SQUARE INTEGRABLE HIGHEST WEIGHT REPRESENTATIONS

by KARL-HERMANN NEEB

(Received 22 April, 1996)

Introduction. If G is the group of holomorphic automorphisms of a bounded symmetric domain, then G has a distinguished class of irreducible unitary representations called the *holomorphic discrete series* of G. These representations have been studied by Harish-Chandra in [**7**]. On the Lie algebra level, the Harish-Chandra modules corresponding to the holomorphic discrete series representations are highest weight modules. Even for G as above, it turns out that not all the unitary highest weight modules belong to the holomorphic discrete series but there exists a condition on the highest weight which characterizes the holomorphic discrete series among the unitary highest weight representations. They can be defined as those unitary highest weight representations with square integrable matrix coefficients.

If $\tilde{G}$ is the simply connected covering group of G, then Harish-Chandra's condition on the highest weight characterizes those representations which belong to the relative holomorphic discrete series, i.e., which have matrix coefficients that are square integrable modulo the infinite center of $\tilde{G}$.

In [**16**] and [**21**] we have studied unitary highest weight representations for general Lie groups. These representations have shown up as exactly those which can be extended holomorphically to a semigroup S containing G and a dense open submanifold on which the semigroup multiplication is holomorphic (cf. [**21**]). We refer to [**21**] for a characterization of those groups which have such representations with discrete kernel. An important example of a highest weight representation for a non semisimple Lie group is the *metaplectic representation* of the 2-fold cover $H_n \rtimes \mathrm{Mp}(n, \mathbb{R})$ of the *Jacobi group* $\mathrm{St}(n, \mathbb{R}) := H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, where H_n denotes the $(2n+1)$-dimensional Heisenberg group.

In this paper we address the question of characterizing those unitary highest weight representations which, in the case of hermitian simple groups mentioned above, correspond to the holomorphic discrete series. We call them *square integrable*. If the adjoint group of G is closed (as is always the case for semisimple groups), the condition of square integrability is the same as the square integrability of the matrix coefficients modulo the center. In general we define it by integrals over G/T, where $T = \exp \mathfrak{t}$ and $\mathfrak{t}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}$.

Sections 1 and 2 contain generalities on representations with matrix coefficients which are square integrable over certain homogeneous spaces. In Section 3 we characterize the square integrable highest weight representations by a condition generalizing Harish-Chandra's condition for the relative holomorphic discrete series. We also calculate the corresponding formal degree in terms of the highest weight. This formula had already been announced in [**14**]. In Section 4 we show that the class of square integrable highest weight representations coincides precisely with those that can be obtained by Duflo's orbit method for general groups.

In Section 5 we generalize an observation made by Wildberger for compact groups (cf. [**26**]). If $(\pi_\lambda, \mathcal{H})$ is a unitary highest weight representation with highest weight λ, then the space $B_2(\mathscr{H})$ of Hilbert-Schmidt operators on $\mathscr{H}$ can be realized in a very natural way on the coadjoint orbit of the functional $i\lambda$ in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. We obtain this result by using a holomorphic extension of the highest weight representation under consideration to a complex semigroup S. This process exhibits the space $B_2(\mathscr{H})$ also as a space of holomorphic functions on the semigroup S so that we obtain a relation between this space of holomorphic functions and a function space on a coadjoint orbit. This interplay between functions on coadjoint orbits and holomorphic functions on S will be taken up in a forthcoming paper where we study Paley-Wiener type theorems in this setting. We note that character formulas resp. realizations of square integrable highest weight representations by square integrable holomorphic sections of certain vector bundles have been discussed in [**20**] resp. [**13**, Sect. VIII].

1. Homogeneous systems of coherent states. Let $(\pi, \mathscr{H})$ be an irreducible continuous unitary representation of the Lie group G. Let $v \in \mathscr{H}$ with $\|v\| = 1$ and $H \subseteq \{g \in G : g.v \in \mathbb{C}.v\}$ a closed subgroup of the stabilizer of the ray $\mathbb{C}v$. Let $\chi : H \to S^1$ be the unitary character of H defined by $\pi(h).v = \chi(h)v$ for all $h \in H$.

We write $M = H\backslash G$ for the right G-space of right cosets Hg of H. The group H acts on the space $G \times \mathbb{C}$ by $h.(g,z) := (hg, \chi(h)z)$. We denote by $E := G \times_H \mathbb{C}$ the set of H-orbits and the orbit of (g,z) by $[g,z] := H.(g,z)$. Then the map $p : E \to H\backslash G$, $[g,z] \mapsto Hg$ is well defined and defines the structure of a complex line bundle over $H\backslash G$. The group G acts on E from the right by $[g',z].g := [g'g, z]$ for $g, g' \in G$, $z \in \mathbb{C}$.

A section of this bundle is a map $\sigma : H\backslash G \to E$ with $p \circ \sigma = \mathrm{id}_{H\backslash G}$. We write $\Gamma(E)$ for the vector space of continuous sections of E and put

$$\Gamma_G(E) := \{f \in C(G) : (\forall h \in H) f(hg) = \chi(h).f(g)\}.$$

Then every $f \in \Gamma_G(E)$ defines a continuous section of E by $\sigma_f(Hg) := [g, f(g)]$ and since the mapping $G \to H\backslash G$ has local sections, it can easily be seen that each continuous section can be written that way.

Now we consider the map

$$\psi : G \to \mathscr{H}, \qquad g \mapsto \pi(g^{-1}).v.$$

Then we obtain a map $\Psi : \mathscr{H} \to \Gamma_G(E)$ by $\Psi(u)(g) := \langle u, \psi(g)\rangle$ for $g \in G$ because

$$\Psi(u)(hg) = \langle u, \psi(hg)\rangle = \langle u, \chi(h)^{-1}.\psi(g)\rangle = \chi(h)\langle u, \psi(g)\rangle.$$

Moreover, the mapping Ψ is G-equivariant with respect to the action of G on $\Gamma_G(E)$ given by $(g.f)(x) := f(xg)$.

So far this is a quite general construction. Now we make the additional assumption that $M = H\backslash G$ has a G-invariant Radon measure μ_M and that there exists $w \in \mathscr{H}$ such that

$$\int_M |\Psi(w)(g)|^2 \, d\mu_M(Hg) < \infty. \tag{1.1}$$

Note that the integrand is a function on G which is constant on the right cosets of H so that it can be interpreted as a function on $H\backslash G$.

Let $\Gamma^2(E)$ denote the Hilbert space of those measurable sections σ of E satisfying

$$\int_M |\sigma(m)|^2 \, d\mu_M(m) < \infty,$$

where $|[g, z]| := |z|$ is well defined. Note also that $\langle [g, z], [g, z'] \rangle := \langle z, z' \rangle$ is a well defined scalar product on each fiber of E and that the scalar product in the Hilbert space $\Gamma^2(E)$ is given by

$$\langle \sigma, \sigma' \rangle = \int_M \langle \sigma(m), \sigma'(m) \rangle \, d\mu_M(m).$$

Moreover, since the measure μ_M is invariant under G and the same holds for the scalar product on the vector bundle E, the action of G on sections given by $(g \,.\, \sigma)(Hx) := \sigma(Hxg) \,.\, g^{-1}$ defines a continuous unitary representation of G on the Hilbert space $\Gamma^2(E)$ (cf. [**25**, pp. 368ff]). We note that we can also identify the Hilbert space $\Gamma^2(E)$ with a space of (equivalence classes) of functions on G which we denote by $\Gamma^2_G(E)$.

Hence (1.1) means that $\Psi(w) \in \Gamma^2_G(E)$ is a square integrable section. Before we draw the main conclusions from this assumption (Theorem 1.2), we need a preparatory lemma which we formulate in a rather general context (cf. [**24**, Proposition 1.2.2] for a statement in the same spirit). We recall that an involutive semigroup S is a semigroup endowed with an involutive antiautomorphism $s \mapsto s^*$ and that a *representation of an involutive semigroup* S on a Hilbert space $\mathcal{H}$ is a semigroup homomorphism $\pi: S \to B(\mathcal{H})$ with $\pi(s^*) = \pi(s)^*$ for all $s \in S$.

LEMMA 1.1. *Let S be an involutive semigroup, $(\pi, \mathcal{H})$ an irreducible representation of S on $\mathcal{H}$, and $(\rho, \mathcal{K})$ another representation of S. Suppose that $T: \mathcal{D}(T) \to \mathcal{K}$ is a non-zero closable operator whose graph $\Gamma(T) \subseteq \mathcal{H} \times \mathcal{K}$ is invariant under S. Then there exists $\lambda > 0$ such that $\lambda T: \mathcal{D}(T) \to \mathcal{K}$ is an S-equivariant isometry which extends to an S-equivariant isometric embedding $\mathcal{H} \to \mathcal{K}$.*

Proof. Let $\bar{T}$ denote the closure of T, i.e., the operator whose graph is the closure $\overline{\Gamma(T)}$ of the graph of T. Then the invariance of $\Gamma(T)$ implies the invariance of $\Gamma(\bar{T})$ so that we may w.l.o.g. assume that T is a closed operator.

Then $\Gamma(T) \subseteq \mathcal{H} \times \mathcal{K}$ is an S-invariant closed subspace and therefore a Hilbert space. Let $p: \Gamma(T) \to \mathcal{H}$ denote the canonical projection. Then $pp^*: \mathcal{H} \to \mathcal{H}$ is a bounded operator commuting with S and Schur's Lemma implies that there exists $\mu \geq 0$ with $pp^* = \mu \mathbf{1}$. We conclude in particular that $p(\Gamma(T)) = \mathcal{D}(T) = \mathcal{H}$ so that the Closed Graph Theorem implies that T is continuous. Now the same argument as above shows that $TT^* = \nu \mathbf{1}$. If $\nu = 0$, then $T^* = 0$ and hence $T = 0$ contrary to our assumptions. If $\nu > 0$, then this means that $\frac{1}{\sqrt{\nu}} T: \mathcal{H} \to \mathcal{K}$ is an isometric embedding. □

For the following theorem we recall the unit vector $v \in \mathcal{H}$.

THEOREM 1.2. *If $(\pi, \mathcal{H})$ is an irreducible unitary representation of G and E as defined above, then if $\Psi(w) \in \Gamma^2_G(E)$ holds for a non-zero $w \in \mathcal{H}$, then $\Psi(\mathcal{H}) \subseteq \Gamma^2_G(E)$ and there exists $d(\pi, v) > 0$ such that $\sqrt{d(\pi, v)}\Psi$ is an isometric G-equivariant embedding $\mathcal{H} \to \Gamma^2_G(E)$.*

Proof. We consider the subspace $\mathcal{D} := \Psi^{-1}(\Gamma^2(E))$ and the unbounded operator $T : \mathcal{D} \to \Gamma^2(E)$, $w \mapsto \Psi(w)$. We claim that T is closed (cf. [**24**, p. 23]). In fact, if $(u_n, T.u_n) \to (u, f)$, then $u_n \to u$ holds in the norm-topology of $\mathcal{H}$. Hence the fact that $\psi(G) = \pi(G).v \subseteq \mathcal{H}$ is bounded implies that the functions $\Psi(u_n)$ on G converge uniformly to the function $\Psi(u)$.

