

LATTICE STRUCTURE OF TORSION CLASSES FOR HEREDITARY ARTIN ALGEBRAS

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Abstract. Let Λ be a connected hereditary artin algebra. We show that the set of functorially finite torsion classes of Λ -modules is a lattice if and only if Λ is either representation-finite (thus a Dynkin algebra) or Λ has only two simple modules. For the case of Λ being the path algebra of a quiver, this result has recently been established by Iyama–Reiten–Thomas–Todorov and our proof follows closely some of their considerations.

Let Λ be a connected hereditary artin algebra. The modules considered here are left Λ -modules of finite length, $\text{mod } \Lambda$ denotes the corresponding category. The subcategories of $\text{mod } \Lambda$ we deal with are always assumed to be closed under direct sums and direct summands (in particular closed under isomorphisms). In this setting, a subcategory is a *torsion class* (the class of torsion modules for what is called a torsion pair or a torsion theory) provided it is closed under factor modules and extensions. The torsion classes form a partially ordered set with respect to inclusion, it will be denoted by $\text{tors } \Lambda$. This poset clearly is a lattice (even a complete lattice). It is easy to see that a torsion class \mathcal{C} in $\text{mod } \Lambda$ is functorially finite if and only if it has a cover (a *cover* for \mathcal{C} is a module C such that \mathcal{C} is the set of modules generated by C), we denote by $\text{f-tors } \Lambda$ the set of functorially finite torsion classes in $\text{mod } \Lambda$.

In a recent paper [IRT], Iyama *et al.* have discussed the question whether the poset $\text{f-tors } \Lambda$ (with the inclusion order) also is a lattice.

THEOREM. *The poset $\text{f-tors } \Lambda$ is a lattice if and only if Λ is representation-finite or Λ has precisely two simple modules.*

Iyama *et al.* have shown this in the special case when Λ is a k -algebra with k an algebraically closed field (so that Λ is Morita equivalent to the path algebra of a quiver). The aim of this note is to provide a proof in general.

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Here is an outline of the essential steps of the proof. Recall that a module is called *exceptional* provided it is indecomposable and has no self-extensions. A pair of modules X, Y will be called an *Ext-pair* provided both X, Y are exceptional, $\text{Hom}(X, Y) = \text{Hom}(Y, X) = 0$ and $\text{Ext}^1(X, Y) \neq 0$, $\text{Ext}^1(Y, X) \neq 0$. We follow the strategy of [IRTT] by establishing the existence of Ext-pairs for any connected hereditary artin algebra which is representation-infinite and has at least three simple modules (Proposition 5). On the other hand, we show directly that the set of functorially finite torsion classes which contain a fixed Ext-pair has no minimal elements (Proposition 4).

§1. Normalization

Let \mathcal{X} be a class of modules. We denote by $\text{add}(\mathcal{X})$ the modules which are direct summands of direct sums of modules in \mathcal{X} . A module M is *generated* by \mathcal{X} provided M is a factor module of a module in $\text{add}(\mathcal{X})$, and M is *cogenerated* by \mathcal{X} provided M is a submodule of a module in $\text{add}(\mathcal{X})$. The subcategory of all modules generated by \mathcal{X} is denoted by $\mathcal{G}(\mathcal{X})$. In case $\mathcal{X} = \{X\}$ or $\mathcal{X} = \text{add } X$, we write $\mathcal{G}(X)$ instead of $\mathcal{G}(\mathcal{X})$, and use the same convention in similar situations. We write $\mathcal{T}(X)$ for the smallest torsion class containing the module X (it is the intersection of all torsion classes containing X , and it can be constructed as the closure of $\{X\}$ using factor modules and extensions).

Since Λ is assumed to be hereditary, we write $\text{Ext}(X, Y)$ instead of $\text{Ext}^1(X, Y)$.

Following Roiter [Ro], we say that a module M is *normal* provided there is no proper direct decomposition $M = M' \oplus M''$ such that M' generates M'' (this means: if $M = M' \oplus M''$ and M' generates M'' , then $M'' = 0$). Of course, given a module M , there is a direct decomposition $M = M' \oplus M''$ such that M' is normal and M' generates M'' and one can show that M' is determined by M uniquely up to isomorphism, thus we call $M' = \nu(M)$ a *normalization* of M . This was shown already by Roiter [Ro], and later by Auslander–Smalø [AS]. It is also a consequence of the following Lemma which will be needed for our further considerations.

