = g^{-1}\varphi(h)^{-1}g\varphi(h).$$

We will rather work with the equivalent notion of pseudo-mitotic groups, introduced by Varadarajan [55]; here an extra homomorphism $H \to \Gamma$ is taken as part of the structure, which leads to more transparent proofs. We refer the reader to the literature [8, Remark 2.3] for a proof of the equivalence, and point out that the terminology *binate* is more commonly used.

DEFINITION 3.2 (Pseudo-mitosis). Let Γ be a a group and let $H \leq \Gamma$ be a subgroup. We say that H has a *pseudo-mitosis* in Γ if there exist homomorphisms $\psi_0 : H \to \Gamma$, $\psi_1 : H \to \Gamma$ and an element $g \in \Gamma$ such that:

(1) for every $h \in H$, we have $h\psi_1(h) = \psi_0(h)$;
(2) for all $h, h' \in H$, we have $[h, \psi_1(h')] = 1$;
(3) for every $h \in H$, we have $\psi_1(h) = g^{-1}\psi_0(h)g$.

Here is an intuitive interpretation of the definition. There exists a homomorphism $\psi_1 : H \to \Gamma$ whose image commutes with H. This induces a homomorphism

$$H \times H \to \Gamma : (h, h') \mapsto h\psi_1(h').$$

Precomposing it with the diagonal inclusion $h \mapsto (h, h)$, we get a second homomorphism $\psi_0 : h \mapsto h\psi_1(h)$. In terms of acyclicity, the crucial condition is the third item: ψ_0 and ψ_1 are conjugate inside Γ.

DEFINITION 3.3 (Pseudo-mitotic group). A group Γ is said to be *pseudo-mitotic* if all finitely generated subgroups of Γ admit a pseudo-mitosis in Γ.

Varadarajan [55] and Berrick [4] independently showed the following fundamental result.

THEOREM 3.4 [55, Theorem 1.7]. *All pseudo-mitotic groups are acyclic.*

In the present article, we prove that pseudo-mitotic groups are also examples of boundedly acyclic groups (Theorem 1.3).

THEOREM 3.5. *All pseudo-mitotic groups are boundedly acyclic.*

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Since the proof is rather technical and it closely follows those of Matsumoto–Morita [42] and Löh [39], we postpone it to Appendix A.

REMARK 3.6. It is an easy consequence of Theorem 3.4 that pseudo-mitotic groups are 2-boundedly acyclic, namely that if Γ is a pseudo-mitotic group, then $\mathrm{H}^2_b(\Gamma;\mathbb{R}) \cong 0$. Indeed, if Γ is a pseudo-mitotic group and $h \in \Gamma$, by definition, there exist homomorphisms $\psi_0, \psi_1 : \langle h \rangle \to \Gamma$ and an element $g \in \Gamma$ such that

$$h = \psi_0(h)\psi_1(h)^{-1} = \psi_0(h)g^{-1}\psi_0(h)^{-1}g = [\psi_0(h)^{-1}, g].$$

In fact, this commutator expression is the one appearing in the definition of binate groups (Definition 3.1). Hence, every element in a pseudo-mitotic group is a commutator and so the second comparison map is injective [2]. This shows that $\mathrm{H}^2_b(\Gamma;\mathbb{R})$ embeds into $\mathrm{H}^2(\Gamma;\mathbb{R})$, which vanishes by Theorem 3.4.

3.1. Examples. We present several examples of pseudo-mitotic groups. A more detailed discussion of these examples can be found in Berrick's work [5].

We start with a combinatorial construction of pseudo-mitotic groups containing a given group.

EXAMPLE 3.7 (Binate tower). Let H be a group. Set $H_0 := H$, and construct H_{i+1} inductively by performing HNN extensions of $H_i \times H_i$ so that the embedding of H_i in the first coordinate is a pseudo-mitosis. More precisely, if

$$H_{i+1} := \langle H_i \times H_i; g_{i+1} \mid g_{i+1}^{-1}(h,h)g_{i+1} = (1,h) : h \in H_i \rangle,$$

then $h \mapsto (h,1)$ is a pseudo-mitotic embedding of H_i in H_{i+1}.

By construction, the direct limit of the H_i is pseudo-mitotic. It is the initial object in a category of pseudo-mitotic groups containing H [4].

This example shows that every group embeds into a pseudo-mitotic group. We see in the next section that a less canonical construction leads to embeddings with more special properties (Proposition 4.4).

The following example allows us to construct new binate groups from old ones.

EXAMPLE 3.8. Let $(\Gamma_i)_{i\in I}$ be a family of binate groups. Then their direct product $\prod_{i\in I} G_i$ is binate [49, Proposition 1.7].

We soon see that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is binate. Therefore, the previous example shows that $\mathrm{Homeo}_c(\mathbb{R}^n)^{\mathbb{N}}$ is binate, whence boundedly acyclic. A direct proof of bounded acyclicity is given by Monod and Nariman [46].

For comparison, note that an arbitrary direct product of amenable groups need not be amenable. For instance, if Γ is a nonamenable residually finite group, such as a nonabelian free group, then Γ embeds into the direct product of its finite quotients, which is therefore not amenable.

3.1.1. Dissipated groups. Let us move to more concrete examples. Varadarajan proved that the group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of compactly supported homeomorphisms of $\mathbb{R}^n$

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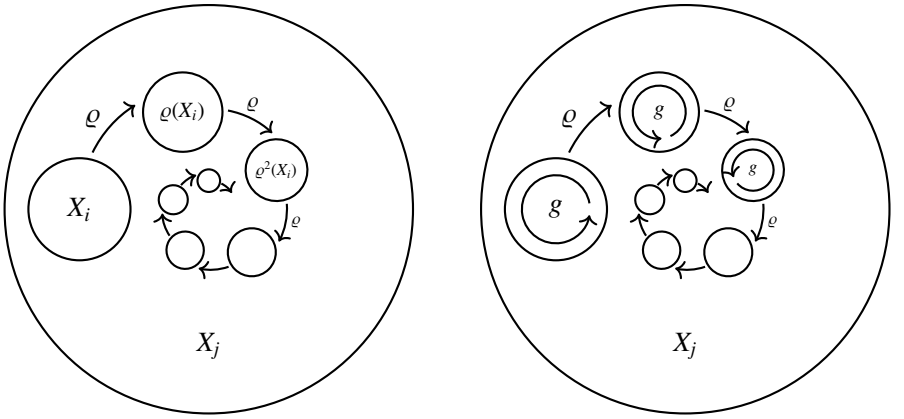

FIGURE 1. Dissipation, schematically: left, the subsets $\varrho^k(X_i)$; right, the action of $\varphi(g)$.

is pseudo-mitotic [55, Theorem 2.2]. Following Berrick [5], we show here that this is just an instance of the behavior of a larger class of groups: dissipated boundedly supported transformation groups.

DEFINITION 3.9 (Boundedly supported group). Let Γ be a group acting faithfully on a set X, which is expressed as a directed union of subsets $(X_i)_{i\in I}$. For each i, let $\Gamma_i := \{g \in \Gamma \mid g \text{ is supported on } X_i\}$. We say that Γ is *boundedly supported* if Γ is the directed union of the Γ_i.

The key property that makes certain boundedly supported groups pseudo-mitotic is the following.

DEFINITION 3.10 (Dissipators). Let $\Gamma \curvearrowright X$ and $(X_i, \Gamma_i)_{i\in I}$ be as in Definition 3.9. Let $i \in I$. A *dissipator* for Γ_i is an element $\varrho_i \in \Gamma$ such that:

(1) $\varrho_i^k(X_i) \cap X_i = \emptyset$ for all $k \geq 1$;
(2) for all $g \in \Gamma_i$, the bijection of X defined by

$$\varphi_i(g) := \begin{cases} \varrho_i^k g \varrho_i^{-k} & \text{on } \varrho^k(X_i), \text{ for every } k \geq 1; \\ \mathrm{id} & \text{elsewhere,} \end{cases} \qquad (*)$$

is in Γ.

If for each $i \in I$ there exists a dissipator for Γ_i, we say that Γ is *dissipated*.

For ϱ_i to be a dissipator, the element $\varphi_i(g)$ needs to belong to Γ, and the boundedly supported hypothesis implies that there exists $j \in I$ such that $\varrho_i^k(X_i) \subset X_j$ for all $k \geq 1$. Figure 1 illustrates this situation.

The presence of dissipators is enough to ensure that the group is pseudo-mitotic.

PROPOSITION 3.11 [5, Section 3.1.6]. *Dissipated groups are pseudo-mitotic.*

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PROOF. Let $\Gamma \curvearrowright X$ and $(X_i, \Gamma_i, \varrho_i)_{i\in I}$ be as in the definition of a dissipated group (Definition 3.10). Let $H \leq \Gamma$ be a finitely generated subgroup. Since Γ is boundedly supported, there exists an $i \in I$ such that $H \leq \Gamma_i$. Notice that H commutes with $\varphi_i(H)$ (as defined in Equation (*)) since their supports are disjoint in X. Hence, if we define $\psi_1 : H \to \Gamma$ as $\psi_1(h) := \varphi_i(h)$, it is immediate to check that ψ_1 is a homomorphism and that $[h', \psi_1(h)] = 1$ for all $h, h' \in H$. We then set $\psi_0 := \varrho_i^{-1}\psi_1\varrho_i : H \to \Gamma$ and $g := \varrho_i^{-1} \in \Gamma$. By construction, this implies that $h\psi_1(h) = \psi_0(h)$ for all $h \in H$. Hence, ψ_0, ψ_1 and g are the witnesses of a pseudo-mitosis of H in Γ. □

A more topological version of this criterion is described by Sankaran and Varadarajan [50, Theorem 1.5]. Many boundedly supported groups are dissipated, and quite surprisingly, this is usually easy to check. We list some examples for which dissipators can be computed directly. More details and further constructions can be found in Berrick's paper [5, Section 3.1.6] and the references therein, as well as in that of Sankaran–Varadarajan [50].

EXAMPLE 3.12 (Dissipated groups). The following groups are dissipated.

(1) The group $\mathrm{Homeo}_c(\mathbb{R}^n)$ of compactly supported homeomorphisms of $\mathbb{R}^n$ is dissipated. This is already contained in a paper of Schreier and Ulam [52], where they study this phenomenon for the (isomorphic) group of homeomorphisms of the n-ball in $\mathbb{R}^n$ fixing a neighborhood of the boundary. Acyclicity was shown by Mather [40], and the proof serves as a model for the proof of acyclicity of pseudo-mitotic groups [55].
(2) The previous example generalizes to certain groups of boundedly supported homeomorphisms of topological manifolds [19] and C^1-manifolds [18].
(3) Let C be the standard Cantor set, embedded in $[0, 1]$. Then the group of homeomorphisms of C that are the identity in a neighborhood of 0 and 1 is dissipated [50, Theorem 2.4].
(4) Let $\mathbb{Q}$ be endowed with the topology as a subspace of $\mathbb{R}$. Then, the group of homeomorphisms of $\mathbb{Q}$ having support contained in some interval $[a, b]$ with $a < b \in \mathbb{Q}$ is dissipated. The same holds for the space of irrational numbers [50, Theorem 1.13].
(5) Forgetting the topology, denote by $\mathrm{Aut}(\mathbb{Q})$ the group of bijections of $\mathbb{Q}$ whose support is contained in some interval (a, b) with $a < b \in \mathbb{Q}$. Then $\mathrm{Aut}(\mathbb{Q})$ is dissipated [49, Theorem 3.2]. The same holds for groups of bijections of infinite sets with similar properties.

