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A CHARACTERIZATION OF LC^n COMPACTA IN TERMS OF GROMOV-HAUSDORFF CONVERGENCE

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ABSTRACT. It is proved that a compactum is locally n-connected if and only if it is the limit (in the sense of Gromov-Hausdorff convergence) of an "equi-locally n-connected" sequence of (at most) $(n+1)$-dimensional compacta.

1. **Introduction.** A compact metric space is called a *compactum* and the set of all compacta is denoted by $\mathcal{CM}$. Gromov [G] introduced a pseudo-metric on $\mathcal{CM}$ which induces a metric on the isometry classes of $\mathcal{CM}$ (called the *Gromov-Hausdorff distance*). It would be an interesting problem to study properties of various subsets of $\mathcal{CM}$ (for example, the set of all ANR compacta, the set of all finite dimensional compacta, *etc.*) with the topology induced by this (pseudo-) metric. In the present paper, we study the set of all LC^n-compacta, denoted by $\mathcal{LC}^n$. Our main theorem (Theorem 3.1) states that a compactum is LC^n if and only if it is the limit of an "equi-LC^n" sequence of (at most) $(n+1)$-dimensional compacta, in the sense of Gromov-Hausdorff convergence.

Here, we outline the proof. Suppose that X is an arbitrary LC^n compactum. By Dranishnikov's resolution theorem [D1, D2], there is a polyhedrally $(n+1)$ soft map (See Section 2 for the definition) $f: D_{n+1} \longrightarrow X$ of an $(n+1)$-dimensional LC^n compactum D_{n+1} onto X. Applying the method of T. Moore [M, Theorem 1] to f instead of cell-like maps, we can see that X is the limit of a sequence of compacta with the required property. Conversely, suppose that X is the limit of a sequence (X_i) of compacta with the property as stated above. By a result of Gromov (Theorem 2.3 in this paper), we can reduce the proof to the case that all of X and X_i's lie in a single compactum. Next, we use an idea of Ferry [F, Proposition 5.6], where it is shown that if, $M = \varprojlim (M_i, f_i: M_{i+1} \longrightarrow M_i)$ is the limit of an inverse sequence of compact ANR's and UV^n bonding maps, then M is LC^n. Ferry used the "approximate lifting property" of UV^n maps (up to dimension $(n+1)$). Although our sequence $(X_i)_{i\geq 1}$ does not have maps $X_{i+1} \longrightarrow X_i$'s with this property, a careful lifting process can be made to apply his argument.

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2. **Preliminaries.**

DEFINITION 2.1. (1) For a metric space (M,d) and its subset A, the ε-neighbourhood of A is denoted by $N^M_\varepsilon(A)$. When there is no confusion, the symbol M will be omitted. The Hausdorff metric induced by d is denoted by d_H.

(2) The set of all compact metric spaces is denoted by $\mathcal{CM}$. For metric spaces (X,d_X) and (Y,d_Y), we define

$$d_{GH}(X,Y) = \inf\big\{d_H\big(i(X),j(Y)\big) \mid i\colon X \longrightarrow M \text{ and } j\colon Y \longrightarrow M \text{ are isometric imbeddings into a metric space } (M,d)\big\}.$$

This defines a pseudo-metric on $\mathcal{CM}$ and it is known [G] that

$$d_{GH}(X,Y) = 0 \quad \text{if and only if } (X,d_X) \text{ and } (Y,d_Y) \text{ are isometric.}$$

Hence d_{GH} defines a metric on $\mathcal{CM}$ modulo isometry classes, and it is called the *Gromov-Hausdorff distance*.

DEFINITION 2.2. (1) The k-dimensional cell is denoted by D^k and $S^{k-1} = \partial D^k$.

(2) A (not necessarily continuous) function $\rho\colon [0,R] \longrightarrow [0,\infty)$ is called a *contractibility function* if $\rho(0) = \lim_{t\to 0}\rho(t) = 0$ and $\rho(t) > t$ for each $t \in (0,R]$.