LEMMA 1.

- (a) *Let $(f_1, \dots, f_t, g) : X \rightarrow X^t \oplus Y$ be an injective map for some natural number t , with all the maps f_i in the radical of $\text{End}(X)$. Then X is cogenerated by Y .*

(b) Let $(f_1, \dots, f_t, g) : X^t \oplus Y \rightarrow X$ be a surjective map for some natural number t , with all the maps f_i in the radical of $\text{End}(X)$, then Y generates X .

Proof. (a) Assume that the radical J of $\text{End}(X)$ satisfies $J^m = 0$. Let W be the set of all compositions w of at most $m - 1$ maps of the form f_i with $1 \leq i \leq t$ (including $w = 1_X$). We claim that $(gw)_{w \in W} : X \rightarrow Y^{|W|}$ is injective. Take a nonzero element x in X . Then there is $w \in W$ such that $w(x) \neq 0$ and $f_i w(x) = 0$ for $1 \leq i \leq t$. Since (f_1, \dots, f_t, g) is injective and $w(x) \neq 0$, we have $(f_1, \dots, f_t, g)(w(x)) \neq 0$. But $f_i w(x) = 0$ for $1 \leq i \leq t$, thus $g(w(x)) \neq 0$. This completes the proof.

(b) This follows by duality. □

COROLLARY [UNIQUENESS OF NORMALIZATION]. *Let M be a module. Assume that $M = M_0 \oplus M_1 = M'_0 \oplus M'_1$ such that both M_0 and M'_0 generate M . Then there is a module N which is a direct summand of both M_0 and M'_0 which generates M .*

Proof. We may assume that M is multiplicity free. Write $M_0 \simeq N \oplus C$, $M'_0 \simeq N \oplus C'$, such that C, C' have no indecomposable direct summand in common. Now, $N \oplus C$ generates $N \oplus C'$, $N \oplus C'$ generates $N \oplus C$, and $N \oplus C$ generates C . We see that $N \oplus C$ generates C , such that the maps $C \rightarrow C$ used belong to the radical of $\text{End}(C)$ (since they factor through $\text{add}(N \oplus C')$ and no indecomposable direct summand of C belongs to $\text{add}(N \oplus C')$). Lemma 1 asserts that N generates C , thus it generates M . □

PROPOSITION 1. *If T has no self-extensions, then T is a cover for the torsion class $\mathcal{T}(T)$. Conversely, if \mathcal{T} is a torsion class with cover C , then $\nu(C)$ has no self-extensions.*

Proof. For the first assertion, one has to observe that $\mathcal{G}(T)$ is closed under extensions, thus equal to $\mathcal{T}(T)$. This is a standard result say in tilting theory. Here is the argument: let $g' : T' \rightarrow M'$ and $g'' : T'' \rightarrow M''$ be surjective maps with T', T'' in $\text{add } T$. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence. The induced exact sequence with respect to g'' is of the form $0 \rightarrow M' \rightarrow Y_1 \rightarrow T'' \rightarrow 0$ with a surjective map $g_1 : Y_1 \rightarrow M$. Since Λ is hereditary and g' is surjective, there is an exact sequence $0 \rightarrow T' \rightarrow Y_2 \rightarrow T'' \rightarrow 0$ with a surjective map $g_2 : Y_2 \rightarrow Y_1$. Since $\text{Ext}(T'', T') = 0$, we see that Y_2 is isomorphic to $T' \oplus T''$, thus in $\text{add } T$. And there is the surjective map $g_1 g_2 : Y_2 \rightarrow M$.

