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ON THE EXPONENTIAL BEHAVIOUR OF NON-AUTONOMOUS DIFFERENCE EQUATIONS

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Abstract Given a sequence of matrices $(A_m)_{m\in\mathbb{N}}$ whose Lyapunov exponents are limits, we show that this asymptotic behaviour is reproduced by the sequences $x_{m+1} = A_m x_m + f_m(x_m)$ for any sufficiently small perturbations f_m. We also consider the general case of exponential rates $\mathrm{e}^{c\rho_m}$ for an arbitrary increasing sequence ρ_m. Our approach is based on Lyapunov's theory of regularity.

Keywords: Lyapunov exponents; non-autonomous difference equations; exponential behavior

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1. Introduction

In this paper, we show that if all Lyapunov exponents associated with a sequence of matrices $(A_m)_{m\in\mathbb{N}}$ are limits, then the asymptotic exponential behaviour persists under sufficiently small perturbations. More precisely, we show that for any sequence

$$x_{m+1} = A_m x_m + f_m(x_m) \tag{1.1}$$

that is not eventually zero, the limit

$$\lambda = \lim_{m\to+\infty} \frac{1}{m} \log \|x_m\|$$

exists and coincides with a Lyapunov exponent of the sequence $(A_m)_{m\in\mathbb{N}}$. We also consider the general case of exponential rates $\mathrm{e}^{c\rho_m}$ for an arbitrary sequence ρ_m. The required smallness of the perturbation is that

$$\sum_{m=1}^{\infty} \mathrm{e}^{\delta m} \sup_{x\neq 0} \frac{\|f_m(x)\|}{\|x\|} < +\infty \tag{1.2}$$

for some $\delta > 0$, or simply that the particular sequence x_m in (1.1) satisfies

$$\sum_{m=1}^{\infty} \mathrm{e}^{\delta m} \frac{\|f_m(x_m)\|}{\|x_m\|} < +\infty$$

for some $\delta > 0$. We note that (1.2) has the advantage that one does not need to know the sequence *a priori*.

Now, we formulate a special case of our main result. Namely, let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $n \times n$ matrices with complex entries such that

$$\sup_{m\in\mathbb{N}} \|A_m\| < +\infty.$$

For each $m, \ell \in \mathbb{N}$, with $m \geqslant \ell$, we set

$$\mathcal{A}(m,\ell) = \begin{cases} A_{m-1}\cdots A_\ell & \text{if } m > \ell, \\ \mathrm{Id} & \text{if } m = \ell, \\ A_m^{-1}\cdots A_{\ell-1}^{-1} & \text{if } m < \ell. \end{cases}$$

The Lyapunov exponent $\lambda\colon \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(x) = \limsup_{m\to+\infty} \frac{1}{m} \log \|\mathcal{A}(m,1)x\|.$$

We assume that the following hold.

(C1) There exists a decomposition

$$\mathbb{C}^n = F_1 \oplus F_2 \oplus \cdots \oplus F_p,$$

with respect to which A_m can be written in the block form

$$A_m = \begin{pmatrix} A_m^1 & & 0 \\ & \ddots & \\ 0 & & A_m^p \end{pmatrix}.$$

(C2) There exist numbers $\lambda_1 < \cdots < \lambda_p$ such that

$$\lim_{m\to+\infty} \frac{1}{m} \log \|\mathcal{A}(m,1)x\| = \lambda_i$$

for each $i = 1, \ldots, p$ and $x \in F_i \setminus \{0\}$.

The following is a particular case of our main result in Theorem 3.1 for the special case of the rates $\rho_m = m$.

Theorem 1.1. *Let x_m be a sequence satisfying (1.1) for some continuous functions $f_m\colon \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\|f_m(x_m)\| \leqslant \gamma_m \|x_m\|, \quad m \in \mathbb{N}, \tag{1.3}$$

where the sequence γ_m satisfies

$$\sum_{m=1}^{\infty} \mathrm{e}^{\delta m} \gamma_m < +\infty$$

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for some $\delta > 0$. Then, one of the following alternatives holds:

(1) *$x_m = 0$ for all sufficiently large m;*

(2) *the limit*

$$\lim_{m\to+\infty} \frac{1}{m} \log \|x_m\|$$

exists and coincides with some Lyapunov exponent of the sequence $(A_m)_{m\in\mathbb{N}}$.

