ON CUPPING AND AHMAD PAIRS

ISKANDER SH. KALIMULLIN, STEFFEN LEMPP ⓘD, KENG NG, AND MARS M. YAMALEEV

Abstract. Working toward showing the decidability of the $\forall\exists$-theory of the Σ_2^0-enumeration degrees, we prove that no so-called Ahmad pair of Σ_2^0-enumeration degrees can join to $\mathbf{0}_e'$.

§1. Introduction. Enumeration reducibility is a natural counterpart to its more famous cousin, Turing reducibility, and arises naturally as a notion of relative computability especially in computable model theory as well as, in slightly modified form, Ziegler reducibility in group theory.

This paper is devoted to the study of enumeration reducibility in terms of a degree structure, more specifically, the degree structure of the enumeration degrees of the Σ_2^0-sets, which can be defined also as those enumeration degrees below the degree $\mathbf{0}_e'$, i.e., the enumeration degree of the complement $\overline{K}$ of the halting problem $K = \{e \mid \varphi_e(e)\downarrow\}$. (By this double characterization, the Σ_2^0-enumeration degrees can be viewed as the counterpart of both the c.e. Turing degrees and the Turing degrees $\leq \mathbf{0}'$, i.e., the Δ_2^0-Turing degrees.)

One of the common questions about a degree structure viewed as a partial order is that of the complexity of its first-order theory as well as the decidability of fragments thereof. For most degree structures commonly being considered, the theory turns out to be as complicated as possible (i.e., equivalent to first-order or second-order arithmetic), while the $\exists$-and often even the $\forall\exists$-fragment is decidable and the $\exists\forall\exists$-fragment is not.

For the Σ_2^0-enumeration degrees, the first of these questions has been completely settled: The full first-order theory was shown to be undecidable by Slaman and Woodin [13], and equivalent to full first-order arithmetic by Ganchev and Soskova [3].

As for the second question, the $\exists$-fragment is easily seen to be decidable, whereas Kent [6] showed the $\exists\forall\exists$-fragment to be undecidable. On the other hand, the decidability of the $\forall\exists$-fragment remains open.

The decidability of the $\forall\exists$-fragment can be rephrased algebraically as (uniformly effectively) deciding the following.

QUESTION 1.1. *For any given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i \leq n$), can any embedding of $\mathcal{P}$ into the Σ_2^0-enumeration degrees be extended to an embedding of $\mathcal{Q}_i$ for some $i \leq n$ (where i may depend on the particular embedding of $\mathcal{P}$)?*

1358

Two major subproblems of Question 1.1 have been shown to be decidable:

- Lempp, Slaman, and Sorbi [9] showed that the above question is decidable for $n = 0$, i.e., given any finite partial orders $\mathcal{P} \subseteq \mathcal{Q}$, it is decidable whether any embedding of $\mathcal{P}$ into the Σ_2^0-enumeration degrees can be extended to an embedding of $\mathcal{Q}$.
- Lempp and Sorbi [10] showed that all finite lattices can be embedded, even preserving 0 and 1. (The lattice embeddings question can be seen as a disjunction of extending embeddings to certain one-point extensions $\mathcal{Q}_i$ of a finite lattice $\mathcal{P}$ viewed as a partial order.)

The Σ_2^0-enumeration degrees are often compared to the c.e. Turing degrees, where the situation is somewhat similar but also quite different in other respects: The full first-order theory is as complicated as first-order arithmetic by Slaman and Woodin (unpublished; see [11]); the $\exists$-fragment is easily seen to be decidable, whereas Lempp, Nies, and Slaman [8] showed the $\exists\forall\exists$-fragment to be undecidable; the lattice embeddings problem for the c.e. Turing degrees remains one of the main open problems dating back to the 1960s (see [7] for the most recent update), and thus the decidability of the $\forall\exists$-theory of the c.e. Turing degrees remains wide open as well.

The main algebraic difference between the c.e. Turing degrees and the Σ_2^0-enumeration degrees was discovered by Ahmad in her Ph.D. thesis [1] (see [2, Corollary 3.2]): There are incomparable Σ_2^0-enumeration degrees $\mathbf{a}, \mathbf{b}$ (called an "Ahmad pair") such that any degree $\mathbf{x} < \mathbf{a}$ is also $< \mathbf{b}$. (This makes $\mathbf{a}$ "non-splitting" (i.e., join-irreducible) and thus cannot happen in the c.e. Turing degrees by the Sacks Splitting Theorem [12].) More interestingly, even Ahmad also showed (see [2, Theorem 3.3]) that this phenomenon is *not* symmetric: For any two incomparable Σ_2^0-enumeration degrees $\mathbf{a}, \mathbf{b}$, there is either a degree $\mathbf{x} < \mathbf{a}$ which is $\not\leq \mathbf{b}$, or there is a degree $\mathbf{y} < \mathbf{b}$ which is $\not\leq \mathbf{a}$.

In the language of Question 1.1, Ahmad's results can be rephrased as stating that not every embedding of an antichain $\mathcal{P} = \{a, b\}$ can be extended to an embedding of $\mathcal{Q}_0 = \{a, b, x\}$ where $x < a$ and $x \not\leq b$, but that every embedding of $\mathcal{P}$ can be extended to an embedding of either $\mathcal{Q}_0$ or of $\mathcal{Q}_1 = \{a, b, y\}$ where $y < b$ and $y \not\leq a$.