On the other hand the sequence $\Psi(u_n)$, considered as elements of the Hilbert space $\Gamma^2_G(E)$ has a subsequence which converges μ_M-almost everywhere to the measurable section $f \in \Gamma^2(E)$. Hence f coincides, as a section of the bundle E, almost everywhere with the continuous section defined by $\Psi(u)$. Thus $\Psi(u) = f$ holds in $\Gamma^2_G(E)$. We conclude that $u \in \mathcal{D}$ and that $(u, f) = (u, \Psi(u))$ is contained in the graph of T. This proves that the graph $\Gamma(T) := \{(u, T.u) : u \in \mathcal{D}\}$ of T is closed.

Since the mapping Ψ is G-equivariant, we see that $\mathcal{D}$ is also G-invariant so that the graph $\Gamma(T) \subseteq \mathcal{H} \times \Gamma^2(E)$ is invariant under G, where G acts on this product space by the direct sum representation. Now Lemma 1.1 implies that $\mathcal{D} = \mathcal{H}$ and that there exists $d(\pi, v) > 0$ such that $\sqrt{d(\pi, v)}\,T : \mathcal{H} \to \Gamma^2(E)$ is a G-equivariant isometry. □

If $\mathcal{H}$ is a Hilbert space we write $B(\mathcal{H})$ for the set of all bounded operators on $\mathcal{H}$, $B_1(\mathcal{H})$ for the set of all *trace class operators*, $\|A\|_1$ for the trace norm, $B_2(\mathcal{H})$ for the set of all *Hilbert-Schmidt operators* and $\|A\|_2$ for the Hilbert-Schmidt norm.

We leave the proof of the following lemma to the reader.

LEMMA 1.3. *For $x, y \in \mathcal{H}$ we write $P_{x,y}$ for the operator given by $P_{x,y}(v) = \langle v, y \rangle x$ and put $P_x := P_{x,x}$. Then the following assertions hold:*

(i) $\operatorname{tr} P_{x,y} = \langle x, y \rangle$.
(ii) $P^*_{x,y} = P_{y,x}$.
(iii) $P_{x,y}P_{z,w} = \langle z, y \rangle P_{x,w}$.
(iv) $AP_{x,y}B^* = P_{Ax,By}$ *for* $A, B \in B(\mathcal{H})$.
(v) $P_{A.x} = AP_xA^*$. □

Theorem 1.2 has the important consequence that we can express the scalar product in the Hilbert space $\mathcal{H}$ by integrating over M. To simplify the notation we simply write $g.v$ instead of $\pi(g).v$. For $w, u \in \mathcal{H}$ we can use Lemma 1.3 to see that

$$\begin{aligned}\frac{1}{d(\pi, v)}\langle w, u \rangle = \langle \Psi(w), \Psi(u) \rangle &= \int_{H\backslash G} \langle w, g^{-1}.v \rangle \langle g^{-1}.v, u \rangle \, d\mu_M(Hg) \\ &= \int_{H\backslash G} \operatorname{tr}(P_{w,u} P_{g^{-1}.v, g^{-1}.v}) \, d\mu_M(Hg) \\ &= \int_{H\backslash G} \operatorname{tr}(P_{w,u} P_{g^{-1}.v}) \, d\mu_M(Hg)\end{aligned}$$

which can be interpreted as

$$\int_{H\backslash G} P_{g^{-1}.v} \, d\mu_M(Hg) = \frac{1}{d(\pi, v)} \mathbf{1} \tag{1.2}$$

in the weak operator topology on $B(\mathcal{H})$ or in the weak topology on $B_2(\mathcal{H})$.

A rather general setup for formulas of that type is provided by the theory of frames.

A nice survey on the ideas and the relations to quantization can be found in [**1**] (cf. also [**22**, p. 43]).

Hilbert-Schmidt Operators. Let $(\pi, \mathcal{H})$, G and H be as above. We have a unitary representation of $G \times G$ on the Hilbert space $B_2(\mathcal{H})$ of Hilbert-Schmidt operators defined by $\pi^c(g_1, g_2) . A := \pi(g_1)A\pi(g_2^{-1})$.

We consider the rank-one projection $P_v \in B_2(\mathcal{H})$. Then $\pi^c(g_1, g_2) . P_v = P_{g_1 . v, g_2 . v}$ (Lemma 1.3) shows that

$$\pi^c(h_1, h_2) . P_v = \chi(h_1)\chi(h_2)^{-1}P_v.$$

We put $\chi^c(h_1, h_2) := \chi(h_1)\chi(h_2)^{-1}$ and, as in the first subsection, we define a vector bundle $E^c \to M \times M$, where $M = H\backslash G$. We write $\Gamma_{G\times G}(E^c)$ for the space of those functions on $G \times G$ which represent the sections of the bundle E^c, i.e., which satisfy

$$f(h_1g_1, h_2g_2) = \chi^c(h_1, h_2) . f(g_1, g_2)$$

for $h_1, h_2 \in H$ and $g_1, g_2 \in G$.

Since the representation $\pi^c = \pi \otimes \bar{\pi}$ of $G \times G$ on $B_2(\mathcal{H}) \cong \mathcal{H} \otimes \bar{\mathcal{H}}$ is irreducible, we are exactly in the setting of the first section. In particular we obtain an injective map $\Psi^c : B_2(\mathcal{H}) \to \Gamma_{G\times G}(E^c)$ defined by

$$\begin{aligned}\Psi^c(A)(g_1, g_2) &= \langle A, \pi^c(g_1, g_2)^{-1} . P_v\rangle = \langle A, P_{g_1^{-1} . v, g_2^{-1} . v}\rangle \\ &= \operatorname{tr}(P_{Ag_2^{-1} . v, g_1^{-1} . v}) = \langle Ag_2^{-1} . v, g_1^{-1} . v\rangle.\end{aligned}$$

We claim that $\Psi^c(P_v) \in \Gamma^2_{G\times G}(E^c)$. In fact, in view of (1.2), we have

$$\begin{aligned}&\int_{M\times M} |\Psi^c(P_v)(g, g')|^2 \, d\mu_M(Hg)\, d\mu_M(Hg') \\ &= \int_{H\backslash G}\int_{H\backslash G} \langle P_v g'^{-1} . v, g^{-1} . v\rangle\langle g^{-1} . v, P_v . g'^{-1} . v\rangle \, d\mu_M(Hg)\, d\mu_M(Hg') \\ &= \frac{1}{d(\pi, v)}\int_{H\backslash G} \langle P_v g'^{-1} . v, P_v g'^{-1} . v\rangle \, d\mu_M(Hg') \\ &= \frac{1}{d(\pi, v)}\int_{H\backslash G} \langle P_v, P_{g'^{-1} . v}\rangle \, d\mu_M(Hg') \\ &= \frac{1}{d(\pi, v)^2}\|v\|^2 = \frac{1}{d(\pi, v)^2}\|P_v\|_2^2.\end{aligned}$$

This proves that $\Psi^c(P_v)$ is a square integrable section of the bundle E^c. Now Theorem 1.2 applies to the representation π^c of $G \times G$ and we obtain the following result.

PROPOSITION 1.4. *The mapping $\Psi^c : B_2(\mathcal{H}) \to \Gamma_{G\times G}(E^c)$ defines an isometric $G \times G$-equivariant embedding $d(\pi, v)\Psi^c : B_2(\mathcal{H}) \to \Gamma^2_{G\times G}(E^c)$.* □

Symbols. Next we define the *symbol* σ_T of an operator $T \in B(\mathcal{H})$ by

$$\sigma_T(Hg) := \Psi^c(T)(g, g) = \langle Tg^{-1} . v, g^{-1} . v\rangle$$

and note that the right hand side depends only on the right coset Hg of g in G.

https://doi.org/10.1017/S0017089500032237 Published online by Cambridge University Press

LEMMA 1.5. *If* $T \in B_1(\mathscr{H})$, *then*

$$\operatorname{tr} T = d(\pi, v) \int_M \sigma_T(m)\, d\mu_M(m).$$

Proof. First let $T = \sum_{j=1}^{n} P_{x_j, y_j}$ be a finite rank operator. Then $\operatorname{tr} T = \sum_{j=1}^{n} \langle x_j, y_j \rangle$. On the other hand

$$\sigma_T(Hg) = \sum_{j=1}^{n} \langle P_{x_j, y_j} g^{-1} . v, g^{-1} . v \rangle = \sum_{j=1}^{n} \langle g^{-1} . v, y_j \rangle \langle x_j, g^{-1} . v \rangle.$$

Therefore (1.2) yields

$$d(\pi, v) \int_M \sigma_T(m)\, d\mu_M(m) = \sum_{j=1}^{n} \langle x_j, y_j \rangle = \operatorname{tr} T.$$

This proves the formula for finite rank operators.

Next we note that σ is complex linear in T and that $\sigma_{T^*} = \overline{\sigma_T}$. Therefore it suffices to prove the formula for symmetric operators. Since every symmetric operator in $B_1(\mathscr{H})$ is the sum of a positive and a negative trace class operator, it even suffices to prove it for positive trace class operators.

Each positive trace class operator T has a representation as $T = \sum_{n=1}^{\infty} P_{v_n}$, where $(v_n)_{n \in \mathbb{N}}$ is an orthogonal sequence with $\operatorname{tr} T = \sum_{n=1}^{\infty} \|v_n\|^2 < \infty$. Set $T_k = \sum_{n=1}^{k} P_{v_n}$. Then $(\sigma_{T_k})_{k \in \mathbb{N}}$ is a monotone sequence of continuous functions which converges to σ_T. In addition, we have for $k < l$ that

$$\begin{aligned} \|\sigma_{T_l} - \sigma_{T_k}\|_1 &= \int_M (\sigma_{T_l} - \sigma_{T_k})(m)\, d\mu_M(m) \\ &= \frac{1}{d(\pi, v)} \sum_{n=k+1}^{l} \|v_n\|^2 = \frac{1}{d(\pi, v)} \operatorname{tr}(T_l - T_k). \end{aligned}$$

Thus $(\sigma_{T_k})_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^1(M)$ and consequently

$$\int_M \sigma_T(m)\, d\mu_M(m) = \frac{1}{d(\pi, v)} \operatorname{tr}(T). \qquad \square$$

2. Relative discrete series representations.

DEFINITION 2.1. (a) An irreducible unitary representation $(\pi, \mathscr{H})$ of a locally compact separable group G with center Z is said to be *square integrable modulo the center* or to belong to the *relative discrete series* if there exist $v, w \in \mathscr{H}$ such that the function $G/Z \to \mathbb{C}$, $gZ \mapsto |\langle g . v, w \rangle|$ is square integrable (cf. [**27**, p. 4], [**28**, pp. 6, 7]).

Let $\chi \in \hat{Z}$ be the central character associated with π, i.e., $\pi(z) = \chi(z)\mathbf{1}$ for all $z \in Z$. Then $E_\chi := G \times_Z \mathbb{C}$ is a homogeneous line bundle over G/Z defined by the representation $z \mapsto \chi(z)$ of Z on $\mathbb{C}$. Note that since Z is a normal subgroup of G, we can also write G/Z for $Z \backslash G$. The sections correspond to those functions on G which satisfy $f(zg) = \chi(z) f(g)$, and the condition from above means that the function $\pi_{v,w}: G \to \mathbb{C}$, $g \mapsto \langle \pi(g) . v, w \rangle$ defines a square integrable section of E_χ. In view of Theorem 1.2, we see that every relative discrete series representation is equivalent to a subrepresentation of the regular representation of G on the space $\Gamma^2(E_\chi)$ of square integrable sections of E_χ. This justifies the terminology "relative discrete series", where "relative" refers to the fact that one has to pass from G to G/Z.