In the particular case of perturbations $x_{m+1} = Ax_m + f(x_m)$ of an autonomous linear difference equation (in which case all Lyapunov exponents of the linear dynamics are limits), the result in Theorem 1.1 was obtained by Coffman [**5**]. For perturbations of a differential equation $x' = Ax$, with constant coefficients, a related result can be found in Coppel's book [**6**]. Earlier results were obtained by Perron [**10**], Lettenmeyer [**8**] and Hartman and Wintner [**7**]. Corresponding results for perturbations of autonomous delay equations were obtained by Pituk [**11**, **12**] (for values in $\mathbb{C}^n$ and finite delay) and Matsui *et al*. [**9**] (for values in a Banach space and infinite delay). We emphasize that all these references consider only perturbations of *autonomous* dynamics.

Our approach is based on Lyapunov's theory of regularity (we refer the reader to [**2**] for a modern exposition), which allows one to obtain precise exponential bounds for the dynamics in terms of the Lyapunov exponents and of the so-called regularity coefficient. This is used to show that the Lyapunov exponent of any sequence satisfying (1.1) is a limit and coincides with some Lyapunov exponent of the sequence $(A_m)_{m\in\mathbb{N}}$. The remaining part of the argument is inspired by the work of Pituk [**11**], where he established a corresponding result for perturbations of a linear delay equation $x' = Lx_t$ (although only autonomous).

We considered earlier, in [**4**], the case of difference equations with infinite delay, although the lack of a general theory of regularity in infinite-dimensional spaces forced us to use a different approach.

2. Preliminaries

Let $(\rho_m)_{m\in\mathbb{N}} \subset \mathbb{R}^+$ be an increasing sequence. Also, let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $n \times n$ matrices with complex entries such that

$$\limsup_{m\to+\infty} \frac{1}{\rho_m} \log \|\mathcal{A}(m,1)\| < +\infty. \tag{2.1}$$

The Lyapunov exponent $\lambda\colon \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(x) = \limsup_{m\to+\infty} \frac{1}{\rho_m} \log \|\mathcal{A}(m,1)x\|,$$

with the convention that $\log 0 = -\infty$ (it follows from (2.1) that λ never takes the value $+\infty$). By the general theory of Lyapunov exponents (see, for example, [**1**]), the

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function λ can take at most n values in $\mathbb{C}^n \setminus \{0\}$, say $-\infty \leqslant \lambda_1 < \cdots < \lambda_p$ for some integer $p \leqslant n$. Furthermore, for $i = 1, \dots, p$ the set

$$E_i = \{x \in \mathbb{C}^n \colon \lambda(x) \leqslant \lambda_i\} \tag{2.2}$$

is a linear subspace over $\mathbb{C}$. We also set $k_i = \dim E_i - \dim E_{i-1}$ (with the convention that $E_0 = \{0\}$).

Now, we assume that each matrix A_m is in block form, with each block corresponding to a Lyapunov exponent. More precisely, we assume the following.

(H1) There exist decompositions

$$\mathbb{C}^n = F_m^1 \oplus F_m^2 \oplus \cdots \oplus F_m^p, \quad m \in \mathbb{N},$$

into subspaces of dimension $\dim F_m^i = k_i$ such that, for each $m, \ell \in \mathbb{N}$ and $i = 1, \dots, p$,

$$\mathcal{A}(m, \ell) F_\ell^i = F_m^i.$$

(H2) For each $i = 1, \dots, p$ and $x \in F_1^i \setminus \{0\}$,

$$\lim_{m \to +\infty} \frac{1}{\rho_m} \log \|\mathcal{A}(m, 1)x\| = \lambda_i.$$

(H3) For each $i, j = 1, \dots, p$, $x \in F_1^i \setminus \{0\}$ and $y \in F_1^j \setminus \{0\}$,

$$\lim_{m \to +\infty} \frac{1}{\rho_m} \log \angle(\mathcal{A}(m, 1)x, \mathcal{A}(m, 1)y) = 0.$$

One can easily verify that

$$E_i = \bigoplus_{j \leqslant i} F_1^j$$

(see (2.2)) for each i.

We also describe some consequences of conditions (H1)–(H3). Given a number $b \in \mathbb{R}$ that is not a Lyapunov exponent, we consider the decompositions

$$\mathbb{C}^n = E_m \oplus F_m, \tag{2.3}$$

where

$$E_m = \bigoplus_{\lambda_i < b} F_m^i \quad \text{and} \quad F_m = \bigoplus_{\lambda_i > b} F_m^i$$

are subspaces for each $m \in \mathbb{N}$. Let P_m and Q_m be the projections associated with the decomposition (2.3). Take also $a < b < c$ such that the interval $[a, c]$ contains no Lyapunov exponent.