One of the two main open questions extending the work of the second author, Slaman, and Sorbi [9] asks whether an Ahmad pair of Σ_2^0-enumeration degrees can join to the greatest Σ_2^0-enumeration degree $\mathbf{0}_e'$. In this paper, we answer this question (communicated to the second author in 2007 by Kent) in the negative:

MAIN THEOREM. *There is no cupping Ahmad pair of Σ_2^0-enumeration degrees; i.e., given any two incomparable Σ_2^0-enumeration degrees $\mathbf{a}$ and $\mathbf{b}$, there is either a Σ_2^0-enumeration degree $\mathbf{x} < \mathbf{a}$ with $\mathbf{x} \not\leq \mathbf{b}$, or else $\mathbf{a} \cup \mathbf{b} < \mathbf{0}_e'$.*

This result is a small part of a proposed framework for deciding the $\forall\exists$-theory of the Σ_2^0-degrees, which is equivalent to deciding Question 1.1.

Given the difficulty of the overall problem of deciding the $\forall\exists$-theory, researchers are currently concentrating on the following question concerning 1-point extensions:

QUESTION 1.2. *Given a finite antichain $\mathcal{P} = \{a_0, \ldots, a_n\}$ and 1-point extensions $\mathcal{Q}_S = \{a_0, \ldots, a_n, x_S\}$ and $\mathcal{Q}^T = \{a_0, \ldots, a_n, x^T\}$ for some sets $S \in \mathcal{S}$ and $T \in \mathcal{T}$ (where $\mathcal{S}, \mathcal{T} \subseteq \mathcal{P}(\{0, \ldots, n\}) - \{\emptyset\}$); $x_S < a_i$ iff $i \in S$; and $x^T > a_i$ iff $i \in T$), does*

any embedding of $\mathcal{P}$ into the Σ_2^0-degrees extend to an embedding into the Σ_2^0-degrees of $\mathcal{Q}_S$ for some $S \in \mathcal{S}$ or to an embedding of $\mathcal{Q}^T$ for some $T \in \mathcal{T}$?

Note that it is always possible to extend an embedding of a finite antichain $\mathcal{P}$ to an embedding of a larger antichain, so the case $S = T = \emptyset$ is not interesting. The subproblem involving only extensions $\mathcal{Q}^T$ is easy to see: We have extendibility iff there is a singleton $T \in \mathcal{T}$.

Recent work of Goh, the second and third authors, and Soskova [4, 5] gives a complete (and quite complicated) characterization of the subproblem involving only extensions $\mathcal{Q}_S$.

We have no working conjecture that combines the $\mathcal{Q}_S$ and the $\mathcal{Q}^T$; our main result is the only result known to us in this direction.

§2. The proof of the Main Theorem.
This section is devoted to the proof of our Main Theorem.

2.1. Requirements. Rather than proving the result directly, we use an indirect approach, trying to "weakly split" the degree of A. Specifically, given two Σ_2^0-sets A and B and an enumeration operator Λ such that $\overline{K} = \Lambda(A \oplus B)$ (K here denotes the usual Halting set), we construct two sets $X_0 = \Phi_0(A)$ and $X_1 = \Phi_1(A)$ with these enumeration operators and an enumeration operator Γ, and meet the following requirements for all enumeration operators Δ:

$$\text{Global} : A = \Gamma(B \oplus X_0 \oplus X_1),$$
$$\mathcal{N}_\Delta^0 : A = \Delta(X_0) \Rightarrow \exists\Omega\,(A = \Omega(B)),$$
$$\mathcal{N}_\Delta^1 : A = \Delta(X_1) \Rightarrow \exists\Omega\,(A = \Omega(B)).$$

Here, the enumeration operators Ω are built locally by $\mathcal{N}$-requirements; indeed, as we will see below, each $\mathcal{N}$-requirement will build different versions of Ω based on different guesses about the higher-priority requirements.

If we succeed with these requirements, then we have that $A \leq_e B$, or that X_0, $X_1 <_e A$ and $A \leq_e B \oplus X_0 \oplus X_1$, and so the degrees of A and B clearly cannot form an Ahmad Pair. This corollary is non-uniform in two ways, in that we do not know whether $A \leq_e B$ or which one (or both) of the X_i is not below B, and even assuming $A \not\leq_e B$, we cannot tell uniformly which of X_0 or X_1 (or both) is now below B (in fact, these non-uniformities are unavoidable, as can be shown).

The construction will be a finite-injury construction. The global requirement should proceed carefully to ensure that we do not end up with a "runaway" Γ-axiom. The other requirements will be ordered in some way in order type ω which determines their priority.

Our construction uses the Recursion Theorem with Parameters in that we will define "agitators" that we enumerate into the Halting set K, which will then force $A \oplus B$ to change. More precisely, we assume that we are given an infinite computable list of indices $i_0, i_1, \ldots$ for which we are able to define φ_{i_y} for every y. This allows us to be able to use any of these indices as an "agitator." For any choice of $\mathcal{N}$-requirement, stage s, and potentially numbers a and strings $\sigma \in 2^{<\omega}$, we may define agitators $p_{a,\sigma}$ and q for the $\mathcal{N}$-requirement at stage s, picked from the aforementioned list of

agitators, and the Recursion Theorem with Parameters allows us to assume that we know what corresponding number will enter K when we enumerate an agitator.

2.1.1. Agitators and the use of the Recursion Theorem. We provide more details about agitators and the use of the Recursion Theorem here; the reader who is familiar with this may skip this subsection. As mentioned above, we assume that we are given an infinite computable list of indices $i_0, i_1, \ldots$ for which we are able to control the value of $K(i_y)$ for every y. What this means is that for any y, we can assume that $i_y \notin K$ until we choose to enumerate it into K (via making a definition $\varphi_{i_y}(i_y)$). If we never choose to enumerate i_y into K, then it will never appear in K_s for any s; otherwise, if we choose to enumerate it into K, then it must appear in K_s for some large s which we can then wait for by speeding up the approximation to K_s.