3.1.2. Flabby groups. Another source of examples are flabby groups.

DEFINITION 3.13 (Flabby group). A group Γ is *flabby* if there exist homomorphisms $\oplus : \Gamma \times \Gamma \to \Gamma$ and $\tau : \Gamma \to \Gamma$ such that for every finitely generated subgroup $H \leq \Gamma$, there exist $a, b, c \in \Gamma$ such that for all $h \in H$:

(1) $h \oplus 1 = a^{-1}ha$;
(2) $1 \oplus h = b^{-1}hb$;
(3) $h \oplus \tau(h) = c^{-1}\tau(h)c$.

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Flabby groups were defined by Wagoner [56], who proved that they are acyclic. In fact, the following stronger result is true.

LEMMA 3.14 [4, Section 3.3]. *Flabby groups are pseudo-mitotic.*

PROOF. Let Γ be a flabby group and $H \leq \Gamma$ a finitely generated subgroup. Let $\oplus, \tau, a, b, c$ be as in the definition of a flabby group. We define $\psi_1(h) := a(1 \oplus \tau(h))a^{-1}$. Then, since $1 \oplus \Gamma$ commutes with $\Gamma \oplus 1$, we have $[h', \psi_1(h)] = 1$ for all $h, h' \in H$. Let $\psi_0(h) := h\psi_1(h)$ for every $h \in H$. Then,

$$\psi_0(h) = ac^{-1}b\psi_1(h)b^{-1}ca^{-1}$$

for every $h \in H$. By setting $g := ac^{-1}b$, we get the thesis. □

The definition of a flabby group is more restrictive than that of a pseudo-mitotic group, since the homomorphisms $\oplus$ and τ impose some uniformity in the choices of the homomorphisms ψ_0 and ψ_1. Still, there are several examples of flabby groups in the literature.

EXAMPLE 3.15 (Flabby groups). The following groups are flabby.

(1) To study the algebraic K theory of a ring R, Wagoner [56] embeds R into another ring CR called the *cone over* R. Then the direct limit general linear group $\mathrm{GL}(CR)$ is shown to be flabby, whence acyclic.
(2) Building on the work of Wagoner, several other examples of flabby groups are exhibited by de la Harpe and McDuff [17], and they all have the following flavor. Let V be an (infinite-dimensional) Hilbert space, and let

$$V = S_0 \supset S_1 \supset \cdots \supset S_i \supset \cdots$$

be a chain of closed subspaces such that S_i/S_{i-1} is isomorphic to V for all i. Let $\mathrm{GL}(V)$ be the group of continuous linear isomorphisms of V, and let us define $\Gamma_i := \{g \in \mathrm{GL}(V) \mid g(S_i^{\perp}) = S_i^{\perp}\}$. Then the direct limit of the Γ_i terms is flabby. The same holds if one restricts to unitary operators.
(3) Certain groups of automorphisms of measure spaces fall into the framework of the previous item, and thus are flabby [17].

3.1.3. Mitotic groups. Pseudo-mitotic groups were introduced by Varadarajan [55] as a generalization of a more restricted class, that of *mitotic groups*, introduced by Baumslag *et al.* [1]. Let us recall the following definition.

DEFINITION 3.16 (Mitosis). Let Γ be a group and let $H \leq \Gamma$ be a subgroup. We say that H has a *mitosis* in Γ if there exist elements $s, d \in \Gamma$ such that:

(1') for every $h \in H$, we have $h \cdot s^{-1}hs = d^{-1}h\,d$;
(2') for all $h, h' \in H$, we have $[h, s^{-1}h's] = 1$.

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DEFINITION 3.17 (Mitotic). A group Γ is said to be *mitotic* if all finitely generated subgroups of Γ admit a mitosis in Γ.

The main examples of mitotic groups are algebraically closed groups [1, Theorem 4.3]; moreover, a functorial embedding analogous to Example 3.7 is also possible for mitotic groups.

If H has a mitosis in Γ, then H also has a pseudo-mitosis in Γ: indeed, we can choose ψ_0 to be a conjugation by d, ψ_1 to be a conjugation by s, and $g := s^{-1}d$. Therefore, every mitotic group is pseudo-mitotic. However, the class of pseudo-mitotic groups is strictly larger, as proved by Sankaran and Varadarajan [49]. Since Theorem 1.3 generalizes the bounded acyclicity of mitotic groups [39] to the class of pseudo-mitotic groups, let us show an explicit example of a group that is pseudo-mitotic but not mitotic.

LEMMA 3.18. *The group* $\mathrm{Homeo}_c(\mathbb{R})$ *is not mitotic.*

PROOF. Let H be a finitely generated group acting minimally on $(0,1)$: for concreteness, one could take H to be the Thompson group F. We embed H inside $\mathrm{Homeo}_c(\mathbb{R})$ by letting H act trivially on $\mathbb{R} \setminus (0,1)$. Let $s \in \mathrm{Homeo}_c(\mathbb{R})$ be an element such that H commutes with $s^{-1}Hs$. Note that $s^{-1}Hs$ is supported on $s^{-1}(0,1)$ and also acts minimally. So, to commute, $s^{-1}(0,1)$ must be disjoint from $(0,1)$.

It follows that the diagonal group $\{hs^{-1}hs \mid h \in H\}$ is supported on the disconnected set $(0,1) \cup s^{-1}(0,1)$, and therefore cannot be conjugate to H by an element in $\mathrm{Homeo}_c(\mathbb{R})$. This shows that $\mathrm{Homeo}_c(\mathbb{R})$ is not mitotic. □

With a little more work, this kind of argument can be applied to many of the groups from Example 3.12. Indeed, it is known that $\mathrm{Homeo}_c(\mathbb{R}^n)$ is not mitotic [49], but it is pseudo-mitotic (Example 3.12).

3.1.4. Amenable examples. Most of the examples that we have seen so far are nonamenable, since they contain free subgroups. In particular, their bounded acyclicity does not follow from the classical result for amenable groups. However, there are some exceptions. The first one is due to Berrick.

EXAMPLE 3.19. Hall's countable universal locally finite group [29] is pseudo-mitotic [4, Section 3.1]. Being locally finite, it is amenable.

Locally finite groups cannot be dissipated, since by construction, the dissipators must have infinite order. In the next example, we construct a dissipated amenable group.

EXAMPLE 3.20. We start with $\Gamma_1 = X_1 = \mathbb{Z}$, with the action given by left translation. Of course, Γ_1 is amenable.

Next, let X_2 be the disjoint union of countably many copies of X_1 indexed by $\mathbb{Z}$, which contains a distinguished copy of X_1, indexed by 0. The direct product $\Gamma_1^{\mathbb{Z}}$ acts on X_2 coordinate-wise. Let ϱ_1 be the bijection of X_2 shifting the copies of X_1. We set Γ_2 to be the group generated by the direct product $\Gamma_1^{\mathbb{Z}}$ and ϱ_1. Note that Γ_2 splits as a semidirect product $\Gamma_1^{\mathbb{Z}} \rtimes \langle \varrho_1 \rangle$, so it is 2-step-solvable. Moreover, given $g \in \Gamma_1$, the

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element $\varphi_1(g)$ from Definition 3.10 is just $(g_n)_{n\in\mathbb{Z}} \in \Gamma_1^{\mathbb{Z}} \leq \Gamma_2$ with $g_n = g$ for $n \geq 0$ and $g_n = 0$ otherwise.

By induction, if Γ_i and X_i have been constructed, we construct X_{i+1} as the disjoint union of $\mathbb{Z}$-many copies of X_i, we let ϱ_i be the shift, and define Γ_{i+1} as the group generated by the direct product of the Γ_i and ϱ_i. Then $\Gamma_{i+1} \cong \Gamma_i^{\mathbb{Z}} \rtimes \mathbb{Z}$ is $(i+1)$-step-solvable. Moreover, given $g \in \Gamma_i$, the element $\varphi_i(g)$ again belongs to $\Gamma_i^{\mathbb{Z}} \leq \Gamma_{i+1}$.

The directed union Γ of the Γ_i acts on the directed union X of the X_i. This action is boundedly supported by definition, and the ϱ_i are dissipators by construction. Finally, Γ is a directed union of solvable groups, so it is amenable.

REMARK 3.21. In the construction in Example 3.20, we cannot pick any amenable group Γ_1, since a direct power of amenable groups need not be amenable in general. Indeed, we strongly use the fact that a direct power of an i-step-solvable group is still i-step-solvable. By the same argument, we could have started with any group satisfying an amenable law.

4. Hereditary properties of boundedly acyclic groups

In this section, we discuss the stability of bounded acyclicity under certain operations. We present new results concerning normal subgroups and directed unions. The case of normal subgroups makes use of pseudo-mitotic groups, showcasing their versatility compared with mitotic groups.

4.1. Extensions. We start with the operation of taking extensions of boundedly acyclic groups. This behaves particularly well.

THEOREM 4.1 [48, Corollary 4.2.2]. *Let $1 \to N \to \Gamma \to Q \to 1$ be an exact sequence of groups, where $\varphi : \Gamma \to Q$ denotes the quotient map, and let $n \in \mathbb{N}$. Suppose that N is n-boundedly acyclic. Then Γ is n-boundedly acyclic if and only if Q is n-boundedly acyclic.*

In particular, the class of n-boundedly acyclic groups is closed under extensions. This generalizes the classical results for extensions with amenable kernels [28] and amenable quotients [35, 47].

A natural question is then whether a 2-out-of-3 property holds.

QUESTION 4.2. In an extension $1 \to N \to \Gamma \to Q \to 1$, suppose that Γ and Q are n-boundedly acyclic. Is N necessarily n-boundedly acyclic?

A characterization of when this occurs is available [48, Corollary 4.2.1], but it is given in terms of a vanishing of bounded cohomology with a larger class of coefficients, and so it does not settle Question 4.2 in either direction. We answer Question 4.2 in the negative in Theorem 4.5.

4.2. Normal subgroups. Pseudo-mitotic groups are varied enough that they allow for several strong embedding constructions. The following is the most relevant one.

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EXAMPLE 4.3 (Cone over a group). If Γ is a group, let $\Gamma^{\mathbb{Q}}$ be the group of functions from $\mathbb{Q}$ to Γ that map all numbers outside some finite interval to the neutral element. The group $C\Gamma := \Gamma^{\mathbb{Q}} \rtimes \mathrm{Aut}(\mathbb{Q})$, where $\mathrm{Aut}(\mathbb{Q})$ (defined in Example 3.12) acts on $\Gamma^{\mathbb{Q}}$ by shifting the coordinates, is pseudo-mitotic, in fact dissipated [4, Section 3.5]. The group $C\Gamma$ is called the *cone over* Γ, and was introduced by Kan and Thurston [36] as a key step in the proof of their celebrated theorem.

PROPOSITION 4.4 [4, Section 3.5]. *Every group embeds 2-step subnormally in a pseudo-mitotic group. More precisely, for every group Γ, there exists a group Γ^0 such that $C\Gamma \cong (\Gamma \times \Gamma^0) \rtimes \mathrm{Aut}(\mathbb{Q})$ is pseudo-mitotic.*

PROOF. By Example 4.3, it suffices to show that $\Gamma^{\mathbb{Q}} \cong \Gamma \times \Gamma^0$, for some group Γ^0. The group Γ embeds normally in $\Gamma^{\mathbb{Q}}$ as the subgroup of functions $\mathbb{Q} \to \Gamma$ that map every nonzero rational to the identity. We then set Γ^0 to be the subgroup of functions that map $0 \in \mathbb{Q}$ to $1 \in \Gamma$. □

In particular, every group embeds 2-step subnormally in a boundedly acyclic group. Embeddings into boundedly acyclic groups have been considered before [22, 39], but Proposition 4.4 goes one step further and provides a strong negative answer to Question 4.2.