Theorem 2.1. *The following properties hold.*

(1)
$$E_1 = \{x \in \mathbb{C}^n \colon \lambda(x) < b\} \quad \textit{and} \quad \lambda(x) > b \quad \textit{for } x \in F_1 \setminus \{0\}.$$

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(2) *Given* $\varepsilon > 0$, *there exists* $L = L(\varepsilon) > 0$ *such that*

$$\|\mathcal{A}(m,\ell) \mid E_\ell\| \leqslant L\mathrm{e}^{a(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}, \quad m \geqslant \ell, \tag{2.4}$$

and

$$\|\mathcal{A}(m,\ell)^{-1} \mid F_m\| \leqslant L\mathrm{e}^{c(\rho_\ell-\rho_m)+\varepsilon\rho_m}, \quad m \geqslant \ell.$$

(3) *Given* $\varepsilon > 0$, *there exists* $M = M(\varepsilon) > 0$ *such that*

$$\|P_m\| \leqslant M\mathrm{e}^{\varepsilon\rho_m} \quad \text{and} \quad \|Q_m\| \leqslant M\mathrm{e}^{\varepsilon\rho_m} \tag{2.5}$$

for every $m \in \mathbb{N}$.

Proof. Property (1) follows readily from (H1) and (H2), and (2) can be obtained as in [**3**, Proof of Theorem 10.6]. For (3), we recall that

$$\frac{1}{\alpha_m} \leqslant \|P_m\| \leqslant \frac{2}{\alpha_m} \quad \text{and} \quad \frac{1}{\alpha_m} \leqslant \|Q_m\| \leqslant \frac{2}{\alpha_m}, \tag{2.6}$$

where α_m is the angle between the subspaces E_m and F_m (see, for example, [**3**]). Also, let α_m^i be the angle between F_m^i and $\bigoplus_{j\neq i} F_m^j$. Clearly, for each i such that $\lambda_i < b$ we have that

$$\alpha_m \geqslant \alpha_m^i \quad \text{for } m \in \mathbb{N}. \tag{2.7}$$

On the other hand, by (H3), given $\varepsilon > 0$, there exists $M' > 0$ such that

$$\alpha_m^i = \min_{j\neq i} \angle(F_m^i, F_m^j) \geqslant M'\mathrm{e}^{-\varepsilon m}$$

for every $m \in \mathbb{N}$. Together with (2.6) and (2.7) this yields (3). □

Since

$$\|\mathcal{A}(m,\ell)P_\ell\| \leqslant \|\mathcal{A}(m,\ell) \mid E_\ell\| \cdot \|P_\ell\|$$

and

$$\|\mathcal{A}(m,\ell)^{-1}Q_m\| \leqslant \|\mathcal{A}(m,\ell)^{-1} \mid F_m\| \cdot \|Q_m\|,$$

it follows from Theorem 2.1 that, given $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,\ell)P_\ell\| \leqslant K\mathrm{e}^{a(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}$$

and

$$\|\mathcal{A}(m,\ell)^{-1}Q_m\| \leqslant K\mathrm{e}^{c(\rho_\ell-\rho_m)+\varepsilon\rho_m}$$

for every $m \geqslant \ell$. In particular, taking $d > \lambda_p$ it follows from (2.4) that, given $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,\ell)\| \leqslant N\mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}, \quad m \geqslant \ell. \tag{2.8}$$

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3. A non-autonomous Perron-type theorem

Now, we consider nonlinear perturbations of the dynamics defined by a sequence of matrices $(A_m)_{m\in\mathbb{N}}$. Namely, we consider the collection of sequences $(x_m)_{m\in\mathbb{N}}$ in $\mathbb{C}^n$ satisfying

$$x_{m+1} = A_m x_m + f_m(x_m), \quad m \in \mathbb{N}, \tag{3.1}$$

for some continuous functions $f_m\colon \mathbb{C}^n \to \mathbb{C}^n$. We show that if a given sequence x_m does not grow too fast, then its Lyapunov exponent (see (3.3)) coincides with some Lyapunov exponent of the unperturbed difference equation (obtained from setting all f_m equal to 0).

Theorem 3.1. *Let $(x_m)_{m\in\mathbb{N}}$ be a sequence such that (3.1) and (1.3) hold for some numbers $\gamma_m \in \mathbb{R}$ satisfying*

$$\sum_{k=1}^{\infty} \mathrm{e}^{(-\lambda_1+\delta)(\rho_{k+1}-\rho_k)+\delta\rho_{k+1}} \gamma_k < \infty \tag{3.2}$$

for some $\delta > 0$. Then, one of the following alternatives hold.