Now it remains to justify the existence of the sequence $i_0, i_1, \ldots$. To do this, we require the use of the Recursion Theorem with Parameters, applied in the following way. We perform infinitely many constructions, rather than just the one described in the proof, and each construction is given a different label $c \in \omega$. These constructions are performed uniformly, and each differs only on the assigned label c. In each construction, we build infinitely many partial computable functions $h_{c,0}, h_{c,1}, \ldots$ representing the list of potential agitators in that construction. The construction will guess that $h_{c,y}$ has index $\varphi_c(c, y)$ for each y, where c is its assigned label. That is, the construction believes that $h_{c,y} = \varphi_{\varphi_c(c,y)}$ for every y. All constructions proceed in exactly the same way, except for its guess on the indices for the sequence $h_{c,0}, h_{c,1}, \ldots$. This allows us to use the s-m-n Theorem to obtain a computable function f such that $h_{c,y} = \varphi_{f(\varphi_c(c,y),c,y)}$ for every c and y. The Recursion Theorem with Parameters gives an index c such that $h_{c,y} = \varphi_{f(\varphi_c(c,y),c,y)} = \varphi_{\varphi_c(c,y)}$ holds for every y. Now the construction which was assigned label c will have the correct guesses on the indices of the agitators.

2.2. Basic strategies for the requirements. We fix a computable enumeration $\{K_s\}$ of K, and let $\overline{K}_s$ be the complement of K_s. By speeding up the enumeration of Λ and the approximations to A and B, we may assume at any stage s that for all numbers $p < s$, if $p \in \overline{K}_s$, then currently $p \in \Lambda(A \oplus B)$, and if $p \notin \overline{K}_s$, then any axiom putting p into $\Lambda(A \oplus B)$ at stage $s - 1$ will no longer apply at stage s.

The global requirement constructing the operator Γ will enumerate axioms into Γ taking into account more and more of the other requirements' action; the basic action of the global requirement is to just enumerate an axiom for a into Γ whenever it sees a enter A. Specifically, for each $a \in A$, we *associate with* a fresh numbers $c_a^i > a$ targeted for X_i for $i < 2$; we then enumerate c_a^i into X_i via the Φ_i-axioms $\langle c_a^i, \{a\}\rangle$, and we enumerate a new axiom $\langle a, B_a \oplus C_a^0 \oplus C_a^1\rangle$ into Γ where B_a is some finite subset of B, and C_a^i is a finite subset of X_i containing c_a^i and possibly other numbers determined later; this will result in $a \in \Gamma(B \oplus X_0 \oplus X_1)$. Now, when a leaves A, this will remove c_a^0 from X_0 and c_a^1 from X_1 unless we introduce additional axioms for either of these numbers. (More generally, ensuring $C_a^0 \not\subseteq X_0$ or $C_a^1 \not\subseteq X_1$ or $B_a \not\subseteq B$ would suffice, as will be required later.)

The basic strategy for the $\mathcal{N}_\Delta^i$-requirement is to try to show that $A = \Omega(B)$ as follows: At each stage, the $\mathcal{N}_\Delta^i$-requirement determines the oldest $a \in A$ (if any) which has not yet been *confirmed* (as defined precisely below) and checks if there

is currently some F such that $a \in \Delta(F)$; if no such F exists, the requirement does nothing for now. If there is such F, the requirement declares a to be *confirmed* and *dumps* all $x \in F$ into X_i by enumerating axioms $\langle x, \emptyset \rangle$ into Φ_i (and thus ensures that $a \in \Delta(X_i)$ permanently); it also chooses a fresh *agitator* p_a, keeps $p_a \in \overline{K}$ for now, and waits for $p_a \in \Lambda(A \oplus B)$ (which must happen by the first stage $s > p_a$) via a Λ-axiom $\langle p_a, F_a \oplus G_a \rangle$, say, for which all $a' \in F_a$ have been confirmed already (which need not ever happen). The $\mathcal{N}_\Delta^i$-requirement then enumerates a into $\Omega(B)$ via the Ω-axiom $\langle a, G_a \rangle$. If p_a leaves $\Lambda(A \oplus B)$ before we enumerate p_a into K (if ever), then we wait again for a to re-enter $\Lambda(A \oplus B)$ via a possibly different Λ-axiom $\langle p_a, F_a' \oplus G_a' \rangle$ (such that again all $a' \in F_a'$ are confirmed); then we will use the (possibly different) set G_a' in redefining $a \in \Omega(B)$, etc.; this can happen only finitely often for this fixed p_a. If a leaves A after a is confirmed, then we enumerate the current agitator p_a into K; if later a re-enters A, then we pick a new, fresh agitator p_a', say, for which we proceed as above (finding sets F_a' and G_a' and enumerating an Ω-axiom $\langle a, G_a' \rangle$, etc.). From now on, whenever some agitator $p_a \in K$ and for any corresponding set G_a, we see $G_a \subseteq B$, then the $\mathcal{N}_\Delta^i$-requirement stops (since it believes that $F_a \nsubseteq A$ while $F_a \subseteq \Delta(X_i)$ by dumping). Note that if a is truly in A, then the final agitator p_a for a will stabilize and be in $\overline{K}$; on the other hand, if $a \notin A$ then a will either appear to be outside A at cofinitely many stages, or the agitator will never stabilize, and any temporary agitator p_a will eventually be in K.