THEOREM 4.5. *There exists a boundedly acyclic group Γ with a normal subgroup H such that Γ/H is boundedly acyclic, but $\mathrm{H}_b^n(H;\mathbb{R})$ is continuum-dimensional for every $n \geq 2$.*

Groups such as H above are said to have *large bounded cohomology*: countable [39] and even finitely presented [22] examples are known to exist.

PROOF. Let H be a group with large bounded cohomology. Then, for every group Λ, the direct product $H \times \Lambda$ also has large bounded cohomology (as it retracts onto a group with large bounded cohomology). By Proposition 4.4 and Example 3.12, this implies that the pseudo-mitotic group $\Gamma := CH$ provides the desired example. □

Even without the additional hypothesis on the quotient, it seems that Theorem 4.5 gives the first example of a nonboundedly acyclic normal subgroup of a boundedly acyclic group. Indeed, subgroups of amenable groups are amenable, and several of the nonamenable examples of boundedly acyclic groups available in the literature are simple, so they cannot provide counterexamples. For instance, $\mathrm{Homeo}_c(\mathbb{R}^n)$ is simple, as are many other groups of boundedly supported homeomorphisms [18].

4.3. Quotients. An intriguing open problem is whether boundedly acyclic groups are closed under passage to quotients [22, Section 3.2]. One of the main difficulties about this problem is that mitotic groups behave extremely well with respect to quotients.

LEMMA 4.6 [1, page 16]. *Mitotic groups are closed under passage to quotients.*

However, the same behavior does not hold for pseudo-mitotic groups.

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LEMMA 4.7 [49, Theorem 3.3]. *Pseudo-mitotic groups are not closed under passage to quotients.*

This suggests that pseudo-mitotic groups might be useful for constructing counterexamples for the problem above. However, the example considered by Sankaran and Varadarajan [49, Theorem 3.3] still produces a boundedly acyclic quotient. Indeed, in this situation, the kernel of the epimorphism is the group of finitely supported permutations of $\mathbb{N}$, which is locally finite and therefore amenable. Hence, by Theorem 4.1 (or simply by Gromov's mapping theorem [28]), the quotient is a boundedly acyclic nonpseudo-mitotic group.

A more interesting situation arises from the context of algebraic K-theory. Indeed, following Berrick [3], given a (unital, associative) ring R, one can embed it into its cone CR as a two-sided ideal. This leads to a short exact sequence

$$1 \to \mathrm{GL}(R) \to \mathrm{GL}(CR) \to Q \to 1,$$

where Q is the direct general linear group over the suspension of R, usually denoted by $\mathrm{GL}(SR)$ [3, page 85]. As discussed in Example 3.15(1), the group $\mathrm{GL}(CR)$ is a flabby group, whence pseudo-mitotic (in fact, this is also a dissipated group as proved by Berrick [3, pages 84–85]). However, since one can compute the K-groups of the original ring R in terms of the plus construction over $\mathrm{GL}(SR)$, in general, $\mathrm{GL}(SR)$ is far from being acyclic [3]. Hence, the following is a natural question.

QUESTION 4.8. Let R be a ring. Is the group $\mathrm{GL}(SR)$ boundedly acyclic?

A negative answer to this question would show that boundedly acyclic groups are not closed under passage to quotients. However, here we prove the following proposition.

PROPOSITION 4.9. *Let R be a ring. Then, the group* $\mathrm{GL}(SR)$ *is* 3*-boundedly acyclic.*

PROOF. Recall that a group extension provides an exact sequence in bounded cohomology in low degrees [43, Corollary 12.4.1 and Example 12.4.3], which in this case gives

$$\begin{aligned} 0 \to \mathrm{H}_b^2(\mathrm{GL}(SR);\mathbb{R}) \to \mathrm{H}_b^2(\mathrm{GL}(CR);\mathbb{R}) \to \mathrm{H}_b^2(\mathrm{GL}(R);\mathbb{R})^{\mathrm{GL}(SR)} \\ \to \mathrm{H}_b^3(\mathrm{GL}(SR);\mathbb{R}) \to \mathrm{H}_b^3(\mathrm{GL}(CR);\mathbb{R}). \end{aligned}$$

Using the fact that $\mathrm{GL}(CR)$ is pseudo-mitotic, whence boundedly acyclic (Theorem 1.3), we then have

$$\mathrm{H}_b^2(\mathrm{GL}(SR);\mathbb{R}) \cong 0$$

and

$$\mathrm{H}_b^2(\mathrm{GL}(R);\mathbb{R})^{\mathrm{GL}(SR)} \cong \mathrm{H}_b^3(\mathrm{GL}(SR);\mathbb{R}).$$

We show that $\mathrm{H}_b^2(\mathrm{GL}(R);\mathbb{R}) \cong 0$, whence the thesis.

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It suffices to show that $\mathrm{GL}(R)$ has *commuting conjugates* [21]; that is, for every finitely generated subgroup $H \leq \mathrm{GL}(R)$, there exists $g \in \mathrm{GL}(R)$ such that H and $g^{-1}Hg$ commute. Now let $H \leq \mathrm{GL}(R)$ be finitely generated. Then there exists some $n \geq 1$ such that $H \leq \mathrm{GL}_n(R)$. Let $g \in \mathrm{GL}_{2n}(R) \leq \mathrm{GL}(R)$ be a permutation matrix that swaps the basis vectors $e_1, \dots, e_n$ with $e_{n+1}, \dots, e_{2n}$. Then $g^{-1}Hg$ acts trivially on the span of $e_1, \dots, e_n$ and H acts trivially on the span of $e_{n+1}, \dots, e_{2n}$; therefore, these subgroups commute. We conclude that $\mathrm{GL}(R)$ has commuting conjugates and so it is 2-boundedly acyclic [21]. This finishes the proof. □

4.4. Directed unions. The operations we have looked at so far are known to preserve amenability. This is not surprising since amenable groups are the most illustrious examples of boundedly acyclic groups. One further operation that preserves amenability is that of directed unions. Here we study the behavior of bounded acyclicity under directed unions, and show that it is preserved under an additional technical requirement.

To proceed with the proof, it is convenient to consider the following dual version of UBC.

DEFINITION 4.10. Let $n \in \mathbb{N}$ and let Γ be a group such that $\mathrm{H}_b^n(\Gamma; \mathbb{R}) \cong 0$. We define the nth *vanishing modulus* of Γ as the minimal $K \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that the following holds:

for each $c \in \ker(\delta_b^n)$, there exists $b \in \mathrm{C}_b^{n-1}(\Gamma; \mathbb{R})$ such that

$$\delta_b^{n-1}(b) = c \quad \text{and} \quad |b|_\infty \leq K \cdot |c|_\infty.$$

EXAMPLE 4.11. Every amenable group Γ has an nth vanishing modulus of 1, for all $n \geq 1$. Indeed, the proof of bounded acyclicity of amenable groups [24, Theorem 3.6] exhibits a contracting chain homotopy τ for the cochain complex $\mathrm{C}_b^*(\Gamma; \mathbb{R})$, which has norm 1 in every degree. Hence, given $c \in \ker(\delta_b^n)$, we can just set $b := \tau^n(c)$ and obtain:

$$\delta_b^{n-1}(b) = \delta_b^{n-1}\tau^n(c) + \tau^{n+1}\delta_b^n(c) = c.$$

In our definition, the vanishing modulus takes values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. It turns out that only finite values are possible.

LEMMA 4.12. *Let $n \in \mathbb{N}$ and let Γ be such that $\mathrm{H}_b^n(\Gamma; \mathbb{R}) \cong 0$. Then the nth vanishing modulus of Γ is finite.*

PROOF. This is implicit in the work of Matsumoto and Morita [42]: because of $\mathrm{H}_b^n(\Gamma; \mathbb{R}) \cong 0$, we have $\operatorname{im} \delta_b^{n-1} = \ker \delta_b^n$. Hence, the bounded linear map δ_b^{n-1} has closed range; by the open mapping theorem, δ_b^{n-1} induces a Banach space isomorphism

$$\overline{\delta}_b^{n-1} : \mathrm{C}_b^{n-1}(\Gamma; \mathbb{R}) / \ker \delta_b^{n-1} \to \ker \delta_b^n.$$

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Let φ^n be the inverse of $\overline{\delta}_b^{n-1}$. If $c \in \ker \delta_b^n$, then the definition of the quotient norm on $\mathrm{C}_b^{n-1}(\Gamma;\mathbb{R})/\ker \delta_b^{n-1}$ shows that there exists a $b \in \mathrm{C}_b^{n-1}(\Gamma;\mathbb{R})$ with

$$\delta_b^{n-1}(b) = c \qquad \text{and} \qquad |b|_\infty \leq 2 \cdot \|\varphi^n\| \cdot |c|_\infty.$$

Thus, the constant $2 \cdot \|\varphi^n\|$ is a finite upper bound for the nth vanishing modulus. □

PROPOSITION 4.13. *Let Γ be a group that is the directed union of a directed family $(\Gamma_i)_{i\in I}$ of subgroups. Moreover, let $n \in \mathbb{N}$ and suppose that $\mathrm{H}_b^n(\Gamma_i;\mathbb{R}) \cong 0$ for all i, and that there is a uniform, finite upper bound for the nth vanishing moduli of all the Γ_i terms. Then $\mathrm{H}_b^n(\Gamma;\mathbb{R}) \cong 0$.*

PROOF. Let $K < +\infty$ be a common upper bound for the nth vanishing moduli of the Γ_i terms. We show that the nth vanishing modulus of Γ is at most K. Let $c \in \mathrm{C}_b^n(\Gamma;\mathbb{R})$ be a bounded cocycle. For each $i \in I$, we set

$$B_i := \{b \in \mathrm{C}_b^{n-1}(\Gamma;\mathbb{R}) \mid \delta_b^{n-1}(b|_{\Gamma_i}) = c|_{\Gamma_i} \text{ and } |b|_\infty \leq K \cdot |c|_\infty\}.$$

It suffices to show that $\bigcap_{i\in I} B_i \neq \emptyset$. To this end, we use the Banach–Alaoglu theorem: by construction, each B_i is a bounded weak $*$-closed subset of $\mathrm{C}_b^{n-1}(\Gamma;\mathbb{R})$ and $B_j \subset B_i$ for all $j \in I$ with $i \leq j$. Moreover, $B_i \neq \emptyset$: by hypothesis, there exists $b_i \in \mathrm{C}_b^{n-1}(\Gamma_i;\mathbb{R})$ with $\delta_b^{n-1}(b_i) = c|_{\Gamma_i}$ and $|b_i|_\infty \leq K \cdot |c|_\infty$. We now extend b_i by 0; this extension lies in B_i.

Because the system is directed, the family $(B_i)_{i\in I}$ satisfies the finite intersection property; by the Banach–Alaoglu theorem, therefore, the whole intersection $\bigcap_{i\in I} B_i$ is nonempty. □

The following special case is used when studying the Thompson group F (Lemma 6.5).

COROLLARY 4.14. *Let Γ be a group that is the directed union of a directed family $(\Gamma_i)_{i\in I}$ of subgroups. Suppose that the Γ_i's are pairwise isomorphic and n-boundedly acyclic. Then Γ is n-boundedly acyclic.*

PROOF. This follows directly from Lemma 4.12 and Proposition 4.13. □

In degree two, we may get rid of the uniformity condition in Proposition 4.13, thanks to the following surprising fact.