For the converse, we may assume that C is normal and have to show that C has no self-extension. Let C_1, C_2 be indecomposable direct summands of C and assume for the contrary that there is a nonsplit exact sequence

$$0 \rightarrow C_1 \rightarrow M \rightarrow C_2 \rightarrow 0.$$

Now M belongs to \mathcal{T} , thus it is generated by C , say there is a surjective map $C' \rightarrow M$ with $C' \in \text{add } C$. Write $C' = C_2^t \oplus C''$ such that C_2 is not a direct summand of C'' . Consider the surjective map $C_2^t \oplus C'' \rightarrow M \rightarrow C_2$. Since the last map $M \rightarrow C_2$ is not a split epimorphism, all the maps $C_2 \rightarrow C_2$ involved belong to the radical of $\text{End}(C_2)$. According to Lemma 1, C'' generates C_2 . This contradicts the assumption that C is normal. \square

REMARK. Proposition 1 provides a bijection between the isomorphism classes of normal modules without self-extensions and torsion classes with covers. This is one of the famous Ingalls–Thomas bijections; see for example [ONFR] or also [R3].

We recall that a torsion class is functorially finite if and only if it has a cover. Of course, if C is a cover of the torsion class \mathcal{T} , then $\mu(C)$ is a minimal cover of \mathcal{T} .

PROPOSITION 2. *Let \mathcal{T} be a nonzero functorially finite torsion class. Then there is an indecomposable module U in \mathcal{T} such that any nonzero map $V \rightarrow U$ with $V \in \mathcal{T}$ is a split epimorphism.*

Proof. Let C be a minimal cover of \mathcal{T} . Since C has no self-extensions, it is a direct summand of a tilting module. In particular, the quiver of $\text{End}(C)$ is directed. It follows that C has an indecomposable direct summand U such that any nonzero map $C \rightarrow U$ is a split epimorphism. Assume now that V belongs to \mathcal{T} and $f : V \rightarrow U$ is a nonzero map. There is a surjective map $g : C^t \rightarrow V$ for some t . Since the composition $fg : C^t \rightarrow U$ is nonzero, it is split epi, thus also f is split epi. \square

REMARK. As we have mentioned, normal modules have been considered by Roiter, but actually, he used a slightly deviating name, calling them “normally indecomposable”.

§2. Inclusions of functorially finite torsion classes

If \mathcal{X} is a class of modules and U is an indecomposable module, we denote by \mathcal{X}_U the class of modules in \mathcal{X} which have no direct summand isomorphic to U .

PROPOSITION 3. *Assume that \mathcal{T} is a torsion class and that U is an indecomposable module in \mathcal{T} . The following assertions are equivalent:*

- (i) *The class \mathcal{T}_U is a torsion class.*
- (ii) *Any nonzero map $V \rightarrow U$ with $V \in \mathcal{T}$ is split epi.*

Proof. (i) \implies (ii). We assume that \mathcal{T}_U is a torsion class. Let $f : V \rightarrow U$ be a nonzero map with $V \in \mathcal{T}$. We claim that f is surjective. Note that $f(V)$ and $U/f(V)$ both belong to \mathcal{T} , since \mathcal{T} is closed under factor modules. If f is not surjective, then $f(V)$ is a factor module of V and a proper nonzero submodule of U , whereas $U/f(V)$ is a proper nonzero factor module of U . It follows that both $f(V)$ and $U/f(V)$ belong to \mathcal{T}_U . Since we assume that \mathcal{T}_U is a torsion class, it is closed under extensions, and therefore U belongs to \mathcal{T}_U , a contradiction.

Write $V = V' \oplus U^t$ for some t with V' in \mathcal{T}_U . If f is not split epi, then Lemma 1(b) asserts that V' generates U . But we assume that \mathcal{T}_U is a torsion class, thus closed under direct sums and factor modules. Therefore, if V' generates U , then U has to belong to \mathcal{T}_U , again a contradiction. Altogether we have shown that f is split epi.

(ii) \implies (i). We assume now that any nonzero map $V \rightarrow U$ with $V \in \mathcal{T}$ is a split epimorphism, and we have to show that \mathcal{T}_U is a torsion class. In order to see that \mathcal{T}_U is closed under factor modules, let V belong to \mathcal{T}_U and let W be a factor module of V . Assume that U is a direct summand of W , thus U is a factor module of V . The projection $p : V \rightarrow U$ is a nonzero map, thus by assumption p is a split epimorphism. But this implies that U is a direct summand of V , whereas V belongs to \mathcal{T}_U . This shows that W belongs to \mathcal{T}_U .