We need to check two things: First of all, note that the $\mathcal{N}_\Delta^i$-requirement can act only finitely often unless $A = \Omega(B)$ (which would contradict our hypotheses): If the requirement acts for the sake of infinitely many distinct a, then, since we always choose a by age, we will have for each $a \in A$ that a will be confirmed and that $a \in \Omega(B)$ by an axiom using the final agitator $p_a \in \overline{K}$ and the B-part G_a of its Λ-use; and for each $a \notin A$ we will have that $a \notin \Omega(B)$ since no Ω-axiom can apply: If $a \in \Omega(B)$ via some axiom $\langle a, G_a \rangle$, say, but $a \notin A$, then G_a is the B-part of the Λ-use of some agitator p_a, but every such agitator p_a will be enumerated into K and thus $G_a \subseteq B$ will eventually stop the $\mathcal{N}_\Delta^i$-requirement permanently. On the other hand, the $\mathcal{N}_\Delta^i$-requirement cannot act infinitely often for some fixed a_0 since that would mean that $a_0 \notin A$ and so the age of a_0 would keep increasing and we would choose other numbers infinitely often (assuming here, of course, that A is infinite).

The second item we need to check is that the $\mathcal{N}_\Delta^i$-requirement satisfies its requirement, so assume that indeed $A = \Delta(X_i)$: Then every a truly in A must be confirmed eventually, but since we may assume A to be infinite, this means that the $\mathcal{N}_\Delta^i$-requirement acts infinitely often and thus ensures $A = \Omega(B)$ as shown in the previous paragraph.

2.3. Conflicts between the $\mathcal{N}$-requirements and the global requirement. A single $\mathcal{N}^i$-requirement will not conflict with the global requirement building and correcting Γ since when a leaves A, Γ can always be corrected by extracting c_a^{1-i} from X_{1-i} (as no requirement has dumped that number into X_{1-i} so far).

So let's consider the case of a higher-priority $\mathcal{N}^0$-requirement above a lower-priority $\mathcal{N}^1$-requirement: The main difficulty is that any number x_0 dumped by the $\mathcal{N}^0$-requirement is associated with some number a potentially in A, which in turn may be associated with a number x_1 that the $\mathcal{N}^1$-requirement may dump into X_1, so

if both x_0 and x_1 are dumped, then we may not be able to correct $\Gamma(B \oplus X_0 \oplus X_1)$ if a leaves A, injuring the global requirement.

So the $\mathcal{N}^1$-requirement, assuming that the $\mathcal{N}^0$-requirement acts only finitely often, will be modified as follows: Each time the $\mathcal{N}^0$-requirement acts, the $\mathcal{N}^1$-requirement's action will be completely undone by enumerating an agitator q (chosen by the $\mathcal{N}^0$-requirement) into K, which will force an $(A \oplus B)$-change undoing all Γ-axioms enumerated since the $\mathcal{N}^0$-requirement was last active (and thus any Γ-axioms enumerated while the $\mathcal{N}^1$-requirement was active since then), where $q \in \Lambda(A \oplus B)$ via a Λ-axiom $\langle q, F_q \oplus G_q \rangle$, say; now the numbers in F_q are associated with sets of numbers F_q^i in X_i for $i < 2$, and so we may assume that each Γ-axiom enumerated since the $\mathcal{N}^0$-requirement was last active contains $B_q \oplus C_q^0 \oplus C_q^1$ in its use; thus we will now ensure that when q enters K, this will entail $B_q \oplus C_q^0 \oplus C_q^1 \not\subseteq B \oplus X_0 \oplus X_1$. This feature will thus prevent the $\mathcal{N}^1$-requirement from interfering with the global requirement as long as the $\mathcal{N}^1$-requirement cannot unconditionally dump numbers into X_1 like the $\mathcal{N}^0$-requirement.

In addition to this extra initialization, we will now modify the strategy for the $\mathcal{N}^1$-requirement as follows: Let r be a strict bound on the largest number ever considered by the $\mathcal{N}^0$-requirement up to this stage, and assume that this bound is reset each time the computation for $q \in \Lambda(A \oplus B)$ changes. (If the $\mathcal{N}^0$-requirement acts only finitely often, then this number will stabilize eventually.)

The $\mathcal{N}^1$-requirement will now be prevented from making any changes to X_1 at any number $< r$ and will work with all possible guesses $\sigma \in 2^r$ about $A \upharpoonright r$. (Recall here that any number x potentially in X_1 is associated with a number $a < x$ potentially in A, or with no number at all, so a guess about $A \upharpoonright r$ will eventually give complete information about $X_1 \upharpoonright r$.) Each version of the strategy for the $\mathcal{N}^1$-requirement (let's call it the $\mathcal{N}_\sigma^1$-strategy) will build its own version of the enumeration operator Ω (call it Ω_σ); each $\mathcal{N}_\sigma^1$-strategy will now act independently, building not only its own Ω_σ but also define its own agitators $p_a = p_{a,\sigma}$. (On the other hand, the agitator q is attached to an entire requirement and so can be joint for all $\mathcal{N}$-strategies working for the same requirement since this is a finite-injury argument.)