PROPOSITION 4.15. *Let Γ be a 2-boundedly acyclic group. Then the second vanishing modulus of Γ is* 1.

PROOF. This is essentially a dual version of a result by Matsumoto and Morita [42, Corollary 2.7]. First, the map δ_b^1 is injective, since the only bounded homomorphism $\Gamma \to \mathbb{R}$ is the trivial one. We consider the map

$$\psi : \ker \delta_b^2 = \operatorname{im} \delta_b^1 \to \mathrm{C}_b^1(\Gamma;\mathbb{R})$$
$$c \mapsto \Big((g_0, g_1) \mapsto \sum_{k=0}^{\infty} 2^{-(k+1)} \cdot c(1, g_{01}^{2^k}, g_{01}^{2^{k+1}})\Big),$$

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where we use the abbreviation $g_{01} := g_0^{-1} \cdot g_1$, and claim that ψ is the inverse of δ_b^1. By definition, $\|\psi\| \leq 1$; moreover, $\|\psi\| = 1$ because δ_b^1 sends constant functions to constant functions.

We are left to prove the claim. Let $c \in \operatorname{im} \delta_b^1$, say $c = \delta_b^1(b)$. We need to show that $\psi(c) = b$: Using Γ-invariance, we obtain for all $g_0, g_1 \in \Gamma$:

$$\begin{aligned}(\psi(c))(g_0, g_1) &= \sum_{k=0}^{\infty} 2^{-(k+1)} \cdot (b(g_{01}^{2^k}, g_{01}^{2^{k+1}}) - b(1, g_{01}^{2^{k+1}}) + b(1, g_{01}^{2^k})) \\ &= \sum_{k=0}^{\infty} 2^{-(k+1)} \cdot (b(1, g_{01}^{2^k}) - b(1, g_{01}^{2^{k+1}}) + b(1, g_{01}^{2^k})) \\ &= \sum_{k=0}^{\infty} 2^{-k} \cdot b(1, g_{01}^{2^k}) - \sum_{k=0}^{\infty} 2^{-(k+1)} \cdot b(1, g_{01}^{2^{k+1}}) \\ &= b(1, g_{01}) = b(g_0, g_1).\end{aligned}$$

Note that all series involved are absolutely convergent because b is bounded, which is what allows us to change the order of summation. □

COROLLARY 4.16. *A directed union of* 2*-boundedly acyclic groups is* 2*-boundedly acyclic.* □

Proposition 4.15 is essentially equivalent to the fact that the canonical semi-norm in degree two is always a norm [42, Corollary 2.7]. This fails already in degree three [23, 53], but such examples also have large bounded cohomology and so are difficult to control. Therefore, we ask the following question.

QUESTION 4.17. Does the analog of Proposition 4.15 hold in higher degrees?

One can use Lemma 4.12 to show that a direct sum of n-boundedly acyclic groups with unbounded vanishing modulus cannot be n-boundedly acyclic. Therefore, a negative answer to this question would imply that, in higher degrees, the uniformity assumption in Proposition 4.13 is necessary.

5. Universal bounded acyclicity

In this section, we show that the bounded acyclicity of pseudo-mitotic groups is not a phenomenon confined to real coefficients. Since several different coefficients are involved in this section, we will be explicit and talk about $\mathbb{Z}$-acyclic groups (Definition 1.2) and $\mathbb{R}$-boundedly acyclic groups (Definition 1.1).

DEFINITION 5.1. Let $\mathbb{K}$ be a complete valued field, and let Γ be a group. We say that Γ is $\mathbb{K}$*-boundedly acyclic* if $\mathrm{H}_b^n(\Gamma; \mathbb{K}) \cong 0$ for all $n \geq 1$. If this holds for all complete valued fields $\mathbb{K}$, we say that Γ is *universally boundedly acyclic*.

We can characterize universal bounded acyclicity in very simple terms.

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THEOREM 5.2. *Let Γ be a group. Then Γ is universally boundedly acyclic if and only if it is $\mathbb{R}$-boundedly acyclic and $\mathbb{Z}$-acyclic.*

REMARK 5.3. In fact, Theorem 5.2 even holds degree-wise. More precisely, for a group Γ and an integer $n \geq 1$, the following are equivalent:

(1) $\mathrm{H}_b^i(\Gamma;\mathbb{R}) \cong 0$ and $\mathrm{H}_i(\Gamma;\mathbb{Z}) \cong 0$ for all $i \in \{1,\dots,n\}$;
(2) $\mathrm{H}_b^i(\Gamma;\mathbb{K}) \cong 0$ for every complete valued field $\mathbb{K}$ and all $i \in \{1,\dots,n\}$.

This will be apparent from the proof, but we prefer to state the theorem in global terms to simplify the notation.

Before giving the proof, we note the following consequence.

COROLLARY 5.4. *Pseudo-mitotic groups are universally boundedly acyclic.*

PROOF. Pseudo-mitotic groups are both acyclic (Theorem 3.4) and boundedly acyclic (Theorem 3.5). Therefore, we can apply Theorem 5.2. □

The proof of Theorem 5.2 is carried out in two steps: the Archimedean and the non-Archimedean case.

LEMMA 5.5. *Let Γ be a group. Then Γ is $\mathbb{R}$-boundedly acyclic if and only if Γ is $\mathbb{C}$-boundedly acyclic.*

PROOF. Because we have $\mathbb{C} \cong \mathbb{R}^2$ as normed $\mathbb{R}$-vector spaces, the cochain complex $\mathrm{C}_b^*(\Gamma;\mathbb{C})$ splits as the direct sum $\mathrm{C}_b^*(\Gamma;\mathbb{R})^{\oplus 2}$. Therefore, we obtain the isomorphism $\mathrm{H}_b^*(\Gamma;\mathbb{C}) \cong \mathrm{H}_b^*(\Gamma;\mathbb{R})^{\oplus 2}$ (over $\mathbb{R}$). The claim easily follows. □

LEMMA 5.6. *Let Γ be a group. Then Γ is $\mathbb{K}$-boundedly acyclic for every non-Archimedean field $\mathbb{K}$ if and only if it is $\mathbb{Z}$-acyclic.*

PROOF. Suppose that Γ is $\mathbb{K}$-boundedly acyclic for every non-Archimedean field $\mathbb{K}$. Endowing an arbitrary field $\mathbb{K}$ with the trivial norm, we deduce that $\mathrm{H}^n(\Gamma;\mathbb{K}) \cong 0$ for every field $\mathbb{K}$. It then follows from the universal coefficient theorem [10, Ch. I] that Γ is $\mathbb{K}$-acyclic for every field $\mathbb{K}$, that is, $\mathrm{H}_n(\Gamma;\mathbb{K}) \cong 0$ for all $n \geq 1$. In particular, Γ is $\mathbb{Q}$-acyclic and $\mathbb{F}_p$-acyclic for every prime p; so Γ is $\mathbb{Z}$-acyclic [30, Corollary 3A.7].

Conversely, let us suppose that Γ is $\mathbb{Z}$-acyclic, and let $\mathbb{K}$ be a non-Archimedean field $\mathbb{K}$. By Lemma 2.1, the comparison map $\mathrm{H}_b^n(\Gamma;\mathbb{K}) \to \mathrm{H}^n(\Gamma;\mathbb{K})$ is injective. So it suffices to show that $\mathrm{H}^n(\Gamma;\mathbb{K}) \cong 0$. This follows immediately from the universal coefficient theorem; hence, Γ is $\mathbb{K}$-boundedly acyclic. □

PROOF OF THEOREM 5.2. By Ostrowski's theorem [16, Ch. 3], every complete valued field is either non-Archimedean or isomorphic to $\mathbb{R}$ or $\mathbb{C}$. So Theorem 5.2 follows from Lemmas 5.5 and 5.6. □ □

REMARK 5.7. Corollary 5.4 provides many examples of groups that are universally boundedly acyclic. One could ask whether something similar could be said for the stronger notion of *universal amenability*, defined analogously using the general notion

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of $\mathbb{K}$-amenability for valued fields defined by Shikhof [51]. However, it turns out that if Γ is $\mathbb{F}_p$-amenable in the sense of Shikhof for every prime p, then Γ is trivial [20, Example 5.5 and Theorem 6.2]. The same holds for the weaker notion of normed $\mathbb{K}$-amenability [20], which also implies bounded $\mathbb{K}$-acyclicity [20, Theorem 1.3].

COROLLARY 5.8. *Let Γ be a universally boundedly acyclic group. Then, for all $n \geq 1$, we have $\mathrm{H}_b^n(\Gamma;\mathbb{Z}) \cong 0$, with the standard absolute value on $\mathbb{Z}$.*

PROOF. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ induces a long exact sequence [24, proof of Proposition 2.13]

$$\cdots \to \mathrm{H}_b^{n-1}(\Gamma;\mathbb{R}) \to \mathrm{H}^{n-1}(\Gamma;\mathbb{R}/\mathbb{Z}) \to \mathrm{H}_b^n(\Gamma;\mathbb{Z}) \to \mathrm{H}_b^n(\Gamma;\mathbb{R}) \to \cdots.$$

By Theorem 5.2, the group Γ is $\mathbb{R}$-boundedly acyclic and $\mathbb{Z}$-acyclic. The universal coefficient theorem and $\mathbb{Z}$-acyclicity give $\mathrm{H}^k(\Gamma;\mathbb{R}/\mathbb{Z}) \cong 0$ for all $k > 0$. Therefore, the long exact sequence and surjectivitiy of the induced map $\mathrm{H}_b^0(\Gamma;\mathbb{R}) \to \mathrm{H}^0(\Gamma;\mathbb{R}/\mathbb{Z})$ show that $\mathrm{H}_b^n(\Gamma;\mathbb{Z}) \cong 0$ for all $n \geq 1$. □

6. Thompson groups and their siblings

The groups F, T, and V were introduced by Richard Thompson in 1965; they are some of the most important groups in geometric and dynamical group theory. These groups can be realized as groups of homeomorphisms of the interval, the circle, and the Cantor set respectively; these realizations exhibit inclusions $F \leq T \leq V$. We refer the reader to the literature [15] for a detailed discussion.

The groups F, T, and V are finitely presented, even of type F_∞. Moreover, T and V are simple (in fact, they were the first examples of infinite finitely presented simple groups). However, F has abelianization $\mathbb{Z}^2$, but its derived subgroup F' is simple, and infinitely generated.

The rational cohomology [9, 26] and, with the exception of T, the integral cohomology [11, 26, 54] of these groups has been computed. However, little is known about their real bounded cohomology. We formulate one question for each group.

QUESTION 6.1. Is the Thompson group F boundedly acyclic?

Question 6.1 is usually attributed to Grigorchuk [27, page 131, Problem 3.19].

QUESTION 6.2. Does the following hold?

The bounded cohomology of the Thompson group T is given by

$$\mathrm{H}_b^n(T;\mathbb{R}) \cong \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{R} & \text{if } n \text{ is even,} \end{cases}$$

where the nontrivial classes are spanned by cup-powers of the bounded real Euler class.

QUESTION 6.3. Is the Thompson group V boundedly acyclic?

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The rest of this section is devoted to discussing these three questions, how they relate to each other, and providing some evidence towards positive answers.

One may also formulate corresponding questions for every degree, namely whether the previous descriptions hold up to degree n. We see that all three questions have a positive answer up to degree two, while to our knowledge, nothing is known from degree three onwards.

6.1. On the bounded cohomology of F. We recall the definition of F.

DEFINITION 6.4. The Thompson group F is the group of orientation-preserving piecewise linear homeomorphisms f of the interval $[0, 1]$ with the following properties:

(1) f has finitely many breakpoints, all of which lie in $\mathbb{Z}[1/2]$;
(2) away from the breakpoints, the slope of f is a power of 2.