PROPOSITION 2.2. *If* $(\pi, \mathcal{H})$ *is square integrable modulo the center,* $\chi \in \hat{Z}$ *the corresponding character, and* G *is unimodular, then every matrix coefficient* $\pi_{v,w} \in \Gamma_G(E_\chi)$ *is contained in* $\Gamma^2(E_\chi)$. *Moreover, there exists an isometric intertwining operator*

$$\Psi : \mathcal{H} \to \Gamma^2(E_\chi)$$

such that $\Psi(\mathcal{H})$ *consists of continuous sections. Here* Ψ *can be obtained by* $\Psi(u)(g) := \langle u, \pi(g^{-1}) . v \rangle = \langle \pi(g) . u, v \rangle$ *for a fixed element* $0 \neq v \in \mathcal{H}$.

Proof. Pick non-zero elements $v, w \in \mathcal{H}$ such that $\pi_{w,v} \in \Gamma^2(E_\chi)$ and define Ψ as above. Then $\pi_{w,v} = \Psi(w) \in \Gamma^2(E_\chi)$ so that Theorem 1.2 implies that $\Psi(\mathcal{H}) \subseteq \Gamma^2(E_\chi)$.

We claim that G/Z is unimodular. In fact, let μ_G be a biinvariant measure on G and $F \in C_c(G/Z)$ a function with compact support which we represent as $F(gZ) := \int_Z f(gz)\, d\mu_Z(z)$ for a function $f \in C_c(G)$ (cf. [**25**, p. 475]). Then $\mu_{G/Z}(F) := \mu_G(f)$ defines a Haar measure on G/Z. Since the assignment $C_c(G) \to C_c(G/Z)$ commutes with left and right shifts, it follows immediately from the biinvariance of μ_G that $\mu_{G/Z}$ is biinvariant, i.e., that G/Z is unimodular.

For a function $f \in C(G)$ we put $f^*(g) := \overline{f(g^{-1})}$. Then the map $f \mapsto f^*$ leaves the subspace $\Gamma_G(E_\chi)$ invariant and since G/Z is unimodular, it even induces an isometry of $\Gamma^2(E_\chi)$. But from $\pi^*_{w,v} = \pi_{v,w}$ we see that $\pi_{v,w} \in \Gamma^2(E_\chi)$ for all $w \in \mathcal{H}$. Applying the argument from above with w instead of v, we see that all the function $\pi_{u,w}$ lie in $\Gamma^2(E_\chi)$. The remainder follows from Theorem 1.2. □

PROPOSITION 2.3. (The Harish-Chandra-Godement orthogonality relations) *Let* G *be a unimodular locally compact group and* $(\pi, \mathcal{H})$ *and* $(\sigma, \mathcal{K})$ *be square integrable modulo the center with the same central character* χ. *Then the following assertions hold:*

(i) *If* σ *and* π *are not equivalent, then*

$$\int_{G/Z} \langle \pi(g) . x, y \rangle \overline{\langle \sigma(g) . z, w \rangle}\, d\mu_{G/Z}(gZ) = 0$$

for all $x, y \in \mathcal{H}$, $z, w \in \mathcal{K}$. *Note that the integrand is in fact a function on* G/Z.

(ii) *There exists a positive real number* $d(\pi) > 0$ *such that if* $x, y, z, w \in \mathcal{H}$, *then*

$$\frac{1}{d(\pi)} \langle w, y \rangle \langle x, z \rangle = \int_{G/Z} \langle \pi(g) . x, y \rangle \overline{\langle \pi(g) . z, w \rangle}\, d\mu_{G/Z}(gZ)$$
$$= \int_{G/Z} \langle w, \pi(g) . z \rangle \langle \pi(g) . x, y \rangle\, d\mu_{G/Z}(gZ).$$

Proof. This is a direct generalization of Proposition 1.3.3 in [**24**] (cf. Remark 1.8.2 in [**24**] and also [**25**, Theorem 4.5.9.3]). Here it follows directly from Proposition 2.2 and Schur's Lemma. □

The constant $d(\pi)$ is called the *formal degree* of the representation π. We note that if π is square integrable modulo the center and we use $H = Z$ in Theorem 1.2, then we see that $d(\pi, v) = d(\pi)$ holds for all unit vectors $v \in \mathcal{H}$.

3. Square integrable highest weight representations. In this section we characterize those unitary highest weight representations of general Lie groups which are square integrable in a sense defined below. If the adjoint group of G is closed, then our condition of square integrability means square integrability modulo the center (cf. Remark 3.5). Moreover we derive a formula for the degree for such representations which generalize on the one hand side Weyl's dimension formula and on the other hand Harish-Chandra's degree formula for the holomorphic relative discrete series.

DEFINITION 3.1. (a) Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be *compactly embedded* if the group generated by $e^{\operatorname{ad} \mathfrak{a}}$ in $\operatorname{Aut}(\mathfrak{g})$ has compact closure. We assume that $\mathfrak{g}$ contains a compactly embedded Cartan subalgebra $\mathfrak{t}$ and recall that there exists a unique maximal compactly embedded subalgebra $\mathfrak{k}$ containing $\mathfrak{t}$ (cf. [**8**, A.2.40]).

Let G be a connected Lie group with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$. We write T and K for the analytic subgroups corresponding to $\mathfrak{t}$ and $\mathfrak{k}$.

(b) Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows ([**9**, Chapter 7]). For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}})[Y, X] = \alpha(Y)X\}$$

and write $\Delta := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \backslash \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$ for the set of roots. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha},$$

$\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, where $X \mapsto \bar{X}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$.

(c) A root α is said to be *compact* if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$ and *non-compact* otherwise. We write Δ_k for the set of compact and Δ_p for the set of non-compact roots. If $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ is a $\mathfrak{k}$-invariant Levi decomposition, i.e., $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a Levi complement, then we set

$$\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{r}_{\mathbb{C}}\} \quad \text{and} \quad \Delta_s := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\}.$$

The Lie algebra $\mathfrak{g}$ is said to have *cone potential* if $i[\overline{X_\alpha}, X_\alpha]$ is non-zero for $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha} \backslash \{0\}$. We recall from [**12**, Corollary II.15] that $\Delta = \Delta_r \dot{\cup} \Delta_s$ is a disjoint union if $\mathfrak{g}$ has cone potential. Note also that if $\mathfrak{u}$ is the nilradical, then $\mathfrak{u} = [\mathfrak{t}, \mathfrak{u}] + \mathfrak{z}(\mathfrak{g})$ ([**9**, Proposition 7.3]) and if $\mathfrak{t} \cap \mathfrak{r} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_1$, then $\mathfrak{l} := \mathfrak{t}_1 \oplus \mathfrak{s}$ is a complementary subalgebra satisfying $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$. Then $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_{\mathfrak{l}}$, where $\mathfrak{t}_{\mathfrak{l}} = \mathfrak{t}_1 \oplus (\mathfrak{t} \cap \mathfrak{s})$ is a compactly embedded Cartan subalgebra of $\mathfrak{l}$.

If $\alpha \in \Delta_s$, then we write $\check{\alpha}$ for the uniquely determined element in the one-dimensional space $[\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}] \subseteq \mathfrak{t}_{\mathbb{C}}$ satisfying $\alpha(\check{\alpha}) = 2$.

(d) A *positive system* Δ^+ of roots is a subset of Δ for which there exists a regular element $X_0 \in i\mathfrak{t}$ with $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$. We put $\Delta_k^+ := \Delta^+ \cap \Delta_k$, $\Delta_p^+ := \Delta^+ \cap \Delta_p$, and $\Delta_{p,s}^+ := \Delta^+ \cap \Delta_p \cap \Delta_s$. We say that a positive system Δ^+ is $\mathfrak{k}$-*adapted* if the set $\Delta_p^+ := \Delta_p \cap \Delta^+$ of positive non-compact roots is invariant under the *Weyl group* $\mathcal{W}_{\mathfrak{k}} := N_K(\mathfrak{t})/Z_K(\mathfrak{t})$.

https://doi.org/10.1017/S0017089500032237 Published online by Cambridge University Press

We recall from [**16**, Proposition II.7] that there exists a $\mathfrak{k}$-adapted positive system if and only if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$, i.e., if $\mathfrak{g}$ is *quasihermitian*.

A functional $\lambda \in i\mathfrak{t}^*$ is said to be *dominant integral* with respect to Δ_k^+ if $\lambda(\check{\alpha}) \in \mathbb{N}_0$ holds for all $\alpha \in \Delta_k^+$.

(e) We associate to a positive system Δ^+ certain convex cones. For a set E in a finite dimensional vector space V we write $\operatorname{cone}(E)$ for the smallest closed convex cone generated by E. We write $E^\star := \{\alpha \in V^* : \alpha(E) \subseteq \mathbb{R}^+\}$ for the dual cone of E. Then we define

$$C_{\min} := \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_p^+\}$$

$$C_{\min, \mathfrak{z}} := \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_r^+\} \subseteq \mathfrak{z},$$

and

$$C_{\min, l} := \operatorname{cone}\{i[\overline{X_\alpha}, X_\alpha] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_{p,s}^+\} \subseteq \mathfrak{t}_{\mathfrak{l}} = \mathfrak{t} \cap \mathfrak{l}.$$

Definition 3.2. Let $\Delta^+ \subseteq \Delta$ denote a positive system.

(a) For a $\mathfrak{g}$-module V and $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$V^\lambda := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \lambda(X)v\}.$$

This space is called the *weight space of weight* λ and λ is called a *weight* of V if $V^\lambda \neq \{0\}$. We write $\mathcal{P}_V$ for the set of weights of V.

(b) Let V be a $\mathfrak{g}_{\mathbb{C}}$-module and $v \in V^\lambda$ a weight vector of weight λ. We say that v is a *primitive element of* V (with respect to Δ^+) if $v \neq 0$ and $\mathfrak{g}_{\mathbb{C}}^\alpha . v = \{0\}$ holds for all $\alpha \in \Delta^+$.

(c) A $\mathfrak{g}_{\mathbb{C}}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ. We recall from [**16**, Proposition II.10] that for each linear function $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, there exists a unique irreducible highest weight module $L(\lambda)$ with highest weight λ.

Definition 3.3. Let $(\pi, \mathcal{H})$ be a unitary representation of the group G, i.e., $\pi : G \to U(\mathcal{H})$ is a continuous homomorphism into the unitary group $U(\mathcal{H})$ of the Hilbert space $\mathcal{H}$.

(a) We write $\mathcal{H}^\infty$ ($\mathcal{H}^\omega$) for the corresponding space of smooth (analytic) vectors, i.e., for the set of all those elements $v \in \mathcal{H}$ for which the mapping $G \to \mathcal{H}$, $g \mapsto \pi(g).v$ is smooth (real analytic). We write $d\pi$ for the derived representation of $\mathfrak{g}$ on $\mathcal{H}^\infty$ given by

$$d\pi(X).v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX).v$$

for $X \in \mathfrak{g}$ and $v \in \mathcal{H}^\infty$. We extend this representation to a representation of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the complex vector space $\mathcal{H}^\infty$.