Each $\mathcal{N}_\sigma^1$-strategy will then, instead of dumping a number $x < r$ into X_1, simply check if its guess about $A \upharpoonright r$ and about Φ_1 puts x into X_1, and this guess will eventually be correct about each such x; for numbers $x \geq r$, the $\mathcal{N}_\sigma^1$-strategy can still simply "dump" numbers into X_1 but only with the set F_q (from the Λ-axiom $\langle q, F_q \oplus G_q \rangle$) in the A-part of the Φ_1-use. (Also, the strategy will stop if it sees a number $a \in A \upharpoonright r$ with $\sigma(a) = 0$.) Note that this is sufficient to allow the global strategy to correct Γ whenever necessary: If any number a leaves A, then the global strategy can either use X_0 to correct Γ if the Γ-axiom for $a \in \Gamma(B \oplus X_0 \oplus X_1)$ was defined after the stage s_r, say, at which the $\mathcal{N}^0$-requirement last acted (since no strategy will have dumped any number $\geq r$ into X_0 up to this point), or it can use X_1 or B if the Γ-axiom for $a \in \Gamma(B \oplus X_0 \oplus X_1)$ was defined before stage s_r (since any number dumped into X_1 before stage s_r must have been dumped by an $\mathcal{N}_\sigma^1$-strategy before r was last increased, and thus the corresponding agitator q of the $\mathcal{N}^0$-requirement from that time was enumerated into K, resulting in a B-change, or in an A- and thus an X_0-change, invalidating any Γ-axiom for a that might involve a number dumped by an $\mathcal{N}_\sigma^1$-strategy in its use).

We now have to verify two things: Firstly, the $\mathcal{N}_\sigma^1$-strategy with the correct guess $\sigma = A \restriction r$ will ensure the satisfaction of the $\mathcal{N}^1$-requirement. And secondly, any $\mathcal{N}_\sigma^1$-strategy will act at most finitely often even if its guess σ about $A \restriction r$ is incorrect.

So suppose first that r is the final value of this parameter and fix $\sigma = A \restriction r$. Then, starting at some stage s_r, any number $x \in X_1 \restriction r$ will forever be in $X_1 = \Phi_1(A)$. So starting at stage s_r, the $\mathcal{N}_\sigma^1$-strategy can act like the $\mathcal{N}$-strategy in isolation except that it cannot dump any number $x < r$ into X_1, but no such number will be truly in X_1 unless it is so by stage s_r, so this does not cause more than finitely many mistakes for Ω_σ in case the $\mathcal{N}_\sigma^1$-strategy acts infinitely often.

Next, fix any $\mathcal{N}_\sigma^1$-strategy (irrespective of whether $\sigma = A \restriction r$ or not). In order to show that this strategy acts at most finitely often, we distinguish two cases: If $A \restriction r \not\subseteq \sigma$, fix the oldest element $a \in A \restriction r$ with $\sigma(a) = 0$; in that case, the strategy clearly stops as soon as a no longer leaves A. On the other hand, if $A \restriction r \subseteq \sigma$, then the strategy will act as in isolation (since it will never want to dump a number into X_1 that it cannot dump) and so must also eventually stop unless $A \leq_e B$.

This concludes the presentation of the intuition behind our construction; we are now ready to describe it formally in full.

2.4. The full construction. Recall that we are given approximations to Σ_2^0-sets A and B and an enumeration operator Λ such that at any stage s, for any number $p < s$, if $p \in \overline{K}_s$ then $p \in \Lambda(A \oplus B)[s]$, and if $p \in K_s$ then p is not in both $\Lambda(A \oplus B)[s-1]$ and $\Lambda(A \oplus B)[s]$ via the same axiom. Furthermore, we may assume to be given a *good approximation* $\{A_s \oplus B_s\}_{s \in \omega}$ (of finite sets) to $A \oplus B$, i.e., there are infinitely many *true stages*, namely, stages s with $A_s \oplus B_s \subseteq A \oplus B$.

Since we have a finite-injury construction with one global requirement, the action at each stage s will consist of a single action for the highest-priority $\mathcal{N}$-requirement requiring attention, followed by some global action. At stage 0, all $\mathcal{N}$-requirements are *initialized* and all functionals and agitators are set to be undefined.

We say that an $\mathcal{N}_{\Delta,\sigma}^i$-strategy (for $\sigma \in 2^r$ and some $r \in \omega$) *requires attention* if:

- there is no $a \in A_s \restriction r$ with $\sigma(a) = 0$; and
- there is no agitator $p_a \in K_s$ with $G_a \subseteq B_s$ for the corresponding Λ-axiom $\langle p_a, F_a \oplus G_a \rangle$ where $p_a = p_{a,\sigma'}$ is any agitator ever used by an $\mathcal{N}_{\Delta,\sigma'}^i$-strategy working for the same $\mathcal{N}_\Delta^i$-requirement; and
- one of the following four conditions holds:
  (1) there is some $a \in \Omega(B)[s] - A_s$ (with the oldest Ω-axiom applicable); or
  (2) there is some (oldest) $a \in A_s - \Omega(B)[s]$ which has already been confirmed by the $\mathcal{N}_{\Delta,\sigma}^i$-strategy (since its last initialization) via F, say, and each $a' \in F$ has also already been confirmed by the $\mathcal{N}_{\Delta,\sigma}^i$-strategy (since its last initialization); or
  (3) there is some (oldest) $a \in A_s$ which has not yet been confirmed by the $\mathcal{N}_{\Delta,\sigma}^i$-strategy (since its last initialization) and for which there is a finite set F (of least canonical index) with $F \restriction r = \Phi_i(\sigma) \restriction r$ and $a \in \Delta_s(F)$; or
  (4) all the strategies for this $\mathcal{N}$-requirement have not acted since they were last initialized.

If an $\mathcal{N}_{\Delta,\sigma}^i$-strategy requires attention, we also say that the corresponding $\mathcal{N}_\Delta^i$-requirement *requires attention*.