(3) A compactum X is said to be $\mathrm{LGC}^n(\rho)$, where ρ is a contractibility function, if for each $k = 0,1,\ldots,n$, each map $\alpha\colon S^k \longrightarrow X$ with $\mathrm{diam}(\mathrm{im}\,\alpha) < t$ has an extension $\bar\alpha\colon D^{k+1} \longrightarrow X$ with $\mathrm{diam}(\mathrm{im}\,\bar\alpha) < \rho(t)$. Clearly, a compactum is LC^n if and only if it is $\mathrm{LGC}^n(\rho)$ for some contractibility function ρ. The class of all $\mathrm{LGC}^n(\rho)$ compacta is denoted by $\mathcal{LGC}^n(\rho)$.

(4) A sequence $(X_i)_{i\geq 1}$ of compacta in a metric space is said to be *equi*-LC^n if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for each $i \geq 1$, any map $\alpha\colon S^k \longrightarrow X_i$ with $\mathrm{diam}(\mathrm{im}\,\alpha) < \delta$ has an extension $\bar\alpha\colon D^{k+1} \longrightarrow X_i$ such that $\mathrm{diam}(\mathrm{im}\,\bar\alpha) < \varepsilon$.

The following theorem is useful in understanding the Gromov-Hausdorff convergence.

THEOREM 2.3 ([G] COMPACTNESS CRITERION P. 64–65). *Suppose that a sequence $(X_i)_{i\geq 1}$ of compacta converges to a compactum X in the sense of Gromov-Hausdorff. Then, there exists a compact metric space (M,d) such that*

(1) there are isometric imbeddings $f_i\colon X_i \longrightarrow M$ and $f\colon X \longrightarrow M$, and

(2) $\lim_{i\to\infty} d_H\big(f_i(X_i), f(X)\big) = 0$.

From the above theorem, it is easy to see the following:

PROPOSITION 2.4. *Suppose that a sequence $(X_i)_{i\geq 1}$ of compacta converges to a compactum X in the sense of Gromov-Hausdorff. Then $(X_i)_{i\geq 1} \subset \mathcal{LGC}^n(\rho)$ for some contractibility function ρ if and only if there exist imbeddings f_i's and f of X_i's and X in a compact metric space (M,d) such that the sequence $\big(f_i(X_i)\big)$ forms an equi-*LC^n *family and* $\lim_{i\to\infty} d_H\big(f_i(X_i), f(X)\big) = 0$.

We need the following result due to Dranishnikov [D_1] and [D_2].

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THEOREM 2.5 ([D_1, D_2]). *For each $n \geq 0$ and for each LC^n compactum X, there is a polyhedrally $(n+1)$-soft map $f_{n+1}: D_{n+1} \longrightarrow X$ of an $(n+1)$-dimensional LC^n compactum D_{n+1} onto X.*

A map $f: X \longrightarrow Y$ between compacta is said to be *polyhedrally n-soft* if it satisfies the following condition.

For each pair (K, L) of polyhedra with $\dim K \leq n$ and for each pair of maps $\phi: K \longrightarrow Y$ and $\gamma: L \longrightarrow X$ such that $\phi|L = f \cdot \gamma$, there is a map $\Phi: K \longrightarrow X$ such that $\Phi|L = \gamma$ and $f \cdot \Phi = \phi$.

$$\begin{array}{ccc} L & \xrightarrow{\gamma} & X \\ \downarrow & \overset{\Phi}{\nearrow} & \downarrow f \\ K & \xrightarrow[\phi]{} & Y \end{array}$$

3. **Results.** Now we can state our main theorem as follows.

THEOREM 3.1. *For a compactum X, the following conditions are equivalent:*

(a) X is LC^n.

(b) There is a sequence $(X_i)_{i \geq 1}$ of compacta and a contractibility function ρ such that

(1) $(X_i)_{i \geq 1} \subset \mathcal{LGC}^n(\rho)$ and $\dim X_i \leq n+1$ for each $i \geq 1$.