(1) *$x_m = 0$ for all sufficiently large m.*

(2) *There exists i such that*

$$\lambda_i = \lim_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\|. \tag{3.3}$$

Proof. Let $b \in \mathbb{R}$ be a number that is not a Lyapunov exponent and set $\varepsilon = \frac{1}{4}\delta$. Also, let $a < b < c$ be as in § 2. We consider the norm

$$\|x\|_m = \sup_{\sigma\geqslant m} (\mathrm{e}^{-a(\rho_\sigma-\rho_m)} \|\mathcal{A}(\sigma, m)P_m x\|) + \sup_{\sigma\leqslant m} (\mathrm{e}^{-c(\rho_\sigma-\rho_m)} \|\mathcal{A}(\sigma, m)Q_m x\|)$$

for each $m \in \mathbb{N}$ and $x \in \mathbb{C}^n$. Clearly,

$$\|x\|_m = \|P_m x\|_m + \|Q_m x\|_m \tag{3.4}$$

and

$$\|x\| \leqslant \|x\|_m \leqslant 2K\mathrm{e}^{\varepsilon\rho_m}\|x\|. \tag{3.5}$$

Lemma 3.2. *We have that*

$$\|\mathcal{A}(m, \ell)P_\ell x\|_m \leqslant \mathrm{e}^{a(\rho_m-\rho_\ell)}\|P_\ell x\|_\ell \quad \text{for } m \geqslant \ell$$

and

$$\|\mathcal{A}(m, \ell)Q_\ell x\|_m \geqslant \mathrm{e}^{c(\rho_m-\rho_\ell)}\|Q_\ell x\|_\ell \quad \text{for } m \geqslant \ell.$$

Proof of the lemma. For $m \geqslant \ell$ we have that

$$\begin{aligned}
\|\mathcal{A}(m,\ell)P_\ell x\|_m &= \sup_{\sigma \geqslant m} (\|\mathcal{A}(\sigma,m)\mathcal{A}(m,\ell)P_\ell x\| \mathrm{e}^{-a(\rho_\sigma - \rho_m)}) \\
&= \mathrm{e}^{a(\rho_m - \rho_\ell)} \sup_{\sigma \geqslant m} (\|\mathcal{A}(\sigma,\ell)P_\ell x\| \mathrm{e}^{-a(\rho_\sigma - \rho_\ell)}) \\
&\leqslant \mathrm{e}^{a(\rho_m - \rho_\ell)} \sup_{\sigma \geqslant \ell} (\|\mathcal{A}(\sigma,\ell)P_\ell x\| \mathrm{e}^{-a(\rho_\sigma - \rho_\ell)}) \\
&\leqslant \mathrm{e}^{a(\rho_m - \rho_\ell)} \|P_\ell x\|_\ell. \qquad (3.6)
\end{aligned}$$

Similarly, for $m \geqslant \ell$ we have that

$$\begin{aligned}
\|\mathcal{A}(m,\ell)Q_\ell x\|_m &= \sup_{\sigma \leqslant m} (\|\mathcal{A}(\sigma,m)\mathcal{A}(m,\ell)Q_\ell x\| \mathrm{e}^{-c(\rho_\sigma - \rho_m)}) \\
&= \mathrm{e}^{c(\rho_m - \rho_\ell)} \sup_{\sigma \leqslant m} (\|\mathcal{A}(\sigma,\ell)Q_\ell x\| \mathrm{e}^{-c(\rho_\sigma - \rho_\ell)}) \\
&\geqslant \mathrm{e}^{c(\rho_m - \rho_\ell)} \sup_{\sigma \leqslant \ell} (\|\mathcal{A}(\sigma,\ell)Q_\ell x\| \mathrm{e}^{-c(\rho_\sigma - \rho_\ell)}) \\
&\geqslant \mathrm{e}^{c(\rho_m - \rho_\ell)} \|Q_\ell x\|_\ell. \qquad (3.7)
\end{aligned}$$