(b) A vector $v \in \mathcal{H}$ is said to be K-finite if it is contained in a K-invariant finite dimensional subspace of $\mathcal{H}$. We write $\mathcal{H}^K$ for the set of K-finite vectors in $\mathcal{H}$. Note that the space $\mathcal{H}^{K,\infty}$ of K-finite smooth vectors is a $\mathfrak{g}_{\mathbb{C}}$-submodule of $\mathcal{H}^\infty$ (cf. [**16**, p. 121]).

(c) The irreducible highest weight module $L(\lambda)$ is said to be *unitarizable* if there exists a unitary representation $(\pi, \mathcal{H})$ of the simply connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ such that $L(\lambda)$ is isomorphic to the $\mathfrak{g}_{\mathbb{C}}$-module $\mathcal{H}^{K,\infty}$ of K-finite smooth vectors in $\mathcal{H}$. According to [**16**, Theorem III.6], if $\mathcal{H}^{K,\infty}$ is a highest weight module, then it is automatically irreducible. For an algebraic characterization of those highest weight modules which occur as spaces of K-finite vectors for unitary representations we refer to [**13**, Sect. X].

We recall from [**21**, Prop. I.6] that if $(\pi_\lambda, \mathscr{H})$ is a unitary highest weight representation with respect to the positive system Δ^+ and with discrete kernel, then Δ^+ is $\mathfrak{k}$-adapted, λ is dominant integral with respect to Δ_k^+, $\Delta_p^+ \subseteq -iC_{\min}^\star$, and $\lambda \in iC_{\min}^\star$ ([**21**, Lemma 1.4]).

Definition 3.4. Let $(\pi_\lambda, \mathscr{H})$ be a unitary highest weight representation of G and $v_\lambda \in \mathscr{H}^K$ a primitive element of weight λ. We call this representation *square integrable* if the function $gT \mapsto |\langle g \,.\, v_\lambda, v_\lambda\rangle|$ is contained in $L^2(G/T)$.

Remark 3.5. We consider some special cases of the preceding situation. Since $\mathfrak{g}$ contains the compactly embedded Cartan subalgebra $\mathfrak{t}$, the group G is unimodular ([**9**, Proposition 7.3(v)]).

If T is compact, then the square integrability of the highest weight representation $(\pi_\lambda, \mathscr{H})$ means that it is square integrable as a representation of G in the usual sense, i.e., a discrete series representation.

Let $Z := Z(G)$ denote the center of G. Then $Z \subseteq T$ and if T/Z is compact, i.e., if $\mathrm{Ad}(G)$ is closed (cf. [**16**, Proposition 1.2]), then the square integrability of $(\pi_\lambda, \mathscr{H})$ means that it is square integrable modulo the center (Definition 2.1).

We note that there are situations where $\mathrm{Ad}(G)$ is not closed but the class of square integrable highest weight representations still plays the role of a rather well behaved class of representations which share a lot of very nice regularity properties such as the existence of character formulas (cf. Theorem 5.2) and that they can be obtained by Duflo's orbit method (cf. Theorem 4.12).

Remark 3.6. If $p: \tilde{G} \to G$ is a group covering and $(\pi_\lambda, \mathscr{H})$ is a highest weight representation of G, then π_λ is square integrable for G if and only if $\pi_\lambda \circ p$ is square integrable for $\tilde{G}$.

In fact, since $\ker p \subseteq Z(\tilde{G}) \subseteq \tilde{T} := \exp_{\tilde{G}} \mathfrak{t}$, it follows that the canonical map $\tilde{G}/\tilde{T} \to G/T$ is a diffeomorphism. Hence the condition for the representation to be square integrable is the same for both groups. ☐

We want to show that if a group G has a square integrable highest weight representation, then the kernel of this representation must be contained in K. We start with a general lemma which will also be useful below.

We recall from Definition 3.1(c) that we have a semidirect decomposition $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$, where $\mathfrak{l}$ is a reductive Lie algebra and $\mathfrak{u}$ is the nilradical. Moreover, $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{t}_\mathfrak{l}$, where $\mathfrak{t}_\mathfrak{l} = \mathfrak{t} \cap \mathfrak{l}$ and $\mathfrak{z}(\mathfrak{g}) = \mathfrak{t} \cap \mathfrak{u}$. We write U and L for the corresponding analytic subgroups of G.

Lemma 3.7. *Let $G = U \rtimes L$, where U is the nilradical and L is reductive. Set $Z := Z(G)_0$ and $K_L := \exp \mathfrak{t}_\mathfrak{l}$. Let $H \subseteq K$ be any closed subgroup containing Z and set $H_L := H \cap K_L = H \cap L$. Then the mapping $\pi: G \to G/H$, $g \mapsto gH$ induces a diffeomorphism*

$$p: U/Z \times L/H_L \to G/H$$

and if $\mu_{U/Z}$ and μ_{L/H_L} are invariant measures on U/Z and L/H_L respectively, then $p^*(\mu_{U/Z} \otimes \mu_{L/H_L})$ is an invariant measure on G/H.

Proof. We have $\pi(ul) = ulH$. For $z \in Z$ and $h \in H_L$ we therefore have $(uz)(lh)H = (ul)(zh)H = ulH$ so that π factors to a map

$$p: U/Z \times L/H_L \to G/H, \qquad (uU, lH_L) \mapsto ulH.$$

We claim that p is a bijection. Surjectivity holds trivially. To see that it is one-to-one, suppose that $ulH = u'l'H$. Then, modulo U, we have $lH_L = l'H_L$ so that we find $h \in H_L$ with $l' = lh$. Now $l^{-1}u'^{-1}ul \in H \cap l^{-1}Ul = H \cap U = Z$ and therefore $u'^{-1}u \in Z$. This proves injectivity.

The smoothness follows from the fact that the product map $U \times L \to G$ is a diffeomorphism which is equivariant with respect to the right action of $H = Z \times H_L$, hence an equivalence of principal bundles. Thus it factors to a diffeomorphism of the orbit spaces $U/Z \times L/H_L \to G/H$.

Let $f_U \in C_c(U)$ and $f_L \in C_c(L)$ denote continuous functions with compact support and set

$$F_U(uZ) := \int_Z f_U(uz)\, d\mu_Z(z) \quad \text{and} \quad F_L(lH_L) := \int_{H_L} f_L(lh)\, d\mu_{H_L}(h).$$

Then the function F defined by $F(ulH) := F_U(uZ)F_L(lH_L)$ is a function of compact support and it suffices to show that the invariant measure on G/H coincides with $p^*(\mu_{U/Z} \otimes \mu_{L/H_L})$ on F.

We calculate

$$p^*(\mu_{U/Z} \otimes \mu_{L/H_L})(F) = \mu_{U/Z}(F_U)\mu_{L/H_L}(F_L) = \mu_U(f_U)\mu_L(f_L) = \mu_G(f),$$

where $f(ul) = f_U(u)f_L(l)$. On the other hand we have

$$\begin{aligned}\int_H f(ulh)\, d\mu_H(h) &= \int_Z \int_{H_L} f(ulzh)\, d\mu_Z(z)\, d\mu_{H_L}(h) \\ &= \int_Z f_U(uz)\, d\mu_Z(z) \int_{H_L} f_L(lh)\, d\mu_{H_L}(h) \\ &= F_U(uZ)F_L(lH_L) = F(ulH).\end{aligned}$$

Therefore $\mu_{G/H}(F) = \mu_G(f) = p^*(\mu_{U/Z} \otimes \mu_{L/H_L})(F)$ and the assertion follows. □

PROPOSITION 3.8. *If the highest weight representation* $(\pi_\lambda, \mathcal{H})$ *is square integrable, then* $\ker \pi_\lambda \subseteq K$.

Proof. In view of Remark 3.6, we may w.l.o.g. assume that G is simply connected. Let $A := \ker \pi_\lambda$ and $\mathfrak{a}$ be its Lie algebra.

We put $B := A_0 T$ and note that B is closed because its Lie algebra $\mathfrak{b} := \mathfrak{a} + \mathfrak{t}$ contains the Cartan subalgebra $\mathfrak{t}$, hence is self-normalizing. Now the function $gT \mapsto |\langle g \cdot v_\lambda, v_\lambda \rangle|$ is constant on the coset gB. Since it is square integrable over G/T, we see that if $q: G \mapsto G/T$, $g \mapsto gT$ denotes the quotient map, then for every compact subset $C \subseteq G$ the set $q(CB) \subseteq G/T$ has finite measure. If we choose C in such a way that it comes from a smooth section of the quotient map $G \to G/B$, $g \mapsto gB$, then we obtain a diffeomorphic direct product decomposition $CB \cong C \times B$. In this decomposition the Haar measure μ_G restricted to this set can be written as $\mu_C \otimes \mu_B$, where μ_C is a measure on C with a smooth density with respect to Lebesgue measure in any given chart and μ_B is a right

Haar measure on B. But $\mathfrak{b}$ contains the compactly embedded Cartan subalgebra $\mathfrak{t}$, so that B is unimodular ([**9**, Proposition 7.3.5]). Hence μ_B is a left-invariant Haar measure on B.

From that we conclude that the invariant measure on G/T, restricted to the set $q(CB) \cong C \times B/T$ can be written as $\mu_C \otimes \mu_{B/T}$, where $\mu_{B/T}$ is an invariant measure on B/T. In fact, if $f \in C_c(G/T)$ is a function with compact support contained in $q(CB)$, then we find a function $F \in C_c(G)$ such that $f(gT) = \int_T F(gt)\, d\mu_T(t)$ ([**25**, p. 475]). Then

$$\mu_{G/T}(f) = \mu_G(F) = \int_{CB} F(g)\, d\mu_G(g) = \int_C \int_B F(cb)\, d\mu_B(b)\, d\mu_C(c)$$

$$= \int_C \int_{B/T} f(cbT)\, d\mu_{B/T}(bT)\, d\mu_C(c) = (\mu_C \otimes \mu_{B/T})(f).$$

Hence $\mu_{G/T}(p(CB)) = \mu_C(C)\mu_{B/T}(B/T) < \infty$ and in particular $\mu_{B/T}(B/T) < \infty$. Let $\tilde{B}$ denote the simply connected covering group of B. Then $\tilde{B}/\tilde{T} \cong B/T$ (cf. Remark 3.6) so that we also see that $\tilde{B}/\tilde{T}$ has finite measure.

Next we apply Lemma 3.7 to the group $\tilde{B} = \tilde{B}_U \rtimes \tilde{B}_L$ and the subgroup $H = \tilde{T}$. So $\mu_{\tilde{B}/\tilde{T}} = \mu_{\tilde{B}_U/\tilde{B}_Z} \otimes \mu_{\tilde{B}_L/\tilde{T}_L}$ implies that $\tilde{B}_U/\tilde{B}_Z$ and $\tilde{B}_L/\tilde{T}_L \cong B_L/T_L$ have finite measure. Since $\tilde{B}_Z$ is a normal subgroup of $\tilde{B}_U$, it follows that $\tilde{B}_U/\tilde{B}_Z$ is compact. From $B_Z \subseteq \ker \mathrm{Ad}_B$ we conclude that $\mathrm{Ad}_B(B_U)$ is compact, so that the Lie algebra $\mathfrak{b}_u$ of B_U is at the same time a nilpotent ideal of $\mathfrak{b}$ and compactly embedded. Hence it must be central and in particular contained in $\mathfrak{t}$. On the other hand the fact B_L/T_L has finite measure implies that B_L/K_L has finite measure and therefore that B_L contains only compact simple factors. In this case we also see that $\mathfrak{b}_l \subset \mathfrak{k}$. Thus we have shown that $\mathfrak{a} \subseteq \mathfrak{k}$.