The map $F \to \mathbb{Z}, f \mapsto \log_2(f_0)$, where f_0 is the slope of f at 0, is a surjective homomorphism, called the *germ at*0. Similarly, there is a germ at 1, leading to a surjective homomorphism $F \to \mathbb{Z}^2$. This is the abelianization of F, so the derived subgroup F' coincides with the subgroup of homeomorphisms that are compactly supported in $(0, 1)$.

The most important open question about F is whether F is amenable or not. Since amenable groups are boundedly acyclic, a negative answer to Question 6.1 would disprove its amenability. The general philosophy is that F is very close to being amenable, and so it is likely to satisfy most properties that are somewhat weaker than amenability.

For example: the group F is 2-boundedly acyclic. This can be deduced from the explicit description of its rational cohomology [26], by using arguments analogous to those of Heuer and Löh for the computation of the second bounded cohomology of T [31] (although a direct approach is possible [21]). To our knowledge, nothing is known about the bounded cohomology of F with trivial real coefficients in higher degrees, although vanishing is known in every degree with mixing coefficients [44].

The connection between pseudo-mitotic groups and F is more transparent when passing to the derived subgroup. The following equivalent formulation is relevant.

LEMMA 6.5. *Let $n \in \mathbb{N}$. Then the Thompson group F is n-boundedly acyclic if and only if F' is n-boundedly acyclic.*

PROOF. If F' is n-boundedly acyclic, then F is also boundedly acyclic by Theorem 4.1, or more simply by co-amenability [47].

Conversely, let us suppose that F is n-boundedly acyclic. Let $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ be sequences of dyadic rationals in $(0, 1)$ that converge to 0 and 1, respectively. Then F' may be expressed as the directed union of the subgroups F_i consisting of elements supported in $[a_i, b_i]$. Since each group F_i is isomorphic to F, the group F' is a directed union of pairwise isomorphic n-boundedly acyclic groups. It follows from Corollary 4.14 that F' is n-boundedly acyclic. □

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The derived subgroup F' is a group of boundedly supported homeomorphisms of the interval. In analogy with Example 3.12, one may ask whether F' is pseudo-mitotic. This is not the case, because F' is not acyclic [26]. Intuitively, F' cannot be dissipated, since a dissipator could not possibly have finitely many breakpoints. However, a *countably singular* analog of F' is dissipated.

DEFINITION 6.6. Let ΩF be the group of orientation-preserving homeomorphisms f of the interval $[0, 1]$ with the following properties:

(1) there exists a closed and countable set $K \subset (0, 1) \cap \mathbb{Z}[1/2]$ such that f is linear on each component of $[0, 1] \setminus K$;
(2) away from K, the slope of f is a power of 2.

Since the set of breakpoints of each element is contained in $(0, 1)$, the germs at 0 and 1 are still defined, and $\Omega F'$ is the subgroup of homeomorphisms that are compactly supported in $(0, 1)$.

PROPOSITION 6.7. *The groups ΩF and $\Omega F'$ are boundedly acyclic.*

PROOF. Once again, the bounded acyclicity of ΩF follows from that of $\Omega F'$ by Theorem 4.1.

We show that $\Omega F'$ is dissipated. Let $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ be sequences of dyadic rationals in $(0, 1)$ converging to 0 and 1, respectively. For every $i \geq 1$, let $H_i \leq \Omega F'$ be the subgroup consisting of homeomorphisms supported in (a_i, b_i). We show that there exists a dissipator $\varrho_i \in \Omega F'$ for H_i, that is:

(1) for every $k \geq 1$, we have $\varrho_i^k((a_i, b_i)) \cap (a_i, b_i) = \emptyset$;
(2) for every $g \in H_i$, the element

$$\varphi_i(g) := \begin{cases} \varrho_i^k g \varrho_i^{-k} & \text{on } \varrho^k(a_i, b_i), \text{ for every } k \geq 1, \\ \mathrm{id} & \text{elsewhere} \end{cases}$$

is in $\Omega F'$.

To this end, let us set $x_0 := a_i$ and $x_1 := b_i$. We then pick a dyadic rational x_{-1} in $(0, x_0)$ such that

$$\frac{x_1 - x_{-1}}{x_0 - x_{-1}}$$

is a power of 2. Moreover, given a dyadic rational $x \in (x_1, 1)$, we can extend x_{-1}, x_0, x_1 to a sequence $(x_j)_{j\geq -1}$ of dyadic rationals converging to x and such that for every $j \geq 1$, the ratio

$$\frac{x_{j+1} - x_j}{x_j - x_{j-1}}$$

is a power of 2.

Now we define $\varrho_i : [0, 1] \to [0, 1]$ piecewise as follows:

$$\varrho_i|_{[0,x_{-1}]\cup[x,1]} := \mathrm{id}\,|_{[0,x_{-1}]\cup[x,1]},$$

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$$\varrho_i([x_{-1}, x_0]) := [x_{-1}, x_1],$$
$$\varrho_i([x_{j-1}, x_j]) := [x_j, x_{j+1}] \quad \text{for } j \geq 1,$$

and let ϱ_i be the unique affine isomorphism on each of these pieces. Notice that ϱ_i is supported in $[x_{-1}, x] \subset (0, 1)$, and the set $\{x, x_{-1}, x_0, x_1, \ldots\}$ of breakpoints is closed, countable, and consists only of dyadic rationals. Since all the slopes are powers of 2, this implies that $\varrho_i \in \Omega F'$.

We claim that ϱ_i is a dissipator for H_i. First, notice that by construction and the definition of x_0 and x_1, we have

$$\varrho_i^k(a_i, b_i) \cap (a_i, b_i) = (x_k, x_{k+1}) \cap (x_0, x_1) = \emptyset$$

for every $k \geq 1$. This shows that ϱ_i satisfies property (1).

Finally, for every $g \in H_i$, the support of the homeomorphism $\varphi_i(g)$ is contained in $[x_0, x]$. Moreover, the set of breakpoints of $\varphi_i(g)$ is still a closed, countable set consisting only of dyadic rationals. Hence, ϱ_i also satisfies property (2).

This shows that $\Omega F'$ is dissipated, whence pseudo-mitotic by Proposition 3.11. The thesis now follows from Theorem 3.5. □

We believe that a careful study of the embedding $F' \hookrightarrow \Omega F'$ could lead to some understanding of the bounded cohomology of F'.

6.2. On the bounded cohomology of T. We recall the definition of T.

DEFINITION 6.8. The Thompson group T is the group of orientation-preserving piecewise linear homeomorphisms f of the circle $\mathbb{R}/\mathbb{Z}$ with the following properties:

(1) f has finitely many breakpoints, all of which lie in $\mathbb{Z}[1/2]/\mathbb{Z}$;
(2) away from the breakpoints, the slope of f is a power of 2;
(3) f preserves $\mathbb{Z}[1/2]/\mathbb{Z}$.

The stabilizer of 0 for the canonical T-action on the circle is canonically isomorphic to the Thompson group F.

Since T acts minimally on the circle, it admits a second bounded cohomology class, namely the *real Euler class* [12, 41]. The bounded Euler class is a refinement of the classical Euler class. All cup-powers of the classical Euler class are nontrivial in cohomology [26]; and thus, also the cup-powers of the bounded Euler class are nontrivial in $\mathrm{H}_b^*(T; \mathbb{R})$; this was first noticed by Burger and Monod [13]. Therefore, Question 6.2 is asking whether these are the only bounded cohomology classes. In degree two, this is known to be true [31], but again, to our knowledge, nothing is known in higher degrees.

The main goal of this section is to show that a positive answer to Question 6.1 implies a positive answer to Question 6.2, and this implication holds degree-wise. To do this, we prove the following general criterion for computing the bounded cohomology of groups acting highly transitively on the circle with boundedly acyclic stabilizers.

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PROPOSITION 6.9. *Let $n \in \mathbb{N}_{\geq 2}$. Let Γ be a group acting orientation-preservingly on the circle, let S be an orbit of Γ with $|S| \geq n+1$. Suppose that the following hold:*

(1) *for all $k \in \{1, \dots, n+1\}$, the action of Γ on the set of circularly ordered k-tuples in S is transitive;*
(2) *for all $k \in \{1, \dots, n\}$, the stabilizer of a circularly ordered k-tuple is n-boundedly acyclic.*

Then $\mathrm{H}^2_b(\Gamma; \mathbb{R})$ is generated by the bounded Euler class of this circle action of Γ and

$$\mathrm{H}^i_b(\Gamma; \mathbb{R}) \cong \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \mathbb{R} & \text{if } i \text{ is even,} \end{cases}$$

for all $i \in \{1, \dots, n\}$, generated by the cup-powers of Euler class.

Recall that a k-tuple $(s_1, \dots, s_k)$ in S^1 is *circularly ordered* if there exists a point $p \in S^1 \setminus \{s_1, \dots, s_k\}$ such that $(s_1, \dots, s_k) \in S^1 \setminus \{p\} \cong (0,1)$ is an ordered k-tuple in the interval. We follow the convention that circularly ordered tuples are *nondegenerate*, that is, they consist of pairwise distinct entries.

For the proof, we follow the general principle of computing bounded cohomology through boundedly acyclic actions. Boundedly acyclic stabilizers lead to boundedly acyclic modules.

LEMMA 6.10. *Let Γ be a group and let $\Gamma \curvearrowright X$ be an action of Γ on a set X that has only finitely many orbits $(X_i)_{i \in I}$. Let $n \in \mathbb{N}$. If each of the orbits has n-boundedly acyclic stabilizer, then we have for all $k \in \{1, \dots, n\}$ that*

$$\mathrm{H}^k_b(\Gamma; \ell^\infty(X)) \cong 0.$$

PROOF. Let $k \in \{1, \dots, n\}$. Because I is finite, we have $\ell^\infty(X) \cong \bigoplus_{i \in I} \ell^\infty(X_i)$ and

$$\mathrm{H}^k_b(\Gamma; \ell^\infty(X)) \cong \mathrm{H}^k_b\Big(\Gamma; \bigoplus_{i \in I} \ell^\infty(X_i)\Big) \cong \bigoplus_{i \in I} \mathrm{H}^k_b(\Gamma; \ell^\infty(X_i)).$$

We show that each of the summands is trivial. Let $i \in I$ and let $H_i \subset \Gamma$ be the stabilizer of a point in X_i. Then, by the Eckmann–Shapiro lemma in bounded cohomology [43, Proposition 10.13], we obtain

$$\mathrm{H}^k_b(\Gamma; \ell^\infty(X_i)) \cong \mathrm{H}^k_b(\Gamma; \ell^\infty(\Gamma)^{H_i}) \cong \mathrm{H}^k_b(H_i; \mathbb{R});$$

the last term is trivial, because H_i is n-boundedly acyclic by hypothesis. □

The effect of boundedly acyclic stabilizers is studied more systematically in a forthcoming article on boundedly acyclic covers and relative simplicial volume [38].

PROOF OF PROPOSITION 6.9. The given Γ-action on S gives a simplicial Γ-resolution $\mathbb{R} \to \ell^\infty(S^{*+1})$ [24, Lemma 4.21].

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Claim 6.11. The Γ-resolution $\mathbb{R} \to \ell^\infty(S^{*+1})$ is boundedly acylic up to degree n, that is, for all $k \in \{0, \dots, n\}$ and all $i \in \{1, \dots, n\}$, we have

$$\mathrm{H}_b^i(\Gamma; \ell^\infty(S^{k+1})) \cong 0.$$

To prove Claim 6.11, first note that the Γ-space S^{k+1} consists only of finitely many Γ-orbits. Indeed, every tuple can be permuted to be circularly ordered (possibly with repetitions), and only finitely many permutations and repetition patterns are possible. Moreover, Γ acts transitively on circularly ordered tuples of every given size $\leq k+1$.