At a stage $s > 0$, we first check if there is a (highest-priority) $\mathcal{N}_\Delta^i$-requirement requiring attention; if so, allow all $\mathcal{N}_{\Delta,\sigma}^i$-strategies working for this $\mathcal{N}_\Delta^i$-requirement to act if they require attention. Each such $\mathcal{N}_{\Delta,\sigma}^i$-strategy will now act according to the first clause of the third bullet that applies.

If (1) applies, then the $\mathcal{N}_{\Delta,\sigma}^i$-strategy enumerates into K the agitator $p_a = p_{a,\sigma}$ which was chosen when the current Ω_σ-axiom for $a \in \Omega_\sigma(B)$ was defined. (Note that $p_a \notin K_s$ since otherwise, the second bullet would have prevented the $\mathcal{N}_\Delta^i$-requirement from requiring attention.)

If (2) applies, then the $\mathcal{N}_{\Delta,\sigma}^i$-strategy, for the current agitator $p_a = p_{a,\sigma}$ and the (oldest) Λ-axiom putting $p_a \in \Lambda(A \oplus B)$ with use $F_a \oplus G_a$, enumerates the axiom $\langle a, G_a \rangle$ into Ω_σ.

If (3) applies, then the $\mathcal{N}_{\Delta,\sigma}^i$-strategy declares a to be *confirmed via F*, chooses a fresh agitator p_a, and enumerates all of $F - [0, r)$ (for the threshold r imposed on this requirement) into $\Phi_i(A)$ with use including all F_q for all agitators $q \in \overline{K}$ defined by strategies for higher-priority requirements (where $q \in \Lambda(A \oplus B)$ via an (oldest) Λ-axiom $\langle q, F_q \oplus G_q \rangle$). If now (2) applies to this a, then continue as for that case, else proceed to the global action for this stage.

If one of (1)–(3) applies, then we also enumerate into K the agitator q of the $\mathcal{N}$-requirement for which we acted at this stage, choose a new fresh agitator q, choose a new threshold r above all numbers mentioned so far which will be imposed on the lower-priority strategies, and initialize all lower-priority $\mathcal{N}$-strategies.

If (4) applies, then the $\mathcal{N}_\Delta^i$-requirement, i.e., the $\mathcal{N}_{\Delta,\sigma}^i$-strategies (for all $\sigma \in 2^r$, where r is the threshold imposed by the higher-priority strategies) simply defines a new threshold r' above all numbers mentioned so far which is imposed on all lower-priority $\mathcal{N}$-strategies.

After the strategies for an $\mathcal{N}$-requirement have acted, the global requirement building Γ will act. First of all, as we will show in the verification below (see Lemma 2.3), since all corrections will be automatic there will never be a number a and a stage s such that $a \in \Gamma(B \oplus X_0 \oplus X_1) - A$. So the global requirement will simply identify the (oldest) $a \in A - \Gamma(B \oplus X_0 \oplus X_1)$ (which must exist since A is infinite), let Q be the set of all current agitators q of $\mathcal{N}$-requirements whose current threshold r is less than $< a$, and then enumerate an axiom $\langle a, B_a \oplus C_a^0 \oplus C_a^1 \rangle$ into Γ where:

- B_a contains all sets G_q such that $q \in \Lambda(A \oplus B)$ via a Λ-axiom $\langle q, F_q \oplus G_q \rangle$ for some $q \in Q$ and some finite set F_q,
- C_a^0 contains all $c_{a'}^0$ for all $a' \in F_q$ such that $q \in \Lambda(A \oplus B)$ via a Λ-axiom $\langle q, F_q \oplus G_q \rangle$ for some $q \in Q$ and some finite sets F_q and G_q, as well as a freshly chosen number c_a^0 for which we enumerate c_a^0 into X_0 via a Φ_0-axiom $\langle c_a^0, \{a\} \rangle$, and
- C_a^1 contains all $c_{a'}^1$ for all $a' \in F_q$ such that $q \in \Lambda(A \oplus B)$ via a Λ-axiom $\langle q, F_q \oplus G_q \rangle$ for some $q \in Q$ and some finite sets F_q and G_q, as well as a freshly chosen number c_a^1 for which we enumerate c_a^1 into X_1 via a Φ_1-axiom $\langle c_a^1, \{a\} \rangle$.

In the last two bullets above, for each a', if there is no $c_{a'}^i$ (for $i < 2$) which has not yet been dumped and for which there is a Φ_i-axiom $\langle c_{a'}^i, \{a'\} \rangle$, then we pick a fresh such number $c_{a'}^i$ for the Γ-axiom, enumerate a corresponding axiom $\langle c_{a'}^i, \{a'\} \rangle$ into

Φ_i, and use it in the above Γ-axiom. Also, recall here that each current agitator $q \in Q$ must be in $\overline{K}$ and thus in $\Lambda(A \oplus B)$.

This concludes the formal description of the construction.

2.5. Verification. We now verify that our construction satisfies the requirements in a number of lemmas. In addition to assuming that $\overline{K} = \Lambda(A \oplus B)$ and that $A \oplus B$ has a good approximation, we will also assume tacitly throughout that $A \not\leq_e B$ and in particular that A is infinite.

We start by verifying that our construction is indeed finite injury:

LEMMA 2.1. *Each $\mathcal{N}$-requirement acts at most finitely often and eventually defines a final threshold r and a final agitator q which is in $\overline{K}$ (unless it does not have an eventual agitator at all).*

PROOF. We proceed by induction on the priority of the $\mathcal{N}$-requirement. So fix an $\mathcal{N}_\Delta^i$-requirement and assume the lemma for all higher-priority $\mathcal{N}$-requirements. Fix a (least) stage s_0 such that no higher-priority $\mathcal{N}$-requirement acts after stage s_0. Let r_0 be the maximum of the final thresholds of all these higher-priority $\mathcal{N}$-requirements, and let Q be the set of the final agitators of all these higher-priority $\mathcal{N}$-requirements.