It remains to show that $A \subseteq K$. We consider the subgroup A/A_0 in the simply connected group G/A_0. It is discrete and normal, hence central and therefore contained in $\exp((\mathfrak{t} + \mathfrak{a})/\mathfrak{a}) = TA_0/A_0$. This shows that $A \subseteq TA_0 \subseteq K$. □

LEMMA 3.9. *If the kernel A of π_λ is contained in K, then the representation π_λ is square integrable if and only if the representation $\bar{\pi}_\lambda$ of G/A defined by $\bar{\pi}_\lambda(gA) := \pi_\lambda(g)$ is square integrable.*

Proof. We consider the mapping $p: G/T \mapsto G/AT \cong (G/A)/(AT/A)$. Let $f \in C_c(G/AT)$ and $F \in C_c(G)$ with

$$f(gAT) = \int_{AT} F(gk)\, d\mu_{AT}(k)$$

([**25**, p. 475]). For each compact subset $C \subseteq G$ and the mapping $q: G \to G/T$ the set $q(CAT) \subseteq q(CK)$ is compact because K/T is compact so that there exists a compact subset $F \subseteq K$ with $K = FT$. Therefore, as a function on G/T, the function defined by $\tilde{f}(gT) := f(gAT)$ has compact support and we see that

$$\mu_{G/T}(\tilde{f}) = \mu_G(F) = \mu_{G/AT}(f).$$

Hence, in this normalization, the mapping $p: G/T \to G/AT$ satisfies $p^*\mu_{G/T} = \mu_{G/AT}$. Now the assertion is immediate. □

The reductive case. Let $\mathfrak{g}$ be a quasihermitian reductive Lie algebra (cf. Definition 3.1(d)) and G an associated simply connected group. We pick a compactly embedded Cartan subalgebra $\mathfrak{t}$ and fix a $\mathfrak{k}$-adapted positive system Δ^+.

Now suppose that $(\pi_\lambda, \mathcal{H})$ is a unitary highest weight representation of G with highest weight λ with respect to Δ^+ and recall from Definition 3.3(c) that this implies in particular that λ is dominant integral with respect to Δ_k^+. We recall that $\mathrm{Ad}(G)$ is always closed for reductive groups ([**16**, Proposition 1.2]). Therefore the representation π_λ is square integrable if and only if it belongs to the relative discrete series (Remark 3.5) which, according to the possibility of realizing such representations as holomorphic functions (cf. [**7**, VI]) is called the *relative holomorphic discrete series*.

Harish-Chandra considers the function $\psi_\lambda(g) := \langle \pi_\lambda(g) . v_\lambda, v_\lambda \rangle$ on G. Actually he uses another definition which turns out to be equivalent (cf. [**13**, proof of Proposition VII.1(1)]). In [**7**, pp. 598–612] Harish-Chandra evaluates the relevant integral which leads to the following explicit characterization of those highest weights belonging to the relative holomorphic discrete series.

THEOREM 3.10. *Let ρ denote the half sum of the positive roots. Then the function ψ_λ is square integrable modulo the center, i.e., the highest weight representation π_λ is square integrable, if and only if for all $\beta \in \Delta_p^+$ we have*

$$(\lambda + \rho)(\check{\beta}) < 0. \tag{HC}$$

Proof. This follows from Lemmas 27 and 29 in [**7**, VI, pp. 604–609]. □

The following lemma makes the condition (HC) very easy to check.

LEMMA 3.11. *If $\mathfrak{g}$ is a simple hermitian Lie algebra and $\lambda \in it^*$ is dominant integral with respect to Δ_k^+, then the following are equivalent:*

(1) *λ satisfies the Harish-Chandra condition* (HC).

(2) *$(\lambda + \rho)(\check{\gamma}) < 0$ holds for the highest root $\gamma \in \Delta^+$.*

Proof. (1)$\Rightarrow$(2). Since the highest root $\lambda \in \Delta^+$ is non-compact (the non-compact simple root occurs in a representation by simple roots), it is clear that (1) implies (2).

(2)$\Rightarrow$(1). The condition that λ is dominant integral with respect to Δ_k^+ means that $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for all $\alpha \in \Delta_k^+$. On the other hand $\rho_p := \sum_{\alpha \in \Delta_p^+} \alpha$ is invariant under $\mathcal{W}_k$, hence $\rho_p(\check{\alpha}) = 0$ for all $\alpha \in \Delta_k$. Since $\rho_k(\check{\alpha}) = 1$ for all compact simple roots, it follows that $\rho_k(\check{\alpha}) > 0$ for all compact roots. Hence

$$(\lambda + \rho)(\check{\alpha}) = \lambda(\check{\alpha}) + \rho_k(\check{\alpha}) > \lambda(\check{\alpha}) \geq 0. \tag{3.1}$$

Now we introduce the standard scalar product $(\cdot \mid \cdot)$ on it so that we can identify it with its dual. With respect to this identification we have $\check{\alpha} = \dfrac{2\alpha}{(\alpha \mid \alpha)}$ for all $\alpha \in \Delta$. Let $w \in \mathcal{W}_{\mathfrak{k}}$. Then the maximality of γ yields

$$w . \gamma = \gamma - \sum_{\alpha \in \Delta_k^+} n_\alpha \alpha$$

and therefore (3.1) implies that $(\lambda + \rho \mid w . \gamma) \leq (\lambda + \rho \mid \gamma) < 0$ for all $w \in \mathcal{W}_{\mathfrak{k}}$. We conclude that (2) is equivalent to $(\lambda + \rho \mid w . \gamma) < 0$ for all $w \in \mathcal{W}_{\mathfrak{k}}$. Since Δ_p^+ generates the same cone as $\mathcal{W}_{\mathfrak{k}}.\gamma$ (cf. [**11**, Proposition 2.15]), it follows that (2) is equivalent to $(\lambda+\rho)(\check{\beta})<0$ for all $\beta \in \Delta_p^+$. This is exactly Harish-Chandra's condition (HC). □

The general case. Since we want to generalize Harish-Chandra's criterion to highest weight representations of general groups, we first have to reformulate the *Harish-Chandra condition* (HC) in such a way that it makes sense for a wider class of groups.

LEMMA 3.12. *For a functional* $\lambda \in i\mathfrak{t}^*$ *with* $\lambda|_{i\mathfrak{z}} \in i \operatorname{int} C^\star_{\min,\mathfrak{z}}$ *the following conditions are equivalent:*
(1) $(\lambda+\rho)(\check{\alpha})<0$ *for all* $\alpha \in \Delta^+_{p,s}$.
(2) $\lambda+\rho \in i \operatorname{int} C^\star_{\min}$.

Proof. We first note that the functional ρ vanishes on the center, hence on the cone $C_{\min,\mathfrak{z}}$. In view of $\lambda|_{i\mathfrak{z}} \in i \operatorname{int} C^\star_{\min,\mathfrak{z}}$, this means that the functional $\lambda+\rho$ is contained in $i \operatorname{int} C^\star_{\min}$ if and only if for all $\beta \in \Delta^+_{p,s}$ and $X_\beta \in \mathfrak{g}^\beta_{\mathbb{C}}$ with $\check{\beta}=[X_\beta, \overline{X_\beta}]$ we have

$$0>(-i)(\lambda+\rho)(i[X_\beta, \overline{X_\beta}])=(-i)(\lambda+\rho)(i\check{\beta})=(\lambda+\rho)(\check{\beta}).$$ □

Let $\mathfrak{g}=\mathfrak{u} \rtimes \mathfrak{l}$ be a $\mathfrak{t}$-invariant semidirect decomposition, where $\mathfrak{u}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{l}$ is a reductive Lie algebra (Definition 3.1(c)). Accordingly we write $G=U \rtimes L$ for the associated simply connected group. Let Δ_r and Δ_s be as in Definition 3.1, Δ^+ be a $\mathfrak{k}$-adapted positive system, $m_\beta := \dim \mathfrak{g}^\beta_{\mathbb{C}}$ the *multiplicity of the root* β, $\rho_r = \frac{1}{2}\sum_{\beta \in \Delta_r^+} m_\beta \beta$, $\rho_s = \frac{1}{2}\sum_{\beta \in \Delta_s^+} \beta$, $\rho_k = \frac{1}{2}\sum_{\beta \in \Delta_k^+} \beta$, and $\rho=\rho_s+\rho_r$. We call the elements of Δ_r the *solvable roots*. Recall that $\Delta_r \subseteq \Delta_p$ and that $m_\beta=1$ for all $\beta \in \Delta_s$.

LEMMA 3.13. *If* $(\pi_\lambda, \mathcal{H})$ *is a highest weight representation with* $\ker d\pi_\lambda \subseteq \mathfrak{k}$, *then* $\lambda \in i \operatorname{int} C^\star_{\min}$. *It follows in particular that the cone* $C_{\min} \subseteq \mathfrak{t}$ *is pointed.*

Proof. Splitting off the commutator algebra of $\ker d\pi_\lambda$, we may w.l.o.g. assume that it is abelian, hence contained in $\mathfrak{t}$ and therefore central. Since, according to [**21**, Lemma I.4], $\lambda \in iC^\star_{\min}$, the assertion follows from [**21**, Proposition I.6(ii)] combined with [**19**, Proposition IV.9]. □

We fix a functional $\alpha_0 \in i\mathfrak{z}(\mathfrak{g})^*$, such that $\alpha_0([\overline{X_\beta}, X_\beta])>0$ holds whenever $0 \neq X_\beta \in \mathfrak{g}^\beta_{\mathbb{C}}$, $\beta \in \Delta_r^+$, i.e., $\alpha_0 \in i \operatorname{int} C^\star_{\min,\mathfrak{z}}$. Put $\mathfrak{m}^+ := \bigoplus_{\beta \in \Delta_r^+} \mathfrak{g}^\beta_{\mathbb{C}}$. Then we define the structure of a complex Hilbert space on $\mathfrak{m}^+$ by $\langle X, Y\rangle := \alpha_0([Y, X])$ for $X, Y \in \mathfrak{m}^+$. Note that, in view of Proposition 3.8 and Lemma 3.13, such a functional α_0 exists if G has square integrable highest weight representations with $\ker d\pi_\lambda \subseteq \mathfrak{k}$.

The mapping $\mathfrak{m}^+ \to \mathfrak{m}$, $X \mapsto X+\bar{X}$ is a linear isomorphism and we obtain a complex structure I on $\mathfrak{m}$ by the prescription $I(X+\bar{X}) := i(X-\bar{X})$. Then $\langle X, Y\rangle := \frac{-i}{2}\alpha_0([X, IY])$ defines a real scalar product on $\mathfrak{m}$ such that $\operatorname{Re}\langle X, Y\rangle = \langle X+\bar{X}, Y+\bar{Y}\rangle$, i.e., the map $\mathfrak{m}^+ \to \mathfrak{m}$ is an isometry of real Hilbert spaces.