The stabilizer groups of the Γ-space S^{k+1} are all n-boundedly acyclic by hypothesis. Thus, Lemma 6.10 shows the claim.

Therefore, we can apply the fact that boundedly acyclic resolutions compute bounded cohomology [48, Proposition 2.5.4] and symmetrization [24, Section 4.10] to conclude that

$$\mathrm{H}_b^i(\Gamma; \mathbb{R}) \cong \mathrm{H}^i(\ell^\infty(S^{*+1})^\Gamma) \cong \mathrm{H}^i(\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma) \tag{6-1}$$

for all $i \in \{1, \dots, n\}$. Here, $\ell^\infty_{\mathrm{alt}}(S^{*+1})$ denotes the subcomplex of alternating cochains, that is, functions f with

$$f(s_{\sigma(0)}, \dots, s_{\sigma(k)}) = \mathrm{sgn}(\sigma) \cdot f(s_0, \dots, s_k)$$

for all $(s_0, \dots, s_k) \in S^{k+1}$ and all permutations σ of $\{0, \dots, k\}$.

Claim 6.12. Let $k \in \{0, \dots, n\}$.

(1) If k is odd, then $\ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma \cong 0$.
(2) If k is even, then $\ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma \cong \mathbb{R}$, generated by the function f_k constructed in the proof below.

To prove Claim 6.12, let $f \in \ell^\infty_{\mathrm{alt}}(S^{k+1})$. We first show that f is determined by its value on a single circularly ordered tuple: indeed, f vanishes on tuples with a repetition; all other tuples may be permuted to be circularly ordered. Moreover, since Γ acts transitively on the set of circularly ordered tuples, f is constant on the set of all circularly ordered tuples. In particular, $\dim_{\mathbb{R}} \ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma \leq 1$.

As $|S| \geq n+1$, there exists a circularly ordered tuple $(s_0, \dots, s_k) \in S^{k+1}$.

Let k be odd. It suffices to show that $f(s_0, \dots, s_k) = 0$. With $(s_0, \dots, s_k)$, also $(s_k, s_0, \dots, s_{k-1})$ is circularly ordered. Because k is odd, these two tuples differ by an odd permutation. As f is both constant on all circulary ordered tuples and alternating, we obtain

$$f(s_0, \dots, s_k) = f(s_k, s_0, \dots, s_{k-1}) = -f(s_0, \dots, s_k)$$

and thus $f(s_0, \dots, s_k) = 0$. Therefore, $\ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma \cong 0$.

Let k be even. We define $f_k : S^{k+1} \to \mathbb{R}$ as follows: on tuples with a repetition, we define f_k to vanish. If $(t_0, \dots, t_k) \in S^{k+1}$ has no repetition, we set

$$f_k(t_0, \dots, t_k) := \mathrm{sgn}(\sigma),$$

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where σ is a permutation such that $(t_{\sigma(0)}, \dots, t_{\sigma(k)})$ is circularly ordered; this permutation σ is only unique up to a $(k+1)$-cycle, but since k is even, $\operatorname{sgn}(\sigma)$ is well defined. Because Γ acts orientation-preservingly and because there exists at least one circularly ordered $(k+1)$-tuple, this gives a well-defined nontrivial element in $\ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma$. Therefore, $\ell^\infty_{\mathrm{alt}}(S^{k+1})^\Gamma \cong \mathbb{R}$. This proves our claim.

In view of Claim 6.12, the cochain complex $\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma$ is (up to degree n) isomorphic to the cochain complex

$$\mathbb{R} \to 0 \to \mathbb{R} \to 0 \to \cdots$$

(whose coboundary operator is necessarily trivial) and if $k \in \{0, \dots, n\}$ is even, then $[f_k]$ is nontrivial in $\mathrm{H}^k(\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma)$. In particular, we obtain

$$\mathrm{H}^i_b(\Gamma; \mathbb{R}) \cong \mathrm{H}^i(\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma) \cong \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \mathbb{R} & \text{if } i \text{ is even,} \end{cases}$$

for all $i \in \{0, \dots, n-1\}$.

As for degree n, under our assumptions, we cannot show that $\ell^\infty_{\mathrm{alt}}(S^{n+1})$ follows the same periodic pattern. However, we still have the following claim.

Claim 6.13. The differential $\ell^\infty_{\mathrm{alt}}(S^{n+1})^\Gamma \to \ell^\infty_{\mathrm{alt}}(S^{n+2})^\Gamma$ is trivial. □

To prove Claim 6.13, first note that this is obvious if n is odd, since then, $\ell^\infty_{\mathrm{alt}}(S^{n+1})^\Gamma \cong 0$ by Claim 6.12.

Suppose instead that n is even, and let $f \in \ell^\infty_{\mathrm{alt}}(S^{n+1})^\Gamma$ be a function that takes the constant value λ on circularly ordered tuples. Then, if $(s_0, \dots, s_{n+1})$ is a circularly ordered tuple, we have

$$\delta^{n+1}_b(f)(s_0, \dots, s_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \cdot \lambda = 0,$$

since n is even. This proves our claim.

Therefore,

$$\mathrm{H}^n_b(\Gamma; \mathbb{R}) \cong \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{R} & \text{if } n \text{ is even,} \end{cases}$$

as well. It remains to deal with the bounded Euler class and its powers.

Claim 6.14. The bounded Euler class $\mathrm{eu}^\Gamma_b \in \mathrm{H}^2_b(\Gamma; \mathbb{R})$ is nontrivial.

To prove Claim 6.14, we first make the isomorphism in Equation (6-1) more explicit. Let $x_0 \in S$. For $k \in \mathbb{N}$, we consider the map

$$\begin{aligned} \varphi^k : \ell^\infty(S^{*+1}) &\to \ell^\infty(\Gamma^{*+1}) \\ f &\mapsto ((\gamma_0, \dots, \gamma_k) \mapsto f(\gamma_0 x_0, \dots, \gamma_k x_0)). \end{aligned}$$

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Then $\varphi^* : \ell^\infty(S^{*+1}) \to \ell^\infty(\Gamma^{*+1})$ is a degree-wise bounded Γ-cochain map that extends the identity on the resolved module $\mathbb{R}$. Because the resolution $\ell^\infty(S^{*+1})$ is strong [24, Lemma 4.21], $(\varphi^*)^\Gamma$ induces an isomorphism $\mathrm{H}_b^i(\Gamma;\mathbb{R}) \cong \mathrm{H}^i(\ell^\infty(S^{*+1})^\Gamma)$ for all $i \in \{0,\dots,n\}$ [48, Proposition 2.5.4 and Remark 2.5.5].

As the inclusion $i^* : \ell^\infty_{\mathrm{alt}}(S^{*+1}) \to \ell^\infty(S^{*+1})$ is a Γ-cochain map that induces an isomorphism $\mathrm{H}^*(\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma) \cong \mathrm{H}^*(\ell^\infty(S^{*+1})^\Gamma)$, we conclude that $\varphi^*_{\mathrm{alt}} := \varphi^* \circ i^*$ induces an isomorphism $\mathrm{H}_b^i(\Gamma;\mathbb{R}) \cong \mathrm{H}^i(\ell^\infty_{\mathrm{alt}}(S^{*+1})^\Gamma)$.

By construction, $(\varphi^2_{\mathrm{alt}})^\Gamma(f_2)$ gives the orientation cocycle or^Γ of the Γ-action. Because of $[f_2] \neq 0$, we know that the bounded Euler class

$$\mathrm{eu}_b^\Gamma = \tfrac{1}{2} \cdot [\mathrm{or}^\Gamma] = \tfrac{1}{2} \cdot \mathrm{H}^2((\varphi^*_{\mathrm{alt}})^\Gamma)[f_2]$$

is nonzero in $\mathrm{H}_b^2(\Gamma;\mathbb{R})$. This proves our claim.

Because of Claim 6.14 and $\mathrm{H}_b^2(\Gamma;\mathbb{R}) \cong \mathbb{R}$, we conclude that $\mathrm{H}_b^2(\Gamma;\mathbb{R})$ is generated by the bounded Euler class.

Claim 6.15. For all $k \in \{0,\dots,n/2\}$, the cup-power $(\mathrm{eu}_b^\Gamma)^{\cup k} \in \mathrm{H}_b^{2k}(\Gamma;\mathbb{R})$ is nontrivial.

To prove Claim 6.15, in view of the relation between eu_b^Γ and $[f_2]$ (proof of Claim 6.14) and the above description of $\mathrm{H}_b^{2k}(\Gamma;\mathbb{R})$, it suffices to show that $\mathrm{alt}({f_2}^{\cup k})$ is nontrivial, where $\cdot \cup \cdot$ denotes the standard cup-product on the cochain level (notice that even if f_2 is alternating, the nontrivial cup-product ${f_2}^{\cup k}$ is not so). Indeed, we have

$$\mathrm{alt}({f_2}^{\cup k}) = \frac{2^k \cdot k!}{(2k)!} \cdot f_{2k}$$

(Appendix B), which is nontrivial (Claim 6.12). This proves our claim.

This completes the proof of Proposition 6.9. □

REMARK 6.16. The second hypothesis in Proposition 6.9 is used to show that the modules $\ell^\infty(S^{k+1})$ are n-boundedly acyclic, which in turn is used to apply the computation of bounded cohomology through acyclic resolutions [48, Proposition 2.5.4]. Note however that this result does not require n-bounded acyclicity of all stabilizers. Indeed, it is enough to ask that the stabilizer of a circularly ordered k-tuple is $(n-k+1)$ boundedly acyclic, for $k \in \{1,\dots,n\}$. To keep the notation simple, we chose to state Proposition 6.9 with the stronger hypothesis.

We apply this to $\Gamma = T$, to show that if F is n-boundedly acyclic, then Question 6.2 has a positive answer up to degree n.

COROLLARY 6.17. *If F is boundedly acyclic, then $\mathrm{H}_b^*(T;\mathbb{R})$ (with the cup-product structure) is isomorphic to the polynomial ring $\mathbb{R}[x]$ with $|x| = 2$, and the bounded Euler class of T is a polynomial generator of $\mathrm{H}_b^*(T;\mathbb{R})$.*

PROOF. For each $k \geq 1$, the group T acts transitively on the set of circularly ordered k-tuples in $\mathbb{Z}[1/2]/\mathbb{Z}$; the stabilizers of this action are isomorphic to direct powers of F [15]. In particular, the stabilizers are boundedly acyclic by Theorem 4.1. Therefore,

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Proposition 6.9 is applicable and we obtain that $\mathrm{H}_b^*(T;\mathbb{R})$ is isomorphic as a graded $\mathbb{R}$-algebra to $\mathbb{R}[x]$ with x corresponding to the bounded Euler class eu_b^T.

Alternatively, in this case, the nontriviality of the powers of the bounded Euler class is already known through the computations of Ghys–Sergiescu and Burger–Monod [13]. □

Corollary 6.17 also holds in a range up to n (with the same proof).