This completes the proof of the lemma. □

Now, let $(x_m)_{m\in\mathbb{N}}$ be a sequence satisfying (3.1). Using the decomposition in (2.3), one can write that $x_m = y_m + z_m$, where

$$y_m = P_m x_m \quad \text{and} \quad z_m = Q_m x_m.$$

Lemma 3.3. *One of the following alternatives holds.*

(1)

$$\limsup_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| < b \qquad (3.8)$$

and

$$\lim_{k\to+\infty} \frac{\|z_k\|_k}{\|y_k\|_k} = 0. \qquad (3.9)$$

(2)

$$\liminf_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| > b \qquad (3.10)$$

and

$$\lim_{k\to+\infty} \frac{\|y_k\|_k}{\|z_k\|_k} = 0. \qquad (3.11)$$

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Proof of the lemma. We have that

$$y_{k+1} = A_k P_k x_k + P_k f_k(x_k) \tag{3.12}$$

and

$$z_{k+1} = A_k Q_k x_k + Q_k f_k(x_k). \tag{3.13}$$

By (3.5) and (3.7), it follows from (3.13) and (2.5) that

$$\begin{aligned}\|z_{k+1}\|_{k+1} &\geqslant \|A_k Q_k x_k\|_{k+1} - \|Q_k f_k(x_k)\|_{k+1} \\ &\geqslant \mathrm{e}^{c(\rho_{k+1}-\rho_k)}\|z_k\|_k - 2K\mathrm{e}^{\varepsilon\rho_{k+1}}\|Q_k f_k(x_k)\| \\ &\geqslant \mathrm{e}^{c(\rho_{k+1}-\rho_k)}\|z_k\|_k - D_1\|x_k\|\delta_k \end{aligned} \tag{3.14}$$

for some constant $D_1 > 0$, where $\delta_k = \mathrm{e}^{\varepsilon\rho_{k+1}}\gamma_k$. By (3.12) and (3.6), it follows from similar estimates that

$$\|y_{k+1}\|_{k+1} \leqslant \mathrm{e}^{a(\rho_{k+1}-\rho_k)}\|y_k\|_k + D_2\|x_k\|_k\delta_k \tag{3.15}$$

for some constant $D_2 > 0$. Inequalities (3.14) and (3.15) yield that

$$\|z_{k+1}\|_{k+1} \geqslant \alpha_k\|z_k\|_k - D\delta_k(\|y_k\|_k + \|z_k\|_k) \tag{3.16}$$

and

$$\|y_{k+1}\|_{k+1} \leqslant \beta_k\|y_k\|_k + D\delta_k(\|y_k\|_k + \|z_k\|_k) \tag{3.17}$$

for all integers k, where

$$D = D_1 + D_2, \quad \alpha_k = \mathrm{e}^{c(\rho_{k+1}-\rho_k)} \quad \text{and} \quad \beta_k = \mathrm{e}^{a(\rho_{k+1}-\rho_k)}. \tag{3.18}$$

Now, we claim that either

$$\|z_k\|_k \leqslant \|y_k\|_k \quad \text{for all large } k \tag{3.19}$$

or

$$\|y_k\|_k < \|z_k\|_k \quad \text{for all large } k. \tag{3.20}$$

We show that if (3.19) fails, then (3.20) holds. We assume that (3.19) does not hold. Then,

$$\|z_k\|_k > \|y_k\|_k \quad \text{for infinitely many } k. \tag{3.21}$$

By (3.16),

$$\|z_{k+1}\|_{k+1} \geqslant (\alpha_k - D\delta_k)\|z_k\|_k - D\delta_k\|y_k\|_k \tag{3.22}$$

and

$$\|y_{k+1}\|_{(k+1)r} \leqslant (\beta_k + D\delta_k)\|y_k\|_k + D\delta_k\|z_k\|_k. \tag{3.23}$$

By (3.21), there exists $k_1 \geqslant 1$ arbitrarily large such that $\|y_{k_1}\|_{k_1} < \|z_{k_1}\|_{k_1}$. We show by induction on k that

$$\|y_k\|_k < \|z_k\|_k \quad \text{for } k \geqslant k_1. \tag{3.24}$$

We assume that $\|y_k\|_k < \|z_k\|_k$ for some $k \geqslant k_1$. By (3.22) and (3.23), this implies that

$$\|z_{k+1}\|_{k+1} \geqslant (\alpha_k - 2D\delta_k)\|z_k\|_k > 0$$

and

$$\|y_{k+1}\|_{k+1} \leqslant (\beta_k + 2D\delta_k)\|z_k\|_k.$$

Now, it follows from (3.2) that

$$\mathrm{e}^{(-\lambda_1+4\varepsilon)(\rho_{k+1}-\rho_k)+4\varepsilon\rho_{k+1}}\gamma_k \to 0$$

when $k \to \infty$. Taking d sufficiently close to λ_p, this implies that $c_k = \delta_k/\alpha_k \to 0$ and $d_k = \delta_k/\beta_k \to 0$ when $k \to \infty$. Therefore,

$$\|y_{k+1}\|_{k+1} \leqslant \frac{\beta_k + 2D\delta_k}{\alpha_k - 2D\delta_k}\|z_{k+1}\|_{k+1} < \|z_{k+1}\|_{k+1},$$

provided that k is sufficiently large. This shows that (3.24) holds. Thus, we have shown that if (3.19) fails, then (3.20) holds. As a consequence, we have the following two cases.