Next we recall a basic construction from [**21**, Section III]. We consider $\mathfrak{g}$ as a direct sum of $\mathfrak{m}=[\mathfrak{t}, \mathfrak{u}]$, $\mathfrak{z} := \mathfrak{z}(\mathfrak{g})$, and $\mathfrak{l}$, and accordingly we write the elements of $\mathfrak{g}$ as triples $(Y, Z, X) \in \mathfrak{m} \times \mathfrak{z}(\mathfrak{g}) \times \mathfrak{l}$. Let $\alpha \in i \operatorname{int} C^\star_{\min,\mathfrak{z}}$. Then $\Omega_\alpha(X, Y) := \alpha([X, Y])$ defines the structure of a symplectic vector space on $\mathfrak{m}$ and since the brackets in $\mathfrak{g}$ can be computed as

$$[(Y,Z,X),(Y',Z',X')]=([X,Y']-[X',Y],[Y,Y'],[X,X']),$$

it is clear that the assignment $\beta(Y,Z,X)=(Y,-i\alpha(Z),\operatorname{ad}X)$ defines a homomorphism $\mathfrak{g}\to\mathfrak{h}_{\mathfrak{m}}\rtimes\mathfrak{sp}(\mathfrak{m},\Omega_\alpha)$, where $\mathfrak{h}_{\mathfrak{m}}=\mathfrak{m}\times\mathbb{R}$ denotes the *Heisenberg algebra* associated to $\mathfrak{m}$ with the bracket $[(Y,t),(Y',t')]=(0,-i\alpha([Y,Y']))$, and $\mathfrak{sp}(\mathfrak{m},\Omega_\alpha)$ is the Lie algebra of the corresponding symplectic group.

Let $\tilde{\nu}$ denote the extended metaplectic representation of $\mathrm{HMp}(\mathfrak{m},\Omega_\alpha)$ on the Fock space $\mathcal{F}_{\mathfrak{m}}$ ([**21**, Proposition II.5]), G a simply connected Lie group with $\mathbf{L}(G)=\mathfrak{g}$, and $\tilde{\beta}:G\to\mathrm{HMp}(\mathfrak{m},\Omega_\alpha)$ the Lie group homomorphism with $d\tilde{\beta}(\mathbf{1})=\beta$. We consider the representation $\nu_\alpha:=\tilde{\nu}\circ\tilde{\beta}$ of the group G.

PROPOSITION 3.14. *Let $\alpha\in i\operatorname{int}C^{\star}_{\min,z}$, ν_α the corresponding representation of the group $G=U\rtimes L$ on the Fock space $\mathcal{F}_{\mathfrak{m}}$, and $A_{\alpha,\beta}:\mathfrak{g}^\beta_{\mathbb{C}}\to\mathfrak{g}^\beta_{\mathbb{C}}$, $\beta\in\Delta_r^+$ the linear map with*

$$\alpha([\bar{Y},X])=\alpha_0([\bar{Y},A_{\alpha,\beta}.X])$$

for $X,Y\in\mathfrak{g}^\beta_{\mathbb{C}}$. Then the representation $\rho_\alpha:=\nu_\alpha\,|_U$ belongs to the relative discrete series of U and its degree is given by $d(\rho_\alpha)=\prod_{\beta\in\Delta_r^+}\det_{\mathbb{C}}A_{\alpha,\beta}$ with respect to a suitable normalization of Haar measure on $U/Z(U)$.

Proof. Write $Z_U:=Z(U)$, $\mathfrak{u}=\mathfrak{z}(\mathfrak{g})\oplus\mathfrak{m}$, and define a Haar measure μ_{U/Z_U} on U/Z_U by the diffeomorphism $\mathfrak{m}\to U/Z_U$, $Y\mapsto\exp(Y)Z_U$ and the Lebesgue measure $\mu_{\mathfrak{m}}$ on $\mathfrak{m}$ normalized by $\int_{\mathfrak{m}}e^{-\|X\|^2}\,d\mu_{\mathfrak{m}}(X)=1$.

Write $E_0\in\mathcal{F}_n$ for the constant function 1 and let $Y\in\mathfrak{m}$. Then it follows from the definition of ν_α and Section II in [**21**] that

$$\langle\nu_\alpha(\exp Y).E_0,E_0\rangle=e^{-\frac{1}{4}\|Y\|^2_\alpha}=e^{\frac{i}{8}\alpha([Y,IY])},$$

where $\|Y\|^2_\alpha=-\dfrac{i}{2}\alpha([Y,IY])$. Since the representation ρ_α is irreducible, we see that this function belongs to the relative discrete series and, using the isomorphism $\mathfrak{m}^+\to\mathfrak{m}$, $Y\mapsto Y+\bar{Y}$, we obtain

$$\begin{aligned}\frac{1}{d(\rho_\alpha)}=\frac{\langle E_0,E_0\rangle}{d(\rho_\alpha)}&=\int_{\mathfrak{m}}e^{\frac{i}{8}\alpha([Y,IY])}\,d\mu_{\mathfrak{m}}(Y)=\int_{\mathfrak{m}^+}e^{-\alpha([\bar{Y},Y])}\,d\mu_{\mathfrak{m}^+}(Y)\\&=\prod_{\beta\in\Delta_r^+}\int_{\mathfrak{g}^\beta_{\mathbb{C}}}e^{-\alpha_0([\bar{Y},A_{\alpha,\beta}.Y])}\,d\mu_{\mathfrak{g}^\beta_{\mathbb{C}}}(Y)=\prod_{\beta\in\Delta_r^+}\int_{\mathfrak{g}^\beta_{\mathbb{C}}}e^{-\langle A_{\alpha,\beta}.Y,Y\rangle}\,d\mu_{\mathfrak{g}^\beta_{\mathbb{C}}}(Y)\\&=\prod_{\beta\in\Delta_r^+}\det_{\mathbb{C}}A^{-1}_{\alpha,\beta}\end{aligned}$$

because our normalization of the measure on $\mathfrak{g}^\beta_{\mathbb{C}}$ resp. $\mathfrak{m}^+$ is such that

$$\int_{\mathfrak{m}^+}e^{-\langle A.Y,Y\rangle}\,d\mu_{\mathfrak{m}^+}(Y)=\det A^{-1}$$

whenever A is positive definite. □

THEOREM 3.15. *Suppose that $(\pi_\lambda,\mathcal{H})$ is a unitary highest weight representation with*

https://doi.org/10.1017/S0017089500032237 Published online by Cambridge University Press

the highest weight λ with respect to the $\mathfrak{k}$-adapted positive system Δ^+ and that $\ker d\pi_\lambda \subseteq \mathfrak{k}$. *Then π_λ is square integrable if and only if*

$$\lambda + \rho \in i \operatorname{int} C^\star_{\min}, \tag{3.2}$$

where $\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} m_\beta \beta$.

Moreover, if v_λ is a normalized primitive element, then $d(\pi, v_\lambda) = d(\rho_{\lambda|_{i\mathfrak{z}}}) d(\pi_{\lambda_l})$, *where π_{λ_l} is the unitary highest weight representation of L with highest weight* $\lambda_l := (\lambda + \rho_r)|_{\mathfrak{t}_\mathfrak{l}}$ *and* $\mathfrak{t}_\mathfrak{l} = \mathfrak{t} \cap \mathfrak{l}$.

Proof. Let $A := \ker \pi_\lambda$ and $\mathfrak{a} = \mathbf{L}(A)$. We identify the dual of $\mathfrak{t}/(\mathfrak{a} \cap \mathfrak{t})$ with the subspace $(\mathfrak{a} \cap \mathfrak{t})^\perp \subseteq \mathfrak{t}^*$. Then the assumption that $\mathfrak{a} \subseteq \mathfrak{k}$ shows that whenever $\mathfrak{g}^\alpha_\mathbb{C} \subseteq \mathfrak{a}_\mathbb{C}$, then this root space is contained in a compact semisimple ideal which is a direct summand. Hence α vanishes on the cone $C_{\min}$. Therefore the condition (3.2) is satisfied for G if and only if it is satisfied for the quotient group G/A. Thus we may w.l.o.g. assume that π_λ has discrete kernel. Moreover, in view of Remark 3.6, we may also assume that G is simply connected.

We write $G = U \rtimes L$, where U is the nilradical and L is reductive and invariant under K (cf. Definition 3.1(c)). Using Theorem III.2 in **[21]**, we obtain a tensor product decomposition $\pi_\lambda = \pi_{\lambda_l} \otimes \nu$, where π_{λ_l} is a highest weight representation of L (considered as a representation of G via $L \cong G/U$) with highest weight $\lambda_l = (\lambda + \rho_r)|_{\mathfrak{t}_\mathfrak{l}}$. The representation ν is an extended metaplectic representation, more precisely, it is a highest weight representation with highest weight $\lambda - \lambda_l$, where λ_l is extended to $\mathfrak{t}$ by 0 on $\mathfrak{z}$, and $\mathscr{H}^K_\nu \cong \operatorname{Pol}(\mathfrak{p}^+_r) \otimes \mathbb{C}$ as T-module, where $\mathbb{C}$ is the one-dimensional T-module corresponding to the highest weight $\lambda - \lambda_l$.

Let $v_\lambda = v_{\lambda_l} \otimes v_\nu \in \mathscr{H}(F)$ denote a unit highest weight vector decomposed according to the decomposition $\pi_\lambda = \pi_{\lambda_l} \otimes \nu$ and put $\psi_\lambda(g) = \langle \pi_\lambda(g) . v_\lambda, v_\lambda \rangle$. Using Proposition 2.3 and Lemma 3.7, we calculate the integral

$$\begin{aligned}
\int_{G/T} |\psi_\lambda(gT)|^2 \, d\mu_{G/T}(gT) &= \int_{U/Z} \int_{L/T_L} |\psi_\lambda(ulT)|^2 \, d\mu_{U/Z}(uZ) \, d\mu_{L/T_L}(lT_L) \\
&= \int_{U/Z} \int_{L/T_L} |\langle \pi_\lambda(ul) . v_\lambda, u_\lambda \rangle|^2 \, d\mu_{U/Z}(uZ) \, d\mu_{L/T_L}(lT_L) \\
&= \int_{U/Z} \int_{L/T_L} |\langle \pi_{\lambda_l}(l) . v_{\lambda_l}, v_{\lambda_l} \rangle \langle \nu(ul) . v_\nu, v_\nu \rangle|^2 \, d\mu_{U/Z}(uZ) \, d\mu_{L/T_L}(lT_L) \\
&= \int_{L/T_L} |\langle \pi_{\lambda_l}(l) . v_{\lambda_l}, v_{\lambda_l} \rangle|^2 \int_{U/Z} |\langle \nu(u)\nu(l) . v_\nu, v_\nu \rangle|^2 \, d\mu_{U/Z}(uZ) \, d\mu_{L/T_L}(lT_L) \\
&= \int_{L/T_L} |\langle \pi_{\lambda_l}(l) . v_{\lambda_l}, v_{\lambda_l} \rangle|^2 \frac{1}{d(\nu|_U)} \|\nu(l) . v_\nu\|^2 \, \|v_\nu\|^2 \, d\mu_{L/T_L}(lT_L) \\
&= \frac{1}{d(\nu|_U)} \int_{L/T_L} |\langle \pi_{\lambda_l}(l) . v_{\lambda_l}, v_{\lambda_l} \rangle|^2 \, d\mu_{L/T_L}(lT_L),
\end{aligned}$$

where $d(\nu|_U)$ is the formal degree of the representation $\nu|_U = \rho_{\lambda|_{i\mathfrak{z}}}$ which is square integrable modulo the center (Proposition 3.14).