(2) $\lim_{i \to \infty} d_{GH}(X_i, X) = 0$.

STEP 1. Proof of (a) $\longrightarrow$ (b). This is essentially the same as [M, Theorem 1], except we use polyhedrally $(n+1)$-soft maps instead of cell-like maps. We give a sketch of the proof for the sake of completeness.

Let X be a LC^n-compactum and take a polyhedrally $(n+1)$-soft map $f: D \longrightarrow X$ of an $(n+1)$-dimensional LC^n compactum D onto X. Let $M(f)$ be the mapping cylinder of f defined by $M(f) = D \times [0,1] \cup X/(x,1) \sim f(x)$, $x \in D$. A map $h: M(f) \longrightarrow [0,1]$ is defined by $h([x,t]) = t$ and $h\big(f(x)\big) = 1$ $(x \in D)$. We may assume that $M(f)$ has a metric d such that X is isometrically imbedded as $h^{-1}(1)$. We identify X with $h^{-1}(1)$.

Define $X_i = h^{-1}(1 - 1/i)$. It is clear that $\mathrm{Lim}_{i \to \infty} d_H(X_i, X) = 0$, hence $d_{GH}(X_i, X) \longrightarrow 0$. As $\dim X_i \leq n+1$ for each i, it remains to prove that $(X_i)_{i \geq 1} \subset \mathcal{LGC}^n(\rho)$ for some contractibility function ρ. In view of Proposition 2.4, it suffices to show that $(X_i)_{i \geq 1}$ forms an equi-LC^n family.

Suppose not. Then, there are an integer $k \leq n$, and $\varepsilon > 0$, and a sequence $(\alpha_i: S^k \longrightarrow X_{n_i})$ such that $\lim n_i = \infty$ and

(1) For each i, $\mathrm{diam}(\mathrm{im}\, \alpha_i) < 1/i$

(2) The image of any extension $\bar{\alpha}_i: D^{k+1} \longrightarrow X_{n_i}$ of α_i has diameter $> \varepsilon$.

For each i, we can define a map $\phi_i: X_{n_i} \longrightarrow X$ by $f_i([x, 1 - 1/i]) = f(x)$. It is clear that each ϕ_i is polyhedrally $(n+1)$-soft and also, we may assume that $d(\phi_i, \mathrm{id}) < 1/2^i$. Since X is LC^n, there is a $\delta > 0$ such that

(3) each map $\beta: S^k \longrightarrow X$ with $\mathrm{diam}(\mathrm{im}\, \beta) < \delta$ has an extension $\bar{\beta}: D^{k+1} \longrightarrow X$ such that $\mathrm{diam}(\mathrm{im}\, \bar{\beta}) < \varepsilon/4$. Take a sufficiently large i such that

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(4) $\operatorname{diam}(\operatorname{im}\alpha_i) < \delta/4$, and $d(\phi_i, \mathrm{id}) < \delta/4$.

Then $\operatorname{diam}(\operatorname{im}\phi_i \cdot \alpha_i) < \delta$ and we obtain an extension $\overline{\phi_i\alpha_i}: D^{k+1} \longrightarrow X$ by (3). Apply the polyhedral $(n+1)$-softness to obtain a lift $\tilde{\alpha}_i$ of $\overline{\phi_i\alpha_i}$ which is an extension of α_i as well. It is easy to see that $\operatorname{diam}(\operatorname{im}\tilde{\alpha}_i) < \varepsilon$ which violates the condition (2).

This completes the proof of (a) $\longrightarrow$ (b).

STEP 2. Proof of (b) $\longrightarrow$ (a). Suppose the sequence of compacta (X_i) converges to X in the sense of Gromov-Hausdorff, satisfying the hypothesis of (b). By Proposition 2.4, there is a compact metric space M and isometric imbeddings of X_i's and X into M such that the images of X_i's converges to the image of X in the sense of Hausdorff metric. Hence it suffices to prove the following theorem to complete the proof of (b) $\longrightarrow$ (a).