In order to show that \mathcal{T}_U is closed under extensions, consider a module M with a submodule V such that both V and M/V belong to \mathcal{T}_U . Since \mathcal{T} is closed under extension, M belongs to \mathcal{T} . Assume that U is a direct summand of M , say $M = U \oplus M'$. If $V \subseteq M'$, then $M/V = U \oplus M'/V$ shows that U is a direct summand of M/V in contrast to our assumption that M/U belongs to \mathcal{T}_U . Thus $V \not\subseteq M'$. It follows that V is not contained in the kernel of the canonical projection $q : M \rightarrow M/M' \simeq U$, thus the restriction of q to V is a nonzero map $V \rightarrow U$. The condition (ii) asserts that this map $V \rightarrow U$ is split epi, therefore V does not belong to \mathcal{T}_U , a contradiction. This shows that M belongs to \mathcal{T}_U . □

PROPOSITION 4. *Let \mathcal{E} be a class of indecomposable modules with the following property: if E belongs to \mathcal{E} , there is E' in \mathcal{E} with $\text{Ext}(E, E') \neq 0$. Then the set of functorially finite torsion classes \mathcal{T} which contain \mathcal{E} has no minimal elements.*

Proof. Let \mathcal{T} be a functorially finite torsion class which contains \mathcal{E} . According to Proposition 2, there is an indecomposable module U in \mathcal{T} such that any nonzero map $V \rightarrow U$ with $V \in \mathcal{T}$ is a split epimorphism. According to Proposition 3, the class \mathcal{T}_U is a torsion class. Since \mathcal{T} is functorially finite, also \mathcal{T}_U is functorially finite.

We claim that \mathcal{E} is contained in \mathcal{T}_U . Thus, let E belong to \mathcal{E} . Since E is indecomposable, we have to show that E is not isomorphic to U . By assumption, there is E' in \mathcal{E} with $\text{Ext}(E, E') \neq 0$. Thus, there is a nonsplit exact sequence $0 \rightarrow E' \rightarrow M \rightarrow E \rightarrow 0$. Since E, E' both belong to $\mathcal{E} \subseteq \mathcal{T}$ and \mathcal{T} is closed under extensions, M belongs to \mathcal{T} . Since the given map $M \rightarrow E$ is not split epi, it follows that E is not isomorphic to U . Thus $\mathcal{E} \subseteq \mathcal{T}_U$. Since \mathcal{T}_U is properly contained in \mathcal{T} , we see that \mathcal{T} is not minimal in the set of functorially finite torsion classes which contain \mathcal{E} . □

§3. Construction of Ext-pairs

The aim of this section is to show the following proposition.

PROPOSITION 5. *A connected hereditary artin algebra which is representation-infinite and has at least three simple modules has Ext-pairs.*

Given a finite-dimensional artin algebra R , we denote by $Q(R)$ its Ext-quiver: its vertices are the isomorphism classes $[S]$ of the simple R -modules S , and given two simple R -modules S, S' , there is an arrow $[S] \rightarrow [S']$ provided $\text{Ext}(S, S') \neq 0$. If R is hereditary, then clearly $Q(R)$ is directed. If necessary, we endow $Q(R)$ with a valuation as follows: Given an arrow $S \rightarrow S'$, consider $\text{Ext}(S, S')$ as a left $\text{End}(S)^{\text{op}}$ -module or as a left $\text{End}(S')$ -module and put

$$v([S], [S']) = (\dim_{\text{End}(S)^{\text{op}}} \text{Ext}(S, S'))(\dim_{\text{End}(S')} \text{Ext}(S, S'))$$

(note that in contrast to [DR], we only need the product of the two dimensions, not the pair). Given a vertex i of $Q(R)$, we denote by $S(i), P(i), I(i)$ a simple, projective or injective module corresponding to the vertex i , respectively.

The valuation of any arrow can be interpreted as follows (τ is the Auslander–Reiten translation).