We conclude that π_λ is square integrable if and only if the same holds for the highest weight representation π_{λ_l} of L, i.e., if

$$\int_{L/T_L} |\langle \pi_{\lambda_l}(l) . v_{\lambda_l}, v_{\lambda_l}\rangle|^2 \, d\mu_{L/T_L}(lT_L) < \infty.$$

Since

$$(\lambda + \rho)([\overline{X_\beta}, X_\beta]) = \lambda([\overline{X_\beta}, X_\beta]) > 0$$

holds for all $0 \neq X_\beta \in \mathfrak{g}_{\mathbb{C}}^\beta$, $\beta \in \Delta_r^+$ (Lemma 3.13), we see that the condition $\lambda + \rho \in i \operatorname{int} C_{\min}^\star$ is equivalent to $\lambda + \rho = \lambda_l + \rho_l \in i \operatorname{int} C_{\min,l}^\star$ which, in view of Lemmas 3.11 and 3.12, is Harish-Chandra's condition for the square integrability of the representation π_{λ_l} of L.

To complete the proof, we note that our calculation gives the following formula for the degree

$$\frac{1}{d(\pi_\lambda, v)} = \int_{G/T} |\psi_\lambda(gT)|^2 \, d\mu_{G/T}(gT) = \frac{1}{d(\rho_{\lambda|_{i\mathfrak{b}}}) d(\pi_{\lambda_l})}. \qquad \square$$

REMARK 3.16. If we assume in Theorem 3.15 instead of $\ker d\pi_\lambda \subseteq \mathfrak{k}$ that $\mathfrak{g}$ has cone potential, then one can show that the condition $\lambda + \rho \in i \operatorname{int} C_{\min}^\star$ even implies that $\ker d\pi_\lambda \subseteq \mathfrak{k}$. In fact, let $\mathfrak{a} = \ker d\pi_\lambda$.

Suppose that $X_\beta \in \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^\beta$ with $\beta \in \Delta_p^+$ is non-zero. Then $i[\overline{X_\beta}, X_\beta] \in C_{\min} \backslash \{0\}$ because $\mathfrak{g}$ has cone potential. Hence $\lambda([\overline{X_\beta}, X_\beta]) = 0$ because $X_\beta \in \mathfrak{a}_{\mathbb{C}}$ and therefore

$$0 < -i(\lambda + \rho)(i[\overline{X_\beta}, X_\beta]) = \rho([\overline{X_\beta}, X_\beta]).$$

If $\beta \in \Delta_s$, then $\check{\beta}$ is a non-zero multiple of $[X_\beta, \overline{X_\beta}]$ and therefore we obtain a contradiction to $\rho(\check{\beta}) \geq \rho_s(\check{\beta}) > 0$ which in turn follows from $\rho_r(\check{\beta}) = 0$ for β compact and $\rho_r(\check{\beta}) = i\rho_r(-i\check{\beta}) \in i\rho_r(C_{\min}) \subseteq \mathbb{R}^+$ (cf. Definition 3.3(c)). Thus $\beta \in \Delta_r^+$ and $[X_\beta, \overline{X_\beta}] \subseteq \mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$. Hence $\rho([\overline{X_\beta}, X_\beta]) = 0$ yields a contradition. $\square$

THEOREM 3.17. *With respect to a suitable normalization of the invariant measure on G/T, the number $d(\pi_\lambda, v_\lambda)$ of the square integrable highest weight representation π_λ of highest weight λ is given by*

$$d(\pi_\lambda, v_\lambda) = \Big(\prod_{\beta \in \Delta_r^+} \det\nolimits_{\mathbb{C}} A_{\lambda,\beta}\Big)\Big(\prod_{\beta \in \Delta_{p,s}^+} \frac{|(\lambda+\rho)(\check{\beta})|}{\rho_s(\check{\beta})}\Big) d(\pi_\lambda^K),$$

where

$$d(\pi_\lambda^K) = \prod_{\beta \in \Delta_k^+} \frac{(\lambda + \rho_k)(\check{\beta})}{\rho_k(\check{\beta})}$$

is the dimension of the irreducible K-representation π_λ^K of highest weight λ.

Proof. According to Theorem 3.15, we have $d(\pi_\lambda) = d(\rho_\lambda) d(\pi_{\lambda_l})$. For the first factor we use Proposition 3.14 and for the second we have Harish-Chandra's degree formula

$$d(\pi_{\lambda_l}) = \prod_{\beta \in \Delta_s^+} \frac{|(\lambda_l + \rho_s)(\check{\beta})|}{\rho_s(\check{\beta})}$$

for the formal degree of the relative discrete series representations of reductive Lie groups (cf. [**7**, VI, p. 612]).

To see that this formula has a factorization as asserted, we write $\rho_s = \rho_k + \rho_{p,s}$. Then the functional $\rho_{p,s}$ is invariant under the Weyl group $\mathcal{W}_t$, hence vanishes on $\check{\beta}$ for $\beta \in \Delta_k$. Thus

$$d(\pi_{\lambda_l}) = \prod_{\beta \in \Delta_{p,s}^+} \frac{|(\lambda_l + \rho_s)(\check{\beta})|}{\rho_s(\check{\beta})} \prod_{\beta \in \Delta_k^+} \frac{(\lambda_l + \rho_k)(\check{\beta})}{\rho_k(\check{\beta})} = \prod_{\beta \in \Delta_{p,s}^+} \frac{|(\lambda + \rho)(\check{\beta})|}{\rho_s(\check{\beta})} d(\pi_{\lambda_l}^K),$$

where

$$d(\pi_{\lambda_l}^K) = \prod_{\beta \in \Delta_k^+} \frac{(\lambda_l + \rho_k)(\check{\beta})}{\rho_k(\check{\beta})} = \prod_{\beta \in \Delta_k^+} \frac{(\lambda + \rho_k)(\check{\beta})}{\rho_k(\check{\beta})} = d(\pi_\lambda^K).$$

□

Note that if $\mathfrak{g}$ is compact then this is Weyl's dimension formula and that, if $\mathfrak{g}$ is reductive, this is Harish Chandra's degree formula for the relative holomorphic discrete series.

4. Highest weight representations via Duflo's orbit method. In this section we explain how the class of square integrable highest weight representations fits into the picture suggested by the orbit method. It will turn out that the square integrable highest weight representations are exactly those which can be constructed with Duflo's orbit method.

We keep the notation introduced in Definition 3.1. In particular $\mathfrak{g}$ always denotes a Lie algebra containing a compactly embedded Cartan subalgebra $\mathfrak{t}$.

Definition 4.1. For $f \in \mathfrak{g}^*$ we write $\mathrm{Ad}^*(g).f := f \circ \mathrm{Ad}(g)^{-1}$ for the coadjoint action and $\mathcal{O}_f := \mathrm{Ad}^*(G).f$ for the coadjoint orbits. Furthermore we define $G_f := \{g \in G : \mathrm{Ad}^*(g).f = f\}$ and

$$\mathfrak{g}_f := \mathbf{L}(G_f) = \{X \in \mathfrak{g} : f \circ \mathrm{ad}\, X = 0\}.$$

We identify the dual $\mathfrak{t}^*$ of $\mathfrak{t}$ with the subspace $[\mathfrak{t}, \mathfrak{g}]^\perp$ of $\mathfrak{g}^*$. A functional $f \in \mathfrak{t}^*$ is said to be *integral* if there exists a character $\chi : T \to \mathbb{C}^*$ with $d_\chi(\mathbf{1}) = if$. It is called *regular* if $\mathfrak{g}_f = \mathfrak{t}$. In view of [**19**, Lemma II.4], this condition means that the coadjoint orbit $\mathcal{O}_f \subseteq \mathfrak{g}^*$ has maximal dimension in $\mathfrak{g}^*$.

In the following we always identify $\mathfrak{g}^*$ with the subset of all complex linear functionals in $\mathfrak{g}_{\mathbb{C}}^*$ which take real values on $\mathfrak{g}$.

Definition 4.2. (a) Let $f \in \mathfrak{g}^*$. A complex subalgebra $\mathfrak{b} \subseteq \mathfrak{g}_{\mathbb{C}}$ is called a (complex) *polarization* in f if

$$f_{\mathbb{C}}([\mathfrak{b}, \mathfrak{b}]) = \{0\} \quad \text{and} \quad \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}} \mathfrak{b} = \frac{1}{2} \dim \mathcal{O}_f.$$

(b) We say that a polarization $\mathfrak{b}$ in f satisfies *Pukanszky's condition* if the group $\mathrm{Ad}(B) := \langle e^{\mathrm{ad}\, \mathfrak{b}} \rangle \subseteq \mathrm{Aug}(\mathfrak{g}_{\mathbb{C}})$ satisfies

$$\mathrm{Ad}(B)^*.f = f + \mathfrak{b}^\perp \subseteq \mathfrak{g}_{\mathbb{C}}^*.$$

(c) An element $f \in \mathfrak{g}^*$ is said to be *well polarizable* if there exists a *good* *polarization* $\mathfrak{b}$ in f, i.e., $\mathfrak{b}$ is solvable and satisfies Pukanszky's condition. For such a polarization we define a linear form $\rho_{\mathfrak{b}}$ on the stabilizer algebra $\mathfrak{g}_f$ by

$$\rho_{\mathfrak{b}}(X) := \frac{1}{2} \operatorname{tr} \operatorname{ad} X \big|_{\mathfrak{b}/(\mathfrak{g}_f)_{\mathbb{C}}}.$$

We say that f is *admissible* if the form $(i \,.\, f\,|_{\mathfrak{g}_f}) + \rho_{\mathfrak{b}}$ is the differential of a character of the group $(G_f)_0$.

Let $\widetilde{G_f}$ denote the metaplectic covering (a central extension by $\mathbb{Z}_2$) of G_f defined by the symplectic action of G_f on the symplectic vector space $V := \mathfrak{g}/\mathfrak{g}_f$ and the pull-back diagram

$$\begin{array}{ccc} \widetilde{G_f} & \longrightarrow & \mathrm{Mp}(V) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ G_f & \longrightarrow & \mathrm{Sp}(V). \end{array}$$

We write $\varepsilon \in \widetilde{G_f}$ for the element $(\mathbf{1}, \varepsilon_0) \in G_f \times \mathrm{Mp}(V)$, where $\varepsilon_0 \neq \mathbf{1}$ is the second element in the kernel of the metaplectic covering of $\mathrm{Sp}(V)$. Then $\{\mathbf{1}, \varepsilon\}$ is the kernel of π. One can show that $\rho_{\mathfrak{b}}$ is always the differential of a character $\widetilde{\rho_{\mathfrak{b}}}$ of $\widetilde{G_f}$ satisfying $\widetilde{\rho_{\mathfrak{b}}}(\varepsilon) = -1$ ([**3**, 2.1]). Consequently f is admissible if and only if there exists a unitary character χ of $\widetilde{(G_f)_0}$ with $\chi(\varepsilon) = -1$. We write $X(G, f)$ for the set of all representations of $\widetilde{G_f}$ for which the restriction to $\widetilde{(G_f)_0}$ is χ. Note that this is a one-element set if G_f is connected and that f is always admissible if $(G_f)_0$ is simply connected.