COROLLARY 6.18. *If F is boundedly acyclic and $r \in \mathbb{N}_{\geq 1}$, then*

$$\begin{aligned} \mathbb{R}[x_1,\dots,x_r] &\to \mathrm{H}_b^*(T^{\times r};\mathbb{R}) \\ x_j &\mapsto \mathrm{H}_b^2(\pi_j;\mathbb{R})(\mathrm{eu}_b^T) \end{aligned}$$

defines an isomorphism of graded $\mathbb{R}$-algebras; here, $\mathrm{H}_b^*(T^{\times r};\mathbb{R})$ carries the cup-product structure, $|x_j| = 2$, and $\pi_j : T^{\times r} \to T$ denotes the projection onto the jth factor for each $j \in \{1,\dots,r\}$. Moreover, the canonical semi-norm on $\mathrm{H}_b^*(T^{\times r};\mathbb{R})$ then is a norm.

PROOF. We combine Corollary 6.17 with suitable Künneth arguments. As usual in bounded cohomology, some care is necessary to execute this.

We first show that the polynomial ring embeds into $\mathrm{H}_b^*(T;\mathbb{R})$: the $\mathbb{R}$-algebra homomorphism $\mathbb{R}[x] \to \mathrm{H}^*(T;\mathbb{R})$ given by

$$x \mapsto \mathrm{eu}^T$$

is injective [26]. Therefore, the Künneth theorem shows that

$$x_j \mapsto \mathrm{H}^2(\pi_j;\mathbb{R})(\mathrm{eu}^T)$$

yields an injective $\mathbb{R}$-algebra homomorphism $\Phi^r : \mathbb{R}[x_1,\dots,x_r] \to \mathrm{H}^*(T^{\times r};\mathbb{R})$. In combination with the universal coefficient theorem, we obtain: for every polynomial $p \in \mathbb{R}[x_1,\dots,x_r] \setminus \{0\}$, there exists a class $\alpha_p \in \mathrm{H}_*(T^{\times r};\mathbb{R})$ with

$$\langle \Phi^r(p), \alpha_p \rangle = 1.$$

These nontrivial evaluations show that also the bounded version

$$\begin{aligned} \Phi_b^r : \mathbb{R}[x_1,\dots,x_r] &\to \mathrm{H}_b^*(T^{\times r};\mathbb{R}) \\ x_j &\mapsto \mathrm{H}_b^2(\pi_j;\mathbb{R})(\mathrm{eu}_b^T) \end{aligned}$$

is injective; even more, for each $p \in \mathbb{R}[x_1,\dots,x_r] \setminus \{0\}$, we have

$$\langle \Phi_b^r(p), \alpha_p \rangle = \langle \Phi^r(p), \alpha_p \rangle = 1$$

and thus $\|\Phi_b^r(p)\|_\infty \neq 0$. So far, we did not use the postulated bounded acyclicity of F.

It remains to show that Φ_b^r is surjective. To this end, it suffices to inductively (on r) establish that $\dim_{\mathbb{R}} \mathrm{H}_b^k(T^{\times r};\mathbb{R}) \leq \dim_{\mathbb{R}}(\mathbb{R}[x_1,\dots,x_r])_k$ holds for all $k \in \mathbb{N}$.

The base case is handled in Corollary 6.17; moreover, the evaluation argument above shows that the canonical semi-norm on $\mathrm{H}_b^*(T;\mathbb{R})$ indeed is a norm.

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For the induction step, let us assume that the claim holds for $r-1$. We recall that for group extensions $1 \to N \to \Gamma \to Q \to 1$, there is a Hochschild–Serre spectral sequence

$$E_2^{pq} = \mathrm{H}_b^p(Q; \mathrm{H}_b^q(N;\mathbb{R})) \Longrightarrow \mathrm{H}_b^{p+q}(\Gamma;\mathbb{R})$$

in bounded cohomology, whenever the canonical semi-norm on $\mathrm{H}_b^*(N;\mathbb{R})$ is a norm [43, Proposition 12.2.1]. Applying this spectral sequence to the trivial product extension

$$1 \to T \to T^{\times r} \to T^{\times(r-1)} \to 1$$

shows that the degree-wise dimensions of $\mathrm{H}_b^*(T^{\times r};\mathbb{R})$ are at most the degree-wise dimensions of $\mathbb{R}[x_1,\dots,x_r]$ (with $|x_i| = 2$ for all $i \in \{1,\dots,r\}$). Hence, Φ_b^r is surjective. In particular, again by the evaluation argument above, the canonical semi-norm on all of $\mathrm{H}_b^*(T^{\times r};\mathbb{R})$ is a norm. □

REMARK 6.19. Analogous results are obtained by Monod and Nariman [46], who computed the full bounded cohomology of the groups of orientation-preserving homeomorphisms of the circle and the 2-disc. In fact, in Proposition 6.9, one can replace orbits by *fat orbits* (that is, orbits of *fat points* [46]). This allows to compute the full bounded cohomology of $\mathrm{Homeo}^+(S^1)$ from the bounded acyclicity of $\mathrm{Homeo}_c(\mathbb{R})$. Moreover, again using fat orbits, one can deduce from Proposition 6.7 a positive answer to Question 6.2 for a natural countably singular analog of Thompson's group T.

6.3. On the bounded cohomology of V. We recall the definition of V.

DEFINITION 6.20. The Thompson group V is the group of piecewise linear right-continuous bijections f of the circle $\mathbb{R}/\mathbb{Z}$ with the following properties:

(1) f has finitely many breakpoints, all of which lie in $\mathbb{Z}[1/2]/\mathbb{Z}$;
(2) away from the breakpoints, f is orientation-preserving and the slope of f is a power of 2;
(3) f preserves $\mathbb{Z}[1/2]/\mathbb{Z}$.

It was recently proved that V is acyclic [54]: this had been conjectured by Brown [9], who already proved that V is rationally acyclic. Moreover, V is uniformly perfect [15]; so, using the same argument as in Remark 3.6, we deduce that V is 2-boundedly acyclic.

While the proof of acyclicity of V is involved, the proof of rational acyclicity is much simpler and only relies on standard arguments in equivariant homology. A bounded analog of equivariant homology theory has recently been developed [37], but it does not seem possible to directly translate Brown's proof to bounded cohomology.

A positive answer to Question 6.3 would make V the first example of a nonamenable boundedly acyclic group of type F_∞. Moreover, it would make V the first tractable

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example of a nonamenable boundedly acyclic finitely presented group: the only known example [22] is very implicit.

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Appendix A. Pseudo-mitotic groups are boundedly acyclic

We prove Theorem 1.3. The proof is an adaption of the original proofs for ordinary cohomology [4, 55] to the setting of bounded cohomology. In bounded cohomology, we keep additional control on primitives, similar to Matsumoto and Morita for certain homeomorphism groups [42] and similarly to the case of mitotic groups [39].

The main ingredient in the proof is the following proposition which is an adaptation for the pseudo-mitotic setting of the mitotic case [39, Proposition 4.6].

PROPOSITION A.1. *Let $n \in \mathbb{N}$ and let $\kappa \in \mathbb{R}_{>0}$. Let*

$$H \xrightarrow{\varphi} H' \xrightarrow{\varphi'} K \xrightarrow{\psi} \Gamma \xrightarrow{i} P$$

be a chain of group homomorphisms such that:

(1) *the homomorphism $i : \Gamma \to P$ is a pseudomitosis of Γ in P;*
(2) *for every $s \in \{1, \dots, n-1\}$, we have $\mathrm{H}_s(\varphi') = 0$;*
(3) *for every $s \in \{0, \dots, n-1\}$, the homomorphisms φ and ψ satisfy (s, κ)-UBC.*

Then, for all $s \in \{1, \dots, n\}$, we have

$$\mathrm{H}_s(i \circ \psi \circ \varphi' \circ \varphi) = 0.$$

Moreover, there exists a constant $c_{n,\kappa} \in \mathbb{R}_{>0}$ (depending only on n and κ) such that the composition $i \circ \psi \circ \varphi' \circ \varphi$ satisfies $(s, c_{n,\kappa})$-UBC for every $s \in \{0, \dots, n\}$.

We give the proof of Proposition A.1 in Appendix A.1. Following the mitotic case [39], we show first how to deduce Theorem 1.3 from this result.

PROOF OF THEOREM 1.3. Let P be a pseudo-mitotic group. According to Theorems 2.3 and 3.4, it is sufficient to show that for every $n \in \mathbb{N}_{>0}$, the group P satisfies n-UBC.

Let $n \in \mathbb{N}_{>0}$ and let $z \in \mathrm{C}_n(P; \mathbb{R})$ be a boundary, that is, $\partial_{n+1} c = z$ for some $c \in C_{n+1}(P; \mathbb{R})$. Since both z and c involve only finitely many elements of P, there exists a finitely generated subgroup Γ_0 of P such that both z and c are supported on Γ_0. We show that the inclusion $\Gamma_0 \hookrightarrow P$ satisfies (n, κ_n)-UBC, where $\kappa_n \in \mathbb{R}_{>0}$ only depends on n. This condition readily implies that P satisfies n-UBC, whence the thesis.

Since P is pseudo-mitotic, Γ_0 has a mitosis into P. Let $\psi_0, \psi_1 : \Gamma_0 \to P$ and $g \in P$ be witnesses of such a pseudo-mitosis. Then, Γ_0 also admits a pseudo-mitosis into the

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following finitely generated subgroup of P:

$$\Gamma_1 = \langle \Gamma_0, \psi_0(\Gamma_0), \psi_1(\Gamma_0), g \rangle \subset P.$$

By iterating this construction, we get a sequence $\Gamma_0 \leq \Gamma_1 \leq \cdots \leq P$ of finitely generated groups such that at each step, the inclusion $\Gamma_j \hookrightarrow \Gamma_{j+1}$ is a pseudo-mitosis.

Following *verbatim* the proof of the mitotic case [39, Theorem 1.2], by induction on $n \geq 1$ and using Proposition A.1, one can show that the inclusion of Γ into a sufficiently large Γ_{j_n} satisfies (n, κ_n)-UBC, where κ_n only depends on n.

Using the fact that $\Gamma_{j_n} \leq P$, this implies that there exists $c' \in \mathrm{C}_{n+1}(P; \mathbb{R})$ with

$$\partial_{n+1} c' = z \quad \text{and} \quad |c'|_1 \leq \kappa_n \cdot |z|_1.$$

This shows that P satisfies n-UBC for all positive degrees n; whence, P is boundedly acyclic (Theorem 2.3). □ □

A.1. Proof of Proposition A.1. This section is devoted to the proof of Proposition A.1. The proof is based on a refinement of Varadarajan's proof [55, Proposition 1.4], additionally taking the norm of the morphisms involved into account. Our approach closely follows the mitotic case [39, Appendix A].

Proof of Proposition A.1. We prove the statement in degree $n \in \mathbb{N}$. For convenience, we write

$$f := \psi \circ \varphi' \circ \varphi.$$

Since $\mathrm{H}_s(\varphi') = 0$ for every $s \in \{1, \dots, n-1\}$, also $\mathrm{H}_s(f) = 0$ in the same degrees. The fact that $\mathrm{H}_s(i \circ f) = 0$ for every $s \in \{1, \dots, n\}$ was already proved by Varadarajan [55, Proposition 1.4]. To adapt the mitosis proof [39, Appendix A] to the case of pseudo-mitoses, it is convenient to recall Varadarajan's argument.

Let $\psi_0, \psi_1 : \Gamma \to P$ and $g \in P$ be witnesses of the pseudo-mitosis $i : \Gamma \to P$. Then, we define the map

$$\begin{aligned} \mu : \Gamma \times \Gamma &\to P \\ (g', g) &\mapsto g' \cdot \psi_1(g). \end{aligned}$$

Notice that μ is a group homomorphism by condition (2) of the definition of pseudo-mitosis [55, proof of Proposition 1.4].