THEOREM 3.2. *Let (X_i) be a sequence of compacta in a compactum M which converges to a compactum X in the sense of Hausdorff metric. Suppose that there is a contractibility function $\rho: [0,R] \longrightarrow [0,\infty)$ such that each X_i is $\mathrm{LGC}^n(\rho)$ and $\dim X_i \leq n+1$. Then X is LC^n.*

REMARK. If X is finite dimensional and $\dim X_i \leq n$ (*i.e.* X_i's are ANR's), then the above result has been proved by Borsuk [B, p. 196].

For the proof of Theorem 3.2, we need some preparations.

LEMMA 3.3. *Let X be $\mathrm{LGC}^n(\rho)$ for some contractibility function ρ and $p: X \longrightarrow Y$ be a map satisfying*

(1) $\left|d_Y\big(p(x_1), p(x_2)\big) - d_X(x_1, x_2)\right| < \alpha$ *for each* $x_1, x_2 \in X$.

Suppose that K is a compact polyhedron with $\dim K \leq n+1$ and L is a subcomplex of K. Further assume that $f: K \longrightarrow Y$ and $f_L: L \longrightarrow X$ satisfy

(2) $d_Y(p \cdot f_L, f|L) < \beta$,

(3) $\operatorname{diam}_Y f(\sigma) < \gamma$ *for each* $\sigma \in K$, *and*

(4) $\operatorname{diam}_X f_L(\tau) < \delta$ *for each* $\tau \in L$.

Inductively, define r_j by

(5) $r_1 = \rho\big(\max(\alpha+\beta+\gamma, \delta)\big)$ *and* $r_j = \rho\big(2\max(r_{j-1}, \delta)\big)$.

Then, there exists a map $\bar{f}: K \longrightarrow X$ such that

(6) $\bar{f}|L = f_L$ *and* $d(p \cdot \bar{f}, f) < r_{n+1} + \alpha + \beta + \gamma$.

PROOF. The proof is a modification of the standard argument. We construct the required map by an induction on the skeleton of K. The i-skeleton of K is denoted by $K^{(i)}$.

Take any vertex $v \in K^{(0)}$ and define $\bar{f}_0(v)$ by

$$\begin{aligned} \bar{f}_0(v) &= f_L(v) \quad \text{if } v \in L^{(0)} \text{ and} \\ &\in p^{-1}\big(f(v)\big) \quad \text{if } v \in (K-L)^{(0)}. \end{aligned}$$

Evidently, $d_Y(p \cdot \bar{f}_0, f|K^{(0)}) < \beta < \alpha+\beta+\gamma$.

Construction of $\bar{f}_1$: Take any 1-simplex $\sigma \in K$ and let $\partial\sigma = \{v_1, v_2\}$. Noticing that

$$d_X\big(\bar{f}_0(v_1), \bar{f}_0(v_2)\big) < d_Y\big(p \cdot \bar{f}_0(v_1), p \cdot \bar{f}_0(v_2)\big) + \alpha \quad \text{by (1)},$$

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it is easy to see that

$$d_X\big(\bar{f}_0(v_1),\bar{f}_0(v_2)\big) < \max(\alpha+\beta+\gamma,\delta).$$

There is a path a_σ from $\bar{f}_0(v_1)$ to $\bar{f}_0(v_2)$ whose diameter $< \rho\big(\max(\alpha+\beta+\gamma,\delta)\big)$. The map $\bar{f}_1|\sigma$ is defined along with this path.