Let's now analyze how the $\mathcal{N}_\Delta^i$-requirement can require attention after stage s_0: First of all, observe that clause (4) of the third bullet can apply at most once after stage s_0.

For the sake of a contradiction, assume that an $\mathcal{N}_{\Delta,\sigma}^i$-strategy acts infinitely often. By the first bullet in the conditions for requiring attention, there can be no $a \in A$ with $\sigma(a) = 0$.

Furthermore, clause (3) of the third bullet must apply infinitely often (and thus infinitely many a will be confirmed by the $\mathcal{N}_{\Delta,\sigma}^i$-strategy), since clauses (1) and (2) of the third bullet can possibly apply to a fixed a only if $a \notin A$, but then the age of a keeps increasing and, since A is infinite, infinitely many other numbers must take precedence in requiring attention. This implies, by age and clause (3) of the third bullet, that each $a \in A$ must eventually be confirmed, and so, since the first bullet fails for σ, $a \in \Delta(X_i)$. But then there will be a final agitator $p_a = p_{a,\sigma}$, which will be in $\overline{K}$ and thus in $\Lambda(A \oplus B)$ via a Λ-axiom $\langle p_a, F_a \oplus G_a \rangle$, say. Thus $a \in \Omega_\sigma(B)$ via the axiom $\langle a, G_a \rangle$. Conversely, if $a \notin A$, then any axiom $\langle a, G_a \rangle$ enumerated into Ω_σ cannot apply and put $a \in \Omega_\sigma(B)$ since otherwise, $G_a \subseteq B$ while the corresponding agitator $p_a \notin \overline{K}$, so the second bullet for requiring attention will eventually prevent any $\mathcal{N}_\Delta^i$-strategy from acting. We have thus shown that an $\mathcal{N}_{\Delta,\sigma}^i$-strategy acting infinitely often will imply $A = \Omega_\sigma(B)$, contrary to assumption.

Thus the $\mathcal{N}_\Delta^i$-requirement will act at most finitely often and eventually define a final threshold r and a final agitator q (if there is an agitator q ever defined).

Note that the action of the global requirement does not interfere with any of the above argument. ⊣

We next verify that each $\mathcal{N}$-requirement is satisfied.

LEMMA 2.2. *Each $\mathcal{N}_\Delta^i$-requirement is satisfied.*

PROOF. Suppose that $A = \Delta(X_i)$. Fix the final threshold r of the higher-priority $\mathcal{N}$-requirements and set $\sigma = A \restriction r$. We will show that the $\mathcal{N}^i_{\Delta,\sigma}$-strategy acts infinitely often, contradicting Lemma 2.1.

First, note that for each a, the first and second bullets of requiring attention will eventually not prevent the $\mathcal{N}^i_{\Delta,\sigma}$-strategy from acting. (This is clear for the first bullet by definition of σ, and for the second bullet, we argue that otherwise, $p_a \in K$ and $G_a \subseteq B$ implies $p_a \notin \Lambda(A \oplus B)$ and so $F_a \not\subseteq A$, say, $a' \in F_a - A$. But a' is confirmed, so $\in \Delta(X_i) - A$, contradicting our assumption.)

But now $A = \Delta(X_i)$ implies that every $a \in A$ will be confirmed, implying that the $\mathcal{N}^i_{\Delta,\sigma}$-strategy acts infinitely often, again contradicting Lemma 2.1. ⊣

Finally, we need to verify that the global requirement is satisfied.

LEMMA 2.3. *The global requirement is satisfied*: $A = \Gamma(B \oplus X_0 \oplus X_1)$.

PROOF. We need to prove two directions. Assume first that $a \in A$, so we need to verify that some Γ-axiom will eventually put a into $\Gamma(B \oplus X_0 \oplus X_1)$. So find a stage s such that all thresholds r of $\mathcal{N}$-requirements with $r < a$, as well as their potential agitators q, have settled down. Then for each such agitator q, $q \in \overline{K}$ and thus there will be a stable Λ-axiom $\langle q, F_q \oplus G_q \rangle$ putting $q \in \Lambda(A \oplus B)$, say, all of this happens by a stage $s' \geq s$. Then by stage s', the global strategy will have enumerated a Γ-axiom $\langle a, B_a \oplus C_a^0 \oplus C_a^1 \rangle$ with $B_a \subseteq B$ and $C_a^i \subseteq X_i$ for $i < 2$, and so $a \in \Gamma(B \oplus X_0 \oplus X_1)$.

Conversely, suppose $a \notin A$, so we need to show that no valid Γ-axiom puts a into $\Gamma(B \oplus X_0 \oplus X_1)$. For the sake of a contradiction, suppose that at some stage s, while $a \in A_s$, the global strategy enumerates a valid axiom $\langle a, B_a \oplus C_a^0 \oplus C_a^1 \rangle$ into Γ. We now distinguish two possibilities:

Assume first that after this stage s, no $\mathcal{N}$-requirement with current threshold $r \leq a$ acts; then the $\mathcal{N}$-requirements with threshold $r \leq a$ cannot dump any numbers $c_{a'}^i$ into X_i involved in the Γ-axiom putting a into $\Gamma(B \oplus X_0 \oplus X_1)$ (for either $i < 2$); and only the highest-priority $\mathcal{N}$-requirement with threshold $r > a$ can dump any such numbers into only one of the X_i, so $a \notin \Gamma(B \oplus X_0 \oplus X_1)$ by an X_{1-i}-change.