Case 1. Assume that (3.19) holds. We show that (3.8) and (3.9) hold. We note that $\|y_k\|_k > 0$ for all large k, since otherwise (3.4) and (3.19) would yield

$$\|x_k\|_k = \|y_k\|_k + \|z_k\|_k \leqslant 2\|y_k\|_k = 0$$

for infinitely many k, contradicting the hypothesis that $\|x_m\|_m \geqslant \|x_m\| > 0$ for all sufficiently large m. Define

$$S = \limsup_{k\to+\infty} \frac{\|z_k\|_k}{\|y_k\|_k}.$$

By (3.19), we have $0 \leqslant S \leqslant 1$. It follows from (3.19) and (3.17) that, for all large k,

$$\|y_{k+1}\|_{(k+1)r} \leqslant (\beta_k + 2D\delta_k)\|y_k\|_k.$$

Together with (3.16), this yields that, for all large k,

$$\frac{\|z_{k+1}\|_{k+1}}{\|y_{k+1}\|_{k+1}} \geqslant \frac{\alpha_k - D\delta_k}{\beta_k + 2D\delta_k} \cdot \frac{\|z_k\|_k}{\|y_k\|_k} - \frac{D\delta_k}{\beta_k + 2D\delta_k}.$$

Since $\alpha_k/\beta_k \to +\infty$ when $k \to \infty$ (see (3.18)), taking lim sup on both sides, we obtain $S \geqslant +\infty \cdot S$. This implies that $S = 0$, and that (3.9) holds. Now, take k_0 so large that $\|z_k\|_k \leqslant \|y_k\|_k$ for all $k \geqslant k_0$. By (3.17), we find that, for $k \geqslant k_0$,

$$\|y_{k+1}\|_{k+1} \leqslant (\beta_k + 2D\delta_k)\|y_k\|_k$$

and, hence,

$$\begin{aligned}\|y_k\|_k &\leqslant \|y_{k_0}\|_{k_0} \prod_{j=k_0}^{k-1}(1+2Dc_j)\prod_{j=k_0}^{k-1}\beta_j \\ &= \|y_{k_0}\|_{k_0} \prod_{j=k_0}^{k-1}(1+2Dc_j)\mathrm{e}^{a(\rho_k-\rho_{k_0})} \end{aligned} \tag{3.25}$$

for $k \geqslant k_0$. On the other hand, it follows from (3.2) that

$$\sum_{j=1}^{\infty}\log(1+2Dc_j) \leqslant 2D\sum_{j=1}^{\infty} c_j < \infty,$$

and, hence, by (3.25),

$$\limsup_{m\to+\infty}\frac{1}{\rho_m}\log\|x_m\| \leqslant a < b.$$

This establishes (3.8).

Case 2. Now assume that (3.20) holds. We show that (3.10) and (3.11) hold. We define

$$R = \limsup_{k\to+\infty}\frac{\|y_k\|_k}{\|z_k\|_k}.$$

By (3.20), we have $0 \leqslant R \leqslant 1$. It follows from (3.20) in (3.16) that, for all large k,

$$\|z_{k+1}\|_{k+1} \geqslant (\alpha_k - 2D\delta_k)\|y_k\|_k.$$

Together with (3.17), this yields that, for all large k,

$$\frac{\|y_{k+1}\|_{k+1}}{\|z_{k+1}\|_{k+1}} \leqslant \frac{\beta_k + D\delta_k}{\alpha_k - 2D\delta_k}\cdot\frac{\|y_k\|_k}{\|z_k\|_k} + \frac{D\delta_k}{\alpha_k - 2D\delta_k}.$$

Since $\beta_k/\alpha_k \to 0$ when $k \to \infty$, taking lim sup on both sides, we obtain $R \leqslant 0 \cdot R$. This implies that $R = 0$ and that (3.11) holds. Now, take k_0 such that $\|y_k\|_k < \|z_k\|_k$ for $k \geqslant k_0$. By (3.16), we find that, for $k \geqslant k_0$,

$$\|z_{k+1}\|_{k+1} \geqslant \alpha_k(1-2Dd_k)\|z_k\|_k,$$

and, hence,

$$\|z_k\|_k \geqslant \|z_{k_0}\|_{k_0}\prod_{j=k_0}^{k-1}(1-2Dd_j)\mathrm{e}^{c(\rho_k-\rho_{k_0})}.$$