Making this process on each 1-simplex of K, we have a map $\bar{f}_1: K^{(1)} \cup L \longrightarrow X$ such that

$$\text{(a-1)} \qquad \operatorname{diam}_X \bar{f}_1(\sigma) < r_1 = \rho\big(\max(\alpha+\beta+\gamma,\delta)\big).$$

Let $x \in \sigma \in K^{(1)}$ and take a vertex v of σ. Since $\operatorname{diam}_Y(p\cdot\bar{f}_1)(\sigma) < r_1 + \alpha$, we have

$$\begin{aligned} d_Y\big(p\cdot\bar{f}_1(x), f(x)\big) &\le d_Y\big(p\cdot\bar{f}_1(x), p\cdot\bar{f}_1(v)\big) + d_Y\big(p\cdot\bar{f}_1(v), f(v)\big) + d_Y\big(f(v), f(x)\big) \\ &< r_1 + \alpha + \beta + \gamma, \end{aligned}$$

and, hence,

$$\text{(b-1)} \qquad d_Y(p\cdot\bar{f}_1, f|K^{(1)}) < r_1 + \alpha + \beta + \gamma.$$

Construction of $\bar{f}_{i+1}$: Suppose that $\bar{f}_i: K^{(i)} \longrightarrow X$ has been defined so as to satisfy

$$\text{(a-i)} \qquad \operatorname{diam}_X \bar{f}_i(\sigma) < \max(r_i,\delta) \quad \text{for } \sigma \in K^{(1)} \text{ and}$$

$$\text{(b-i)} \qquad d_Y(p\cdot\bar{f}_i, f|K^{(i)}) < r_i + \alpha + \beta + \gamma.$$

Take any $(i+1)$-simplex σ of K and consider $\bar{f}_i(\partial\sigma)$. By (a-i), it is easy to see that $\operatorname{diam}_X \bar{f}_i(\partial\sigma) < 2\max(r_i,\delta)$. There is an extension $\bar{f}^\sigma_{i+1}: \sigma \longrightarrow X$ such that $\operatorname{diam}_X \bar{f}^\sigma_{i+1}(\sigma) < \rho\big(2\max(r_i,\delta)\big) = r_{i+1}$. Repeating this process on each $(i+1)$-simplex, we obtain a map $\bar{f}_{i+1}: K^{(i+1)} \longrightarrow X$. A similar estimation can be applied to see that $\bar{f}_{i+1}$ satisfies (a-(i+1)) and (b-(i+1)).

The induction step can be continued until $i = n+1$. Then the required map is $\bar{f}_{n+1}$. This completes the proof.

The following lemma was essentially proved by Petersen ([P], Proposition on p. 390).

LEMMA 3.4. *Let $\rho: [0,R] \longrightarrow [0,\infty)$ be a contractibility function and define $\rho_j(\varepsilon)$ inductively by $\rho_1(\varepsilon) = \varepsilon + \rho(\varepsilon)$, and $\rho_j(\varepsilon) = \varepsilon + \rho\big(\rho_{j-1}(\varepsilon)\big)$ (so far as it is defined, i.e. $\rho_{j-1}(\varepsilon) < R$). Suppose that $\rho_{n-1}(4\varepsilon) < R$. Then the following holds:*

Let X and Y be compacta in a metric space (M,d) such that $\dim X \le n+1$ *and Y is* LGC$^n(\rho)$. *If $X \subset N_\varepsilon(Y)$, then there exists a map $f: X \longrightarrow Y$ such that $d(f, i_X) < 2\varepsilon + \rho_{n+1}(4\varepsilon)$, where i_X is the inclusion of X into M.*

PROOF OF THEOREM 3.2. By the Hausdorff metric extension theorem (See [T] for a simple proof), M can be isometrically imbedded in the Hilbert cube with some compatible metric.

Take a map $\alpha: S^k \longrightarrow X$, where $0 \le k \le n$. In the sequel, we construct an extension $\bar{\alpha}$ of α to D^{k+1} and estimate the diameter of its image.

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Fix the following notation:

NOTATION. (1) $d_H(X, X_i) = \varepsilon_i$, $d_H(X_i, X_j) = \varepsilon_{ij}$ (d_H denotes the Hausdorff metric with respect to the above metric on the Hilbert cube). We may assume that $\rho_n(4\varepsilon_{ij}) < R$ for each i, j.