On the other hand, suppose that some $\mathcal{N}$-requirement with current threshold $r \leq a$ acts, say, the first such requirement is an $\mathcal{N}^j$-requirement acting at a stage $s' > s$. In that case, the $\mathcal{N}^j$-requirement's current agitator q will be enumerated into K, and so the Λ-axiom involving $F_q \oplus G_q$ will no longer apply. Now if $G_q \not\subseteq B$, then the Γ-axiom will clearly no longer apply for a. If only $F_q \not\subseteq A$, then we will need to argue that $C_a^0 \not\subseteq X_0$ or $C_a^1 \not\subseteq X_1$ to invalidate the Γ-axiom for a, but by the choice of the numbers $c_{a'}^i$ in the definition of the Γ-axiom for a, both $c_{a'}^0$ and $c_{a'}^1$ will leave X_0 and X_1, respectively, unless they will be dumped. However, since any $\mathcal{N}$-requirement of lower priority than the $\mathcal{N}^j$-requirement is prevented from dumping a number in C_a^{1-j} into X_{1-j} after the $\mathcal{N}^j$-requirement acts at stage s', we know that $C_a^{1-j} \not\subseteq X_{1-j}$ as long as no $\mathcal{N}$-requirement of higher priority acts later. If that should happen, we repeat the above argument until there is no even higher-priority $\mathcal{N}$-requirement acting to dump even smaller numbers. So eventually, we see that this Γ-axiom cannot apply to a anymore, as desired.

This establishes that the global requirement does indeed succeed as desired, completing the proof. ⊣

REFERENCES

[1] S. AHMAD, *Some results on the structure of the Σ_2 enumeration degrees*, Ph.D. thesis, Simon Fraser University, Canada, 1989.

[2] S. AHMAD and A. H. LACHLAN, *Some special pairs of Σ_2 e-degrees*. **Mathematical Logic Quarterly**, vol. 44 (1998), no. 4, pp. 431–449.

[3] H. A. GANCHEV and M. I. SOSKOVA, *Interpreting true arithmetic in the local structure of the enumeration degrees*, this JOURNAL, vol. 77 (2012), pp. 1184–1194.

[4] J. L. GOH, K. M. NG, S. LEMPP, and M. I. SOSKOVA, *Extensions of two constructions of Ahmad*. **Computability**, vol. 11 (2022), no. 3-4, pp. 269–297.

[5] ———, *Extensions of embeddings of 1-point extensions of finite antichains in the Σ_2^0-enumeration degrees*, in preparation.

[6] T. F. KENT, *The Π_3-theory of the Σ_2^0-enumeration degrees is undecidable*, this JOURNAL, vol. 71 (2006), pp. 1284–1302.

[7] S. LEMPP, M. LERMAN, and D. REED SOLOMON, *Embedding finite lattices into the computably enumerable degrees—A status survey*, **Logic Colloquium'02** (Z. Chatzidakis, P. Koepke and W. Pohlers, editors), Lecture Notes in Logic, vol. 27, Association for Symbolic Logic, La Jolla, 2006, pp. 206–229.

[8] S. LEMPP, A. NIES, and T. A. SLAMAN, *The Π_3-theory of the computably enumerable Turing degrees is undecidable*. **Transactions of the American Mathematical Society**, vol. 350 (1998), pp. 2719–2736.

[9] S. LEMPP, T. A. SLAMAN, and A. SORBI, *On extensions of embeddings into the enumeration degrees of the Σ_2^0-sets*. **Journal of Mathematical Logic**, vol. 5 (2005), pp. 247–298.

[10] S. LEMPP and A. SORBI, *Embedding finite lattices into the Σ_2^0-enumeration degrees*, this JOURNAL, vol. 67 (2002), pp. 69–90.

[11] A. NIES, R. A. SHORE, and T. A. SLAMAN, *Interpretability and definability in the recursively enumerable degrees*. **Proceedings of the London Mathematical Society (3)**, vol. 77 (1998), pp. 241–291.

[12] G. E. SACKS, *On the degrees less than $\mathbf{0}'$*. **Annals of Mathematics. Second Series**, vol. 77 (1963), pp. 211–231.

[13] T. A. SLAMAN and W. H. WOODIN, *Definability in the enumeration degrees*. **Archive for Mathematical Logic**, vol. 36 (1997), nos. 4–5, pp. 255–267.

N.L. LOBACHEVSKY INSTITUTE OF MATHEMATICS AND MECHANICS
    KAZAN FEDERAL UNIVERSITY
        KAZAN 420008, RUSSIA
E-mail: iskander.kalimullin@kpfu.ru
URL: https://kpfu.ru/Iskander.Kalimullin?p_lang=2

E-mail: marsiam2@yandex.ru
URL: https://kpfu.ru/Mars.Yamaleev?p_lang=2

DEPARTMENT OF MATHEMATICS
    UNIVERSITY OF WISCONSIN
        MADISON, WI 53706-1325, USA
E-mail: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/∼lempp/

DIVISION OF MATHEMATICAL SCIENCES
SCHOOL OF PHYSICAL & MATHEMATICAL SCIENCES
COLLEGE OF SCIENCE
NANYANG TECHNOLOGICAL UNIVERSITY, SINGAPORE
E-mail: kmng@ntu.edu.sg
URL: http://www.ntu.edu.sg/home/kmng/