On the other hand, it follows from (3.2) that

$$-\sum_{j=1}^{\infty}\log(1-2Dd_j) \leqslant \sum_{j=1}^{\infty}\log\frac{1}{1-2Dd_j} \leqslant \sum_{j=1}^{\infty}\frac{2Dd_j}{1-2Dd_j} < \infty.$$

Therefore,

$$\liminf_{m\to\infty}\frac{1}{\rho_m}\log\|x_m\| \geqslant c > b.$$

This establishes (3.10). □

Now we establish an auxiliary result.

Lemma 3.4. *There exists $C > 0$ such that*

$$\|x_m\| \leqslant C\|x_\ell\| \mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell} \tag{3.26}$$

for all $m \geqslant \ell$.

Proof of the lemma. For each $m \geqslant \ell$ we have that

$$x_m = \mathcal{A}(m,\ell)x_\ell + \sum_{j=\ell}^{m-1} \mathcal{A}(m,j+1)f_j(x_j).$$

Therefore, by (2.8) and (1.3),

$$\begin{aligned}\|x_m\| &\leqslant N\mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}\|x_\ell\| + N\sum_{j=\ell}^{m-1} \mathrm{e}^{d(\rho_m-\rho_{j+1})+\varepsilon\rho_{j+1}}\gamma_j\|x_j\| \\ &\leqslant N\mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}\|x_\ell\| + N\sum_{j=\ell}^{m-1} \mathrm{e}^{d(\rho_m-\rho_j)+\varepsilon\rho_{j+1}}\gamma_j\|x_j\|,\end{aligned}$$

where in the last inequality we have used that ρ is increasing. Hence,

$$\mathrm{e}^{-d(\rho_m-\rho_\ell)}\|x_m\| \leqslant N\mathrm{e}^{\varepsilon\rho_\ell}\|x_\ell\| + N\sum_{j=\ell}^{m-1} \mathrm{e}^{-d(\rho_j-\rho_\ell)+\varepsilon\rho_{j+1}}\gamma_j\|x_j\|.$$

One can use induction to show that

$$\mathrm{e}^{-d(\rho_m-\rho_\ell)}\|x_m\| \leqslant N\mathrm{e}^{\varepsilon\rho_\ell}\|x_\ell\| \prod_{j=\ell}^{m-1}(1 + N\mathrm{e}^{\varepsilon\rho_{j+1}}\gamma_j)$$

for $m \geqslant \ell$. Hence,

$$\begin{aligned}\|x_m\| &\leqslant N\mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}\|x_\ell\| \exp\biggl(\sum_{j=\ell}^{m-1} N\mathrm{e}^{\varepsilon\rho_{j+1}}\gamma_j\biggr) \\ &\leqslant N\mathrm{e}^{d(\rho_m-\rho_\ell)+\varepsilon\rho_\ell}\|x_\ell\|\mathrm{e}^{NS},\end{aligned}$$

where

$$S = \sum_{j=1}^{\infty} \mathrm{e}^{\varepsilon\rho_{j+1}}\gamma_j < +\infty.$$

This completes the proof of the lemma. □

We proceed with the proof of Theorem 3.1. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence satisfying (3.1). If $x_k = 0$ for some k, then it follows from (3.26) that $x_m = 0$ for all $m \geqslant k$, and, hence, the first alternative in the theorem holds. Now, we assume that $x_m \neq 0$ for all $m \geqslant \ell$.

Also, let $\lambda_1 < \cdots < \lambda_p$ be the Lyapunov exponents of the sequence $(A_m)_{m\in\mathbb{N}}$. Take real numbers b_j such that

$$\lambda_j < b_j < \lambda_{j+1} \quad \text{for } 1 \leqslant j < p.$$

Also, take $b_0 < \lambda_1$ (when $\lambda_1 \neq -\infty$) and $b_p > \lambda_p$. By Lemma 3.3 applied to each $b = b_j$, there exists $j \in \{1, \ldots, p\}$ such that

$$\limsup_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| < b_j$$

and

$$\liminf_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| > b_{j-1}.$$

Letting $b_j \searrow \lambda_1$ and $b_{j-1} \nearrow \lambda_j$, we find that

$$\lim_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| = \lambda_j.$$

This completes the proof of the theorem. □

Now, we show that any sequence satisfying (3.1) and the second alternative in Theorem 3.1 is essentially asymptotically tangent to the spaces F_m^i, with i as in (3.3). We consider the decompositions

$$\mathbb{C}^n = E_m \oplus F_m \oplus F_m^i,$$

where

$$E_m = \bigoplus_{j<i} F_m^j \quad \text{and} \quad F_m = \bigoplus_{j>i} F_m^j$$

for each $m \in \mathbb{N}$. Also, let P_m, Q_m and R_m be the projections associated with this decomposition.