(2) $\phi_i : X \longrightarrow P_i$ is an η_i-translation onto a compact polyhedron P_i.

We may assume that $\rho_n(4\eta_i) + 4\varepsilon_i < R$ for each i.

(3) $\operatorname{diam} \alpha(S^k) < \delta$.

(4) $\beta_i : S^k \longrightarrow P_i$ is a simplicial approximation of $\phi_i \cdot \alpha$ and $d(\phi_i \cdot \alpha, \beta_i) < \xi_i$. Notice that $\dim(\operatorname{im} \beta_i) \leq k \leq n$.

Further, we define:

$$A_i = 2\rho(4\varepsilon_{i\,i+1}) + 4\varepsilon_{i\,i+1}$$

$$C_i = \rho_n\big(4(\varepsilon_i + \eta_i)\big) + 2\varepsilon_i + 3\eta_i + \xi_i, \quad \text{where } \rho_n \text{ is as in Lemma 3.4}$$

$$B_i = A_i + C_i + C_{i+1}, \quad \text{and}$$

$$D_i(\delta) = \delta + 2\xi_i + 4\varepsilon_i + 6\eta_i + 2\rho\big(4(\varepsilon_i + \eta_i)\big).$$

It should be observed that A_i, C_i, B_i and $D_i(\delta)$ converge to 0 if $i \longrightarrow \infty$, $\eta_i \longrightarrow 0$, $\xi_i \longrightarrow 0$, and $\delta \longrightarrow 0$.

Applying Lemma 3.4 to X_{i+1} and X_i, we obtain a map $f_i : X_{i+1} \longrightarrow X_i$ such that

$$d(f_i, \mathrm{id}_{X_{i+1}}) < 2\varepsilon_{i\,i+1} + \rho_n(4\varepsilon_{i\,i+1}) < A_i. \tag{5}$$

Since $d_H(P_i, X_i) < \eta_i + \varepsilon_i$, we have $\operatorname{im} \beta_i \subset N_{\eta_i+\varepsilon_i}(X_i)$. Applying Lemma 3.4 to $\operatorname{im} \beta_i$ and X_i, we have a map $p_i : \beta_i \longrightarrow X_i$ such that

$$d(p_i, \mathrm{id}_{\operatorname{im} \beta_i}) < 2(\varepsilon_i + \eta_i) + \rho\big(4(\varepsilon_i + \eta_i)\big). \tag{6}$$

Define $\alpha_i = p_i \cdot \beta_i : S^k \longrightarrow X_i$. We have the following estimation:

$$\begin{aligned} \operatorname{diam}(\operatorname{im} \beta_i) &< \operatorname{diam}\big(\operatorname{im}(\phi_i \cdot \alpha)\big) + 2\xi_i \quad \text{by (4)} \\ &< \operatorname{diam}\big(\operatorname{im}(\alpha)\big) + 2\eta_i + 2\xi_i \quad \text{by (2)} \\ &< \delta + 2\eta_i + 2\xi_i \quad \text{by (3)}. \end{aligned}$$

Combining the above with (6), we have

$$\begin{aligned} \operatorname{diam}(\operatorname{im} p_i) &< \delta + 2\eta_i + 2\xi_i + 4(\varepsilon_i + \eta_i) + 2\rho\big(4(\varepsilon_i + \eta_i)\big) \\ &= \delta + 2\xi_i + 4\varepsilon_i + 6\eta_i + 2\rho\big(4(\varepsilon_i + \eta_i)\big) = D_i(\delta). \end{aligned} \tag{7}$$

Taking a sufficiently large i, sufficiently "small" translation ϕ_i and sufficiently close approximation β_i, we may assume that $D_i(\delta) < R/2$. Since X_i is $\mathrm{LGC}^n(\rho)$, we have an extension $\bar{\alpha}_i : D^{k+1} \longrightarrow X_i$ of α_i such that

$$\operatorname{diam}(\operatorname{im} \bar{\alpha}_i) < \rho\big(D_i(\delta)\big). \tag{8}$$