LEMMA 2. *If $Q(\Lambda) = (1 \rightarrow 2)$, then the arrow $1 \rightarrow 2$ has valuation at least 2 if and only if $I(2)$ is not projective if and only if $P(1)$ is not injective. If the arrow $1 \rightarrow 2$ has valuation at least 3, then $\tau S(1)$ is neither projective, nor a neighbor of $P(1)$ in the Auslander–Reiten quiver, consequently $\text{Hom}(P(1), \tau^2 S(1)) \neq 0$, thus $\text{Ext}(\tau S(1), P(1)) \neq 0$. \square*

In the proof of Proposition 5, we have to construct some exceptional modules. Two general results will be needed.

LEMMA 3. *Let e be an idempotent of the artin algebra Λ and $\langle e \rangle$ the two-sided ideal generated by e . Let M be a Λ -module with $eM = 0$. Then M is exceptional as a Λ -module if and only if M is exceptional when considered as a $\Lambda/\langle e \rangle$ -module.*

Proof. Of course, if $0 \rightarrow M \rightarrow M' \rightarrow M \rightarrow 0$ is an exact sequence in $\text{mod } \Lambda$, then $eM' = 0$, thus it is an exact sequence in $\text{mod } \Lambda/\langle e \rangle$. \square

A Λ -module M is said to be sincere provided there is no nonzero idempotent $e \in \Lambda$ with $eM = 0$.

LEMMA 4. *Any connected hereditary artin algebra Λ has sincere exceptional modules.*

(Let us add that sincere exceptional modules are even faithful, see for example [R2, Corollary 2.3].)

Proof, using induction on the number n of vertices of $Q(\Lambda)$. If $n = 1$, then any simple Λ -module is a sincere exceptional module.

Now assume that $n \geq 2$. Up to duality, we can assume that there exists a simple injective module S such that the full subquiver Q' of $Q(\Lambda)$ whose vertices are the isomorphism classes $[S']$ of the simple modules S' which are not isomorphic to S is connected. Let Λ' be the restriction of Λ to Q' . By induction, there is a sincere exceptional Λ' -module M' . We form the universal extension M of M' by S , thus there is an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow S^t \rightarrow 0$$

with $t \geq 1$ such that S is not a direct summand of M and $\text{Ext}(S, M) = 0$. It is well-known (and easy to see) that M is indecomposable and has no self-extensions. \square

The proof of Proposition 5 requires to look at four special cases.

Case 1. The algebra Λ is tame.

We use the structure of the Auslander–Reiten quiver of Λ as presented in [DR]. Since we assume that Λ has at least 3 vertices, there is a tube of rank $r \geq 2$. The simple regular modules in this component form an Ext-cycle of cardinality r , say X_1, \dots, X_r . There is a unique indecomposable module Y with a filtration $Y = Y_0 \supset Y_1 \supset \dots \supset Y_{r-1} = 0$ such that $Y_{i-1}/Y_i = X_i$ for $1 \leq i \leq r - 1$. Clearly, the pair Y, X_r is an Ext-pair.

Case 2. The quiver $Q = Q(\Lambda)$ is not a tree.

Deleting, if necessary, vertices, we may assume that the underlying graph of Q is a cycle. Let w be a path from a source i to a sink j of smallest length, let Q' be the subquiver of Q given by the vertices and the arrows which occur in w . Not every vertex of Q belongs to Q' , since otherwise Q is obtained from Q' by adding just arrows, thus by adding a unique arrow, namely an arrow $i \rightarrow j$. But then this arrow is also a path from a sink to a source, and it has length 1. By the minimality of w , we see that also w has length 1 and therefore Q has just the two vertices i, j . But then Q can have only one arrow, thus is a tree. This is a contradiction.

Let Q'' be the full subquiver given by all vertices of Q which do not belong to Q' . Of course, Q'' is connected (it is a quiver of type \mathbb{A}). According to Lemma 4, there is an exceptional module X with support Q' and an exceptional module Y with support Q'' . Since Q', Q'' have no vertex in common, we see that $\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X)$.

There is an arrow $i \rightarrow j''$ with j'' a vertex of Q'' . This arrow shows that $\text{Ext}(X, Y) \neq 0$. Similarly, there is an arrow $i'' \rightarrow j$ with i'' a vertex of Q'' . This arrow shows that $\text{Ext}(Y, X) \neq 0$.