Theorem 3.5. *Let $(x_m)_{m\in\mathbb{N}}$ be a sequence such that (3.1) and (1.3) hold for some numbers $\gamma_m \in \mathbb{R}$ satisfying (3.2) for some $\delta > 0$. If (3.3) holds, then*

$$\lim_{m\to+\infty} \frac{\|P_m x_m\|_m}{\|R_m x_m\|_m} = 0$$

and

$$\lim_{m\to+\infty} \frac{\|Q_m x_m\|_m}{\|R_m x_m\|_m} = 0.$$

Proof. We write that

$$x_m = y_m + z_m + w_m,$$

where

$$y_m = P_m x_m, \quad z_m = Q_m x_m \quad \text{and} \quad w_m = R_m x_m.$$

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Take $b < \lambda_i$ such that the interval $[b, \lambda_i)$ contains no Lyapunov exponent of the sequence $(A_m)_{m\in\mathbb{N}}$. Then,

$$\lim_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| = \lambda_i > b,$$

and it follows from Lemma 3.3 that

$$\lim_{m\to+\infty} \frac{\|y_m\|_m}{\|z_m + w_m\|_m} = 0. \tag{3.27}$$

Now, take $c > \lambda_i$ such that the interval $(\lambda_i, c]$ contains no Lyapunov exponent of the cocycle $(A_m)_m$. Then,

$$\lim_{m\to\infty} \frac{1}{\rho_m} \log \|x_m\| = \lambda_i < c,$$

and it follows from Lemma 3.3 that

$$\lim_{m\to+\infty} \frac{\|z_m\|_m}{\|y_m + w_m\|_m} = 0. \tag{3.28}$$

Given $\delta > 0$, take $\eta \in (0, 1)$ such that $\eta(1 + \eta)(1 - \eta^2)^{-1} < \delta$. By (3.28), for all large m we have that

$$\|z_m\|_m \leqslant \eta\|y_m + w_m\|_m.$$

Furthermore, (3.27) implies that, for all large m,

$$\|y_m\|_m \leqslant \eta\|z_m + w_m\|_m$$

and, hence,

$$\begin{aligned}\|z_m\|_m &\leqslant \eta(1 + \eta)\|w_m\|_m + \eta^2\|z_m\|_m \\ &\leqslant \eta(1 + \eta)(1 - \eta^2)^{-1}\|w_m\|_m \leqslant \delta\|w_m\|_m.\end{aligned}$$

Since δ is arbitrary, this yields the identity

$$\lim_{m\to+\infty} \frac{\|z_m\|_m}{\|w_m\|_m} = 0.$$

Reversing the roles of P and Q, we find that

$$\lim_{m\to+\infty} \frac{\|y_m\|_m}{\|z_m\|_m} = 0.$$

This completes the proof of the theorem. □

Finally, we formulate two non-trivial results that are consequences of Theorem 1.1. We first consider perturbations of linear dynamics with negative Lyapunov exponents.

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Theorem 3.6. *Let* $f_m\colon \mathbb{C}^n \to \mathbb{C}^n$ *be continuous functions such that*

$$\|f_m(x)\| \leqslant \gamma_m \|x\|, \quad m \in \mathbb{N},\ x \in \mathbb{C}^n,$$

for a sequence γ_m *satisfying (3.2) for some* $\delta > 0$. *If all values of the Lyapunov exponent* λ *of the sequence* $(A_m)_{m\in\mathbb{N}}$ *are negative, then all solutions* x_m *of (1.1) satisfy*

$$\lim_{m\to+\infty} \frac{1}{\rho_m} \log \|x_m\| < 0.$$

Now, we consider the particular case of linear perturbations.

Theorem 3.7. *If* B_m *are* $n \times n$ *matrices with complex entries such that the sequence* $\gamma_m = \|B_m\|$ *satisfies (3.2) for some* $\delta > 0$, *then the Lyapunov exponents of the sequences* $(A_m)_{m\in\mathbb{N}}$ *and* $(A_m + B_m)_{m\in\mathbb{N}}$ *have the same values.*

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