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We have the following estimation:

$$\begin{aligned} d(\alpha, \bar{\alpha}_i | S^k) &= d(\alpha, \alpha_i) = d(\alpha, p_i \cdot \beta_i) \\ &\leq d(\alpha, \beta_i) + d(\beta_i, p_i \cdot \beta_i) \\ &\leq d(\alpha, \phi_i \cdot \alpha) + d(\phi_i \cdot \alpha, \beta_i) + d(\beta_i, p_i \cdot \beta_i) \\ &< \eta_i + \xi_i + 2(\varepsilon_i + \eta_i) + \rho_n\big(4(\varepsilon_i + \eta_i)\big) \\ &= \xi_i + 2\varepsilon_i + 3\eta_i + \rho_n\big(4(\varepsilon_i + \eta_i)\big) = C_i. \end{aligned} \tag{9}$$

In what follows, we construct a sequence of maps $(\bar{\alpha}_{i+j} : D^{k+1} \longrightarrow X_{i+j})_{j \geq 1}$ each of which is an extension of α_{i+j}.

$j = 1$: First we estimate the distance $d(f_i \cdot \alpha_{i+1}, \bar{\alpha}_i | S^k)$.

$$\begin{aligned} d(f_i \cdot \alpha_{i+1}, \bar{\alpha}_i | S^k) &= d(f_i \cdot \alpha_{i+1}, \alpha_i) \\ &\leq d(f_i \cdot \alpha_{i+1}, \alpha_{i+1}) + d(\alpha_{i+1}, \alpha) + d(\alpha, \alpha_i) \\ &< A_i + C_i + C_{i+1} = B_i. \end{aligned} \tag{10}$$

Take a sufficiently small triangulation T_{i+1} of D^{k+1} and let

$$\begin{aligned} &\operatorname{diam} \bar{\alpha}_i(\sigma) < \gamma_i \quad \text{for any } \sigma \in T_{i+1}, \quad \text{and} \\ &\operatorname{diam} \alpha_{i+1}(\tau) < \delta_{i+1} \quad \text{for any } \tau \in T_{i+1} | S^k. \end{aligned} \tag{11}$$

Applying Lemma 3.3 to $p = f_i$, $(K, L) = (D^{k+1}, S^k)$, $f = \bar{\alpha}_i$, $f_L = \alpha_{i+1}$, $\alpha = A_i$, $\beta = B_i$, $\gamma = \gamma_i$, and $\delta = \delta_{i+1}$, we have a map $\bar{\alpha}_{i+1} : D^{k+1} \longrightarrow X_{i+1}$ such that

$$\bar{\alpha}_{i+1} | S^k = \alpha_{i+1} \quad \text{and} \tag{1-1}$$

$$d(f_i \cdot \bar{\alpha}_{i+1}, \bar{\alpha}_i) < r_n^i + A_i + B_i + \gamma_i \quad (= \text{denoted by } F_i), \text{ where} \tag{2-1}$$

r_n^i is defined as in Lemma 3.3 in the above situation. From (9) and (1-1), it follows that

$$d(\alpha, \bar{\alpha}_{i+1} | S^k) < C_{i+1}. \tag{3-1}$$

Combining (5) with (2-1), we have that

$$d(\bar{\alpha}_{i+1}, \bar{\alpha}_i) < A_i + F_i \quad (= \text{denoted by } E_i). \tag{4-1}$$