We consider now algebras Λ with Ext-quiver $1 \rightarrow 2 \rightarrow 3$. We denote by Λ' the restriction of Λ to the subquiver with vertices 1, 2, and by Λ'' the restriction of Λ to the subquiver with vertices 2, 3. Given a representation M , let M_3 be the sum of all submodules of M which are isomorphic to $S(3)$, then M/M_3 is a Λ' -module.

LEMMA 5. *Let X, Y be Λ -modules. If $X_3 = 0$ and $\text{Ext}(Y/Y_3, X) \neq 0$, then also $\text{Ext}(Y, X) \neq 0$.*

Proof. The exact sequence $0 \rightarrow Y_3 \rightarrow Y \rightarrow Y/Y_3 \rightarrow 0$ yields an exact sequence

$$\text{Hom}(Y_3, X) \rightarrow \text{Ext}(Y/Y_3, X) \rightarrow \text{Ext}(Y, X).$$

The first term is zero, since Y_3 is a sum of copies of $S(3)$ and $X_3 = 0$. Thus, the map $\text{Ext}(Y/Y_3, X) \rightarrow \text{Ext}(Y, X)$ is injective.

Case 3. $Q(\Lambda) = (1 \rightarrow 2 \rightarrow 3)$, and $v(1, 2) \geq 2, v(2, 3) \geq 2$.

Let $X = S(2)$ and let Y be the universal extension of X using the modules $S(1)$ and $S(3)$ (thus, we form the universal extension from above using copies of $S(1)$ and the universal extension from below using copies of $S(3)$). Clearly, Y is exceptional. Since the socle of Y consists of copies of $S(3)$, we have $\text{Hom}(S(2), Y) = 0$. Since the top of Y consists of copies of $S(1)$, we have $\text{Hom}(Y, S(2)) = 0$.

Since $v(1, 2) \geq 2$, the module Y/Y_3 is not a projective Λ' -module. As a consequence, $\text{Ext}(Y/Y_3, S(2)) \neq 0$. Lemma 5 shows that also $\text{Ext}(Y, S(2)) \neq 0$. By duality, we similarly see that $\text{Ext}(S(2), Y) \neq 0$.

Case 4. $Q(\Lambda) = (1 \rightarrow 2 \rightarrow 3)$, and $v(1, 2) \geq 3, v(2, 3) = 1$.

Let $X = P(1)/P(1)_3$ (thus X is the projective Λ' -module with top $S(1)$). Let $Y = \tau X$, where $\tau = D \text{Tr}$ is the Auslander–Reiten translation in $\text{mod } \Lambda$. Of course, both modules X, Y are exceptional. Since $Y = \tau X$, we know already that $\text{Ext}(X, Y) \neq 0$.

We claim that $Y/Y_3 = \tau' S(1)$, where τ' is the Auslander–Reiten translation of Λ' . Since $P(1)_3 = S(3)^a$ for some $a \geq 1$, a minimal projective presentation of X has the form

$$(*) \quad 0 \rightarrow S(3)^a \rightarrow P(1) \rightarrow X \rightarrow 0,$$

thus the defining exact sequences for $Y = \tau X$ is of the form

$$0 \rightarrow Y \rightarrow I(3)^a \rightarrow S(1) \rightarrow 0.$$

In order to obtain $\tau' S(1)$, we start with a minimal projective presentation

$$(**) \quad 0 \rightarrow S(2)^a \rightarrow P'(1) \rightarrow S(1) \rightarrow 0,$$

where $P'(1)$ is the projective cover of $S(1)$ as a Λ' -module (actually, $P'(1) = X$). Since $\nu(2, 3) = 1$, the number a in $(*)$ and $(**)$ is the same. The defining exact sequences for $Y = \tau X$ and $\tau' S(1)$ are part of the following

commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \tau' S(1) & \longrightarrow & I(2)^a & \longrightarrow & S(1) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & Y & \longrightarrow & I(3)^a & \longrightarrow & S(1) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & S(3)^a & \xlongequal{\quad} & S(3)^a & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The left column shows that $Y/Y_3 = \tau' S(1)$.