Let $j_1 : H \to H \times H$ and $j_2 : H \to H \times H$ be the inclusions into the first and the second factor, respectively. Let γ_g denote the conjugation with respect to g, that is, $\gamma_g(g') = g g' g^{-1}$ for every $g' \in P$. Moreover, let $\Delta_H : H \to H \times H$ denote the diagonal homomorphism. Then, for every $h \in H$,

$$\begin{aligned} \gamma_g \circ \psi_1 \circ f(h) = \psi_0 \circ f(h) &= \psi_0(f(h)) \\ &= f(h) \cdot \psi_1(f(h)) \\ &= \mu(f(h), f(h)) \\ &= \mu \circ (f \times f) \circ \Delta_H(h). \end{aligned}$$

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However,

$$\mu \circ (f \times f) \circ j_1 = \mu(f, 1) = f = i \circ f \quad \text{and}$$
$$\mu \circ (f \times f) \circ j_2 = \mu(1, f) = \psi_1(f) = \psi_1 \circ f.$$

Hence, the Künneth formula (and its naturality) together with the assumption that $\mathrm{H}_s(f) = 0$ for all $s \in \{1, \dots, n-1\}$ imply that the following diagram commutes (similarly to the mitotic case [39, page 729]):

$$\begin{array}{ccc}
\mathrm{H}_n(H \times H; \mathbb{R}) & \xrightarrow{\mathrm{H}_n(p_1) \oplus \mathrm{H}_n(p_2)} & \mathrm{H}_n(H; \mathbb{R}) \oplus \mathrm{H}_n(H; \mathbb{R}) \\
\downarrow{\scriptstyle \mathrm{H}_n(f \times f)} & & \downarrow{\scriptstyle \mathrm{H}_n(f) \oplus \mathrm{H}_n(f)} \\
\mathrm{H}_n(\Gamma \times \Gamma; \mathbb{R}) & \xleftarrow[\mathrm{H}_n(i_1) + \mathrm{H}_n(i_2)]{} & \mathrm{H}_n(\Gamma; \mathbb{R}) \oplus \mathrm{H}_n(\Gamma; \mathbb{R}) \\
\downarrow{\scriptstyle \mathrm{H}_n(\mu)} & & \downarrow{\scriptstyle \mathrm{H}_n(i) \oplus \mathrm{H}_n(\psi_1)} \\
\mathrm{H}_n(P; \mathbb{R}) & \xleftarrow[\mathrm{id} + \mathrm{id}]{} & \mathrm{H}_n(P; \mathbb{R}) \oplus \mathrm{H}_n(P; \mathbb{R}),
\end{array}$$

where $p_1, p_2 : H \times H \to H$ denote the projections and $i_1, i_2 : \Gamma \to \Gamma \times \Gamma$ the inclusions. The commutativity of the previous diagram leads to

$$\begin{aligned}
\mathrm{H}_n(\gamma_g) \circ \mathrm{H}_n(\psi_1 \circ f) &= \mathrm{H}_n(\mu \circ (f \times f) \circ \Delta_H) \\
&= \mathrm{H}_n(i \circ f) + \mathrm{H}_n(\psi_1 \circ f).
\end{aligned}$$

Since the conjugation γ_g is trivial in homology, that is, $\mathrm{H}_n(\gamma_g) = \mathrm{id}$, we obtain

$$\mathrm{H}_n(i \circ f) = 0.$$

We are thus reduced to showing that the previous construction can be controlled in such a way that $i \circ f$ satisfies the required UBC condition.

Let $z \in \partial_{n+1}(\mathrm{C}_{n+1}(H; \mathbb{R}))$. We construct a controlled ∂_{n+1}-primitive for $\mathrm{C}_n(i \circ f)$. Following the mitotic case [39, page 731], we have

$$(f \times f)_* \circ \Delta_{H*}(z) = (f \times f)_* \circ j_{1*} + (f \times f)_* \circ j_{2*} + \partial_{n+1} E'(z) \tag{A-1}$$

on the chain level, where E' is bounded and $\|E'\|$ admits a bound that only depends on the given $\kappa \in \mathbb{R}_{>0}$ and n (the proof uses hypothesis (3) in the statement).

To complete the construction of a controlled ∂_{n+1}-primitive for $\mathrm{C}_n(i \circ f)$, we consider the following chain homotopy:

$$\begin{aligned}
\Theta_n : \mathrm{C}_n(P; \mathbb{R}) &\to \mathrm{C}_{n+1}(P; \mathbb{R}) \\
(g_1, \dots, g_n) &\mapsto \sum_{j=1}^{n+1} (-1)^{j+1} \cdot (g_1, \dots, g_{j-1}, g, g^{-1} g_j g, \dots, g^{-1} g_n g)
\end{aligned}$$

between $\mathrm{C}_*(\gamma_g)$ and the identity. The previous map is in fact such a chain homotopy and $\|\Theta_n\| \leq n + 1$ is proved as in the mitotic case [39, Lemma A.2] (notice that for

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convenience we changed the sign of Θ). We then have:

$$\begin{aligned}(i \circ f)_*(z) &= (\mu \circ (f \times f) \circ j_1)_*(z)\\ &= (\mu \circ (f \times f) \circ \Delta_H)_*(z)\\ &\quad - (\mu \circ (f \times f) \circ j_2)_*(z) - \mu_* \circ \partial_{n+1} \circ E'(z)\\ &= (\gamma_g)_* \circ (\psi_1 \circ f)_*(z)\\ &\quad - (\psi_1 \circ f)_*(z) - \partial_{n+1} \circ \mu_* \circ E'(z)\\ &= (\partial_{n+1} \circ \Theta + \Theta \circ \partial_n) \circ (\psi_1 \circ f)_*(z) - \partial_{n+1} \circ \mu_* \circ E'(z)\\ &= \partial_{n+1}(\Theta \circ (\psi_1 \circ f)_*(z) - \mu_* \circ E'(z)),\end{aligned}$$

where we moved from the first line to the second one using the Equation (A-1) and the last equality holds because z is a cycle. Moreover, using the fact that group homomorphisms induce norm nonincreasing chain maps, we have that the norm

$$\|\Theta \circ (\psi_1 \circ f)_*(z) - \mu_* \circ E'(z)\| \leq \|\Theta\| + \|E'\| \leq n + 1 + \|E'\|$$

is bounded from above by a quantity $c_{n,\kappa} \in \mathbb{R}_{>0}$ depending only on n and κ (since this is true for E' and Θ). This shows that $i \circ f$ satisfies $(n, c_{n,\kappa})$-UBC, as claimed. □ □

Appendix B. Computation of cup-powers

In the following, we give the combinatorial part of the proof of Claim 6.15 in the proof of Proposition 6.9. We use the notation established in the proof of Proposition 6.9.

LEMMA B.1. *For all $k \in \{0, \dots, n/2\}$, we have*

$$\operatorname{alt}({f_2}^{\cup k}) = \frac{2^k \cdot k!}{(2k)!} \cdot f_{2k}.$$

PROOF. Proceeding inductively, it suffices to show that

$$\operatorname{alt}(f_2 \cup f_{2(k-1)}) = \frac{1}{2k-1} \cdot f_{2k}.$$

This is a purely combinatorial statement. Let $(s_0, \dots, s_{2k})$ be a circularly ordered $(2k+1)$-tuple over S. To simplify notation, we write $f(i_0, \dots, i_p)$ for $f(s_{i_0}, \dots, s_{i_p})$, and so on. In this notation, since $(s_0, \dots, s_{2k})$ is circularly ordered, it suffices to show that $A := \operatorname{alt}(f_2 \cup f_{2(k-1)})(0, \dots, 2k)$ is equal to

$$\frac{1}{2k-1}.$$

By definition of alt and the cup-product on simplicial cochains, we have

$$A = \frac{1}{(2k+1)!} \cdot \sum_{\sigma \in \Sigma(0,\dots,2k)} \operatorname{sgn}(\sigma) \cdot f_2(\sigma(0), \sigma(1), \sigma(2)) \cdot f_{2(k-1)}(\sigma(2), \dots, \sigma(2k)).$$

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Because $2k$ is even, we can fix one position and obtain via cyclic permutations that

$$A = \frac{2k+1}{(2k+1)!} \cdot \sum_{\sigma \in \Sigma(0,\dots,2k-1)} \operatorname{sgn}(\sigma) \cdot f_2(\sigma(0), \sigma(1), 2k) \cdot f_{2(k-1)}(2k, \sigma(2), \dots, \sigma(2k-1)).$$

Flipping $\sigma(0)$ and $\sigma(1)$ changes both the sign of σ and that of $f_2(\sigma(0), \sigma(1), 2k)$. Therefore, we obtain

$$\begin{aligned} A &= \frac{2}{(2k)!} \cdot \sum_{i=0}^{2k-2} \sum_{j=i+1}^{2k-1} \sum_{\sigma \in \Sigma(X_{i,j})} \operatorname{sgn}([i,j] * \sigma) \cdot f_2(i,j,2k) \cdot f_{2(k-1)}([2k] * \sigma) \\ &= \frac{2}{(2k)!} \cdot \sum_{i=0}^{2k-2} \sum_{j=i+1}^{2k-1} \sum_{\sigma \in \Sigma(X_{i,j})} \operatorname{sgn}([i,j] * \sigma) \cdot 1 \cdot \operatorname{sgn}(\sigma). \end{aligned}$$

Here, we use the following notation: $X_{i,j} := \{0, \dots, 2k-1\} \setminus \{i, j\}$; the permutation/tuple $[i,j] * \sigma$ on $\{0, \dots, 2k-1\}$ is obtained by using i, j in the first two positions, and then filling up with σ, and so on.

Let $[X_{i,j}]$ be the sequence of elements in $X_{i,j}$, in order. Then

$$\operatorname{sgn}(\sigma) \cdot \operatorname{sgn}([i,j] * \sigma) = \operatorname{sgn}([i,j] * [X_{i,j}]) = \operatorname{sgn}(j-1) \cdot \operatorname{sgn}(i)$$

for all $i \in \{0, \dots, 2k-2\}$, $j \in \{i+1, \dots, 2k-1\}$; here, we set $\operatorname{sgn}(x)$ of $x \in \mathbb{N}$ to $+1$ if x is even, and to -1 if x is odd. We distinguish two cases.

- If i is odd, then $\sum_{j=i+1}^{2k-1} \operatorname{sgn}(j-1)$ is zero, because there are equally many even and odd numbers in $\{i+1, \dots, 2k-1\}$.
- If i is even, then $\{i+1, \dots, 2k-1\}$ contains one more odd number than even numbers, whence

$$\operatorname{sgn}(i) \cdot \sum_{j=i+1}^{2k-1} \operatorname{sgn}(j-1) = 1.$$

Therefore, since there are k even numbers inside $\{0, \dots, 2k-2\}$, we obtain

$$A = \frac{2}{(2k)!} \cdot k \cdot |X_{i,j}|! = \frac{2}{(2k)!} \cdot k \cdot (2k-2)! = \frac{1}{2k-1},$$

as claimed. □

Similar computations can also be found in the literature [34].

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FRANCESCO FOURNIER-FACIO, Department of Mathematics,
ETH Zürich, Zürich, Switzerland
e-mail: francesco.fournier@math.ethz.ch

CLARA LÖH, Fakultät für Mathematik,
Universität Regensburg, Regensburg, Germany
e-mail: clara.loeh@ur.de

MARCO MORASCHINI, Fakultät für Mathematik,
Universität Regensburg, Regensburg, Germany
and
Department of Mathematics,
University of Bologna, Bologna, Italy
e-mail: marco.moraschini2@unibo.it

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