Having constructed $\bar{\alpha}_{i+1}, \ldots, \bar{\alpha}_{i+j-1}$, $E_{i+1}, \ldots, E_{i+j-1}$, and $F_{i+1}, \ldots, F_{i+j-1}$ satisfying

$$\bar{\alpha}_{i+s} | S^k = \alpha_{i+s} \quad \text{and} \tag{1-s}$$

$$d(f_{i+s} \cdot \bar{\alpha}_{i+s}, \bar{\alpha}_{i+s-1}) < F_{i+s-1} \quad (s = 1, \ldots, j-1), \tag{2-s}$$

we proceed to the construction of $\bar{\alpha}_{i+j}$. As in (10), we have

$$d(f_{i+j} \cdot \alpha_{i+j}, \alpha_{i+j-1}) < A_{i+j-1} + C_{i+j-1} + C_{i+j} = B_{i+j}. \tag{12}$$

Take a sufficiently small triangulation T_{i+j} of D^{k+1} and let

$$\begin{aligned} &\operatorname{diam} \bar{\alpha}_{i+j-1}(\sigma) < \gamma_{i+j-1} \quad \text{for any } \sigma \in T_{i+j} \text{ and} \\ &\operatorname{diam} \alpha_{i+j}(\tau) < \delta_{i+j-1} \quad \text{for any } \tau \in T_{i+j} | S^k. \end{aligned} \tag{13}$$

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Applying Lemma 3.3 in a manner similar to that in the case $j = 1$, we obtain a map $\bar{\alpha}_{i+j}: D^{k+1} \longrightarrow X_{i+j}$ such that

(1-j) $$\bar{\alpha}_{i+j}|S^k = \alpha_{i+j} \quad \text{and}$$

(2-j) $$d(f_{i+j} \cdot \bar{\alpha}_{i+j}, \bar{\alpha}_{i+j-1}) < r_n^{i+j-1} + A_{i+j-1} + B_{i+j-1} + \gamma_{i+j-1} \quad (= \text{denoted by } F_{i+j-1}).$$

This completes the inductive step. By (1-j) and (9), we have that

(3-j) $$d(\alpha, \bar{\alpha}_{i+j}|S^k) < C_{i+j} \quad \text{for } j \geq 0.$$

Further by (2-j) and (5), we have

(4-j) $$d(\bar{\alpha}_{i+j}, \bar{\alpha}_{i+j-1}) < A_{i+j-1} + F_{i+j-1} \quad (= \text{denoted by } E_{i+j-1}).$$

Note that $C_{i+j}, E_{i+j} \longrightarrow 0$ as $j \longrightarrow \infty$, $\eta_i \longrightarrow 0$, $\xi_i \longrightarrow 0$, $\gamma_i \longrightarrow 0$ and $\delta_i \longrightarrow 0$.

To complete the proof of Theorem 3.2, take any $\varepsilon > 0$. Take a sufficiently small $\delta > 0$, sufficiently large i, sufficiently small translation ϕ_{i+j}'s, sufficiently close approximations β_{i+j}'s, and sufficiently small triangulations T_{i+j}'s, so that $\rho\big(D_i(\delta)\big) < \varepsilon/4$, $\sum_{j=0}^{\infty} E_{i+j} < \varepsilon/4$, and $C_{i+j} \longrightarrow 0$ as $j \longrightarrow \infty$.

When a map $\alpha: S^k \longrightarrow X$ is given so that $\operatorname{diam}(\operatorname{im}\alpha) < \delta$, we obtain a sequence $(\bar{\alpha}_{i+j}: D^{k+1} \longrightarrow X_{i+j})$ of maps by the above construction. By the choice of E_{i+j}'s and (4-j), this forms a Cauchy sequence. Let $\bar{\alpha}: D^{k+1} \longrightarrow Q$ be the limit map. Clearly, $\operatorname{im}\bar{\alpha} \subset X$ and by (3-j), $\bar{\alpha}|S^k = \alpha$. Finally,

$$\operatorname{diam}(\operatorname{im}\bar{\alpha}) < \rho\big(D_i(\delta)\big) + \sum_{j=0}^{\infty} E_{i+j} < \varepsilon.$$

Therefore $\bar{\alpha}$ is the required extension. This completes the proof.

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