As we have mentioned in Lemma 2, $v(1, 2) \geq 3$ implies that $\text{Ext}(\tau' S(1), P'(1)) \neq 0$. According to Lemma 5, we see that $\text{Ext}(Y, X) \neq 0$.

Finally, let us show that X, Y are orthogonal. Since $Y = \tau X$ and X is exceptional, we see that $\text{Hom}(X, Y) = 0$. On the other hand, any homomorphism $Y \rightarrow X$ vanishes on Y_3 , since X has no composition factor $S(3)$. Now Y/Y_3 is indecomposable and not projective as a Λ' -module, whereas X is a projective Λ' -module, thus $\text{Hom}(Y, X) = \text{Hom}(Y/Y_3, X) = 0$.

REMARK. Concerning the cases 3 and 4, there is an alternative proof which uses dimension vectors and the Euler form on the Grothendieck group $K_0(\Lambda)$. But for this approach, one needs to deal with the valuation of $Q(\Lambda)$ as in [DR], attaching to any arrow $i \rightarrow j$ a pair (a, b) of positive numbers instead of the single number $v(i, j) = ab$.

Proof of Proposition 5. Let Λ be connected, hereditary, representation-infinite, with at least 3 simple modules. Case 2 shows that we can assume that $Q(\Lambda)$ is a tree.

Assume that there is a subquiver Q' such that at least two of the arrows have valuation at least 2, choose such a Q' of minimal length. We are to construct an Ext-pair for the restriction of Λ to Q' . Using reflection functors (see [DR]), we can assume that Q' has orientation $1 \rightarrow 2 \rightarrow \dots \rightarrow n-1 \rightarrow n$. If $n = 3$, then this is case 3. Thus assume $n \geq 4$. The minimality of Q' asserts that $\nu(i, i + 1) = 1$ for $2 \leq i \leq n - 2$. If we denote by Λ' the restriction of Λ to Q' , then Λ' has a full exact abelian subcategory \mathcal{U} which is equivalent

to the module category of an algebra as discussed in case 3 (namely the subcategory of all Λ' -modules which do not have submodules of the form $S(i)$ with $2 \leq i \leq n - 2$ and no factor modules of the form $S(i)$ with $3 \leq i \leq n - 1$). Since \mathcal{U} has Ext-pairs, also $\text{mod } \Lambda$ has Ext-pairs.

Thus, we can assume that at most one arrow $i \rightarrow j$ has valuation greater than 2. If $v(i, j) \geq 3$, then we take a connected subquiver Q' with 3 vertices containing this arrow $i \rightarrow j$. If necessary, we use again reflection functors in order to change the orientation so that we are in case 4.

Thus we are left with the representation-infinite algebras Λ with the following properties: $Q(\Lambda)$ is a tree, there is no arrow with valuation greater than 2 and at most one arrow with valuation equal to 2. It is easy to see that $Q(\Lambda)$ contains a subquiver Q' such that the restriction of Λ to Q' is tame, thus we can use case 1. \square

Proof of Theorem. Let Λ be connected and hereditary. If Λ is representation-finite, then $\text{tors } \Lambda = \text{f-tors } \Lambda$, thus $\text{f-tors } \Lambda$ is a lattice. If Λ has precisely two simple modules, then $\text{f-tors } \Lambda$ can be described easily (see [IRTT, proof of Proposition 2.2] which works in general), thus $\text{f-tors } \Lambda$ obviously is a lattice.

On the other hand, if Λ is representation-infinite and has at least three simple modules, then Proposition 5 asserts that Λ has an Ext-pair, say X, Y . Since X, Y are exceptional modules, Proposition 1 shows that $\mathcal{T}(X) = \mathcal{G}(X)$ and $\mathcal{T}(Y) = \mathcal{G}(Y)$ both belong to $\text{f-tors } \Lambda$. The join of $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ in $\text{tors } \Lambda$ is $\mathcal{T}(X, Y)$. According to Proposition 4, $\mathcal{T}(X, Y)$ cannot belong to $\text{f-tors } \Lambda$. \square

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