LEMMA 4.3. *If* Δ^+ *is a positive system and* $f \in \mathfrak{t}^*$ *is regular, then the subalgebra* $\mathfrak{b} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}$ *is a complex polarization in* f *satisfying Pukanszky's condition.*

Proof. Let $h \in f + \mathfrak{b}^{\perp}$ and $X, Y \in \mathfrak{b}$. Then

$$(e^{\operatorname{ad}^* X} . h - h)(Y) = \sum_{n=1}^{\infty} (\operatorname{ad}^* X)^n . h(Y) \in h([\mathfrak{b}, \mathfrak{b}]) = f([\mathfrak{b}, \mathfrak{b}]) = \{0\}$$

shows that the set $f + \mathfrak{b}^{\perp} \subseteq \mathfrak{g}_{\mathbb{C}}^*$ is invariant under the group $\operatorname{Ad}(B)^*$ and in particular that $\operatorname{Ad}(B)^* . f \subseteq f + \mathfrak{b}^{\perp}$.

It is clear that $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{b}$ and that $\operatorname{ad}^*(\mathfrak{t}_{\mathbb{C}}) . f = \{0\}$. Hence Theorem I.11 in [**15**] implies that the orbit $\operatorname{Ad}(B)^* . f \subseteq f + \mathfrak{b}^{\perp}$ is closed. We claim that it is also open in $f + \mathfrak{b}^{\perp}$. To see this, note the tangent space of this orbit is $f + \operatorname{ad}^*(\mathfrak{b}) . f$. Since $(\mathfrak{g}_f)_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}$, it follows that

$$\dim \operatorname{ad}^*(\mathfrak{b}) . f = \dim \mathfrak{b} - \dim \mathfrak{t}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{C}} - \dim \mathfrak{b} = \dim \mathfrak{b}^{\perp}.$$

Therefore $f + ad^*(\mathfrak{b}) . f = f + \mathfrak{b}^{\perp}$ and consequently $\operatorname{Ad}(B)^* . f$ is an open orbit. Since it is also closed, it follows that $\operatorname{Ad}(B)^* . f = f + \mathfrak{b}^{\perp}$. □

We have shown so far that f is well polarizable because $\mathfrak{b}$ is a solvable Lie algebra. Now we have to deal with the admissibility condition. To do this we assume from now on that G is a simply connected group with $\mathbf{L}(G) = \mathfrak{g}$ and that Δ^+ is a $\mathfrak{k}$-adapted positive system, i.e., the set of positive non-compact roots is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$ (cf. Definition 3.1(d)).

LEMMA 4.4. *Suppose that G is simply connected and $T = \exp \mathfrak{t}$. Then for a functional $f \in \mathfrak{t}^*$ the following are equivalent:*

(1) *f is integral, i.e, $X \mapsto e^{if(X)}$ factors to a character of the group T.*

(2) *$if(\check{\alpha}) \in \mathbb{Z}$ for all $\alpha \in \Delta_k^+$.*

Proof. [**21**, Proposition I.14]. □

LEMMA 4.5. *We have*

$$\rho_{\mathfrak{b}}(X) = \frac{1}{2} \operatorname{tr} \operatorname{ad} X \mid_{\mathfrak{b}/(\mathfrak{g}_f)_{\mathbb{C}}} = \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha(X)$$

and $\rho_{\mathfrak{b}}(\check{\alpha}) = 1$ for all simple roots $\alpha \in \Delta_k^+$.

Proof. If s_α is the involution on $\mathfrak{t}^*$ coming from the root α, then

$$s_\alpha(\beta) = \beta - \beta(\check{\alpha})\alpha \tag{4.1}$$

for all $\beta \in \mathfrak{t}^*$ and if, in addition, α is a simple root, then $s_\alpha(\Delta^+) = (\Delta^+ \backslash \{\alpha\}) \cup \{-\alpha\}$ implies that $s_\alpha(\rho_{\mathfrak{b}}) = \rho_{\mathfrak{b}} - \alpha$. Therefore $\rho_{\mathfrak{b}}(\check{\alpha}) = 1$ follows from (4.1). □

PROPOSITION 4.6. *The functional f is admissible for the simply connected group G if and only if f is integral.*

Proof. Since the coroots $\check{\alpha}$ for the simple roots $\alpha \in \Delta_k^+$ form a basis of the dual root system [(**2**, Ch. VI, §1, no. 1.5, Rem. 5]), Lemma 4.5 shows that $\rho_{\mathfrak{b}}$ is integral so that it always integrates to a character of the group T. Therefore $if + \rho_{\mathfrak{b}}$ integrates to a character of T if and only if f is integral (Lemma 4.4). Now the assertion follows from the connectedness of the group G_f ([**15**, Theorem I.18]) which therefore must be equal to T. □

Since $G_f = T$ is connected, there exists exactly one $\tau \in X(G, f)$. Now Duflo's orbit method ([**3**], [**6**, Theorem 4.1]) provides for each f a unitary representation $T_f := T_{f,\tau}^G$ with *infinitesimal character* $\chi_{if}: \mathfrak{z}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$.

REMARK 4.7. If G is connected reductive and $f \in \mathfrak{t}^*$, where $\mathfrak{t}$ is a compactly embedded Cartan subalgebra, then the construction of the representations T_f^G in [**5**, p. 118] shows that T_f^G belongs to the relative discrete series. This means that, for G reductive, the only highest weight representations which we can expect to be obtained by this method are those which belong to the relative discrete series, i.e., which are square integrable.

REMARK 4.8. (a) Let G be compact semisimple and $f \in \mathfrak{g}^*$ well polarizable and admissible, i.e., $\mathfrak{g}_f$ is a Cartan subalgebra. Let $\mathfrak{t} = \mathfrak{g}_f$ denote the corresponding Cartan subalgebra. Let $\Delta^+ := \{\alpha : if(\check{\alpha}) > 0\}$ and ρ the half sum of the roots in Δ^+. Then the representation T_f^G associated to f is a representation of highest weight $if - \rho$.

(b) Let G be hermitian simple and simply connected, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan subalgebra and $f \in \mathfrak{t}^*$ regular and integral. Set $\lambda := if \in i\mathfrak{t}^*$. We use [**3**, pp. 115–118] to explain what T_f^G is. First we choose a positive system of roots $\Delta^+(\lambda) := \{\alpha : \lambda(\check{\alpha}) > 0\}$ and write $\Delta_p^+(\lambda)$ and $\Delta_k^+(\lambda)$ for the corresponding systems of positive non-compact and compact roots. We define

$$\rho^\lambda := \frac{1}{2} \sum_{\alpha \in \Delta_p^+(\lambda)} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_k^+(\lambda)} \alpha = \rho(\Delta_p^+(\lambda)) - \rho(\Delta_k^+(\lambda)).$$

Since the stabilizer G_f is connected, hence equal to $T = \exp \mathfrak{t}$, there exists a character $\Lambda \in \hat{T}$ such that $d\Lambda = \lambda + \rho^\lambda$. Now the representation T_f^G associated to f is the relative discrete series representation, where $\lambda + \rho^\lambda$ is the highest weight with respect to Δ_k^+ of the lowest K-type (cf. [23]). Hence, for all compact simple roots, we have

$$(\lambda + \rho^\lambda)(\check{\alpha}) = (\lambda - \rho(\Delta_k^+(\lambda)))(\check{\alpha}) = \lambda(\check{\alpha}) - 1 \geq 0$$

whenever $\Delta_p^+(\lambda)$ is $\mathcal{W}_\mathfrak{k}$-invariant because the integrality of $\lambda(\check{\alpha}) > 0$ implies that $\lambda(\check{\alpha}) \geq 1$.

PROPOSITION 4.9. *Suppose that G is a simply connected quasihermitian reductive Lie group and that $f \in \mathfrak{t}^*$ is integral such that $\lambda := if$ satisfies*

$$\lambda(\check{\alpha}) \begin{cases} <0, & \text{for } \alpha \in \Delta_p^+ \\ >0, & \text{for } \alpha \in \Delta_k^+, \end{cases}$$

where Δ^+ is a $\mathfrak{k}$-adapted positive system with $\Delta_p^+ \subseteq -iC_{\min}^\star$. Then T_f^G is the highest weight representation with highest weight $\lambda - \rho$ with respect to Δ^+.

Proof. In view of Remark 4.8(b), we know that the highest weight of the lowest K-type in $\mathcal{H}$ (the corresponding Hilbert space) is the dominant integral functional $\lambda + \rho^\lambda$. On the other hand $\lambda(\check{\beta}) < 0$ for all $\beta \in \Delta_p^+$ implies that $\Delta^+(\lambda) = \Delta_k^+ \cup (-\Delta_p^+)$. Therefore $\rho^\lambda = -\rho_k - \rho_p = -\rho$ and $\lambda + \rho^\lambda = \lambda - \rho$.

To see that the representation T_f^G is in fact a highest weight representation, note that

$$(\lambda - \rho + \rho)(\check{\beta}) = \lambda(\check{\beta}) < 0$$

for all $\beta \in \Delta_p^+$, $(\lambda - \rho)(\check{\alpha}) \geq 0$ for $\alpha \in \Delta_k^+$ (Remark 4.8(b)), and $(\lambda - \rho)(\check{\beta}) < \lambda(\check{\beta}) < 0$ for $\beta \in \Delta_p^+$. Hence $\lambda - \rho$ satisfies the Harish-Chandra condition for the relative holomorphic discrete series. This means that T_f^G must be a relative holomorphic discrete series representations, in particular it is a highest weight representation. □

REMARK 4.10. Let $G = U \rtimes L$ denote a semidirect decomposition of G, where U is the nilradical and L is a reductive subgroup which is T-invariant (cf. Definition 3.1). We use [5, pp. 121ff] to analyze how the representation T_f for $f \in \mathfrak{t}^*$ is adapted to this decomposition.

Let $f_u := f|_\mathfrak{u}$ and extend this function by 0 on $\mathfrak{l}$ to a functional on $\mathfrak{g}$. Then

$$f_u([\mathfrak{l}, \mathfrak{g}]) \subseteq f_u([\mathfrak{l}, \mathfrak{l}] + [\mathfrak{l}, \mathfrak{u}]) = \{0\}$$

because $\mathfrak{m} := [\mathfrak{t}, \mathfrak{u}]$ is invariant under $\mathfrak{l}$ (cf. Definition 3.1). Therefore $L \subseteq G_{f_u}$. It follows in particular that $UG_{f_u} = G$. According to [5, p. 123], the representation T_f is a tensor product

$$T_f^G = T_{f_1}^{G_1} \otimes S_{f_u} T_{f_u}^U.$$

We explain the different ingredients of this decomposition. Let $\mathfrak{q} := \ker f \cap \mathfrak{u}_{f_u}$. Then $\mathfrak{q}$ is an ideal in $\mathfrak{g}_{f_u}$ because $\mathfrak{u}_{f_u}$ and $\ker f$ are invariant under $\mathfrak{g}_{f_u}$. We set

$$\mathfrak{g}_1 := \mathfrak{g}_{f_u}/\mathfrak{q} = (\mathfrak{l} + \mathfrak{u}_{f_u})/\mathfrak{q} = (\mathfrak{l} + \mathfrak{z})/(\ker f \cap \mathfrak{z}).$$

Therefore $\mathfrak{g}_1$ is a reductive Lie algebra and f_1 is obtained by factorization of $f|_{\mathfrak{g}^{f_u}}$ to $\mathfrak{g}_1$.

The representation $T_{f_u}^U$ is the Kirillov representation associated to f_u. Since in our case U is a central extension of a Heisenberg algebra, $T_{f_u}^U$ is exactly the Schrödinger representation associated to f_u (cf. [21, Section II]). The representation $S_{f_u} T_{f_u}^G$ is a representation of the semidirect product $U \rtimes L$ given by $(u,l) \mapsto T^U_{f_u}(u) S_{f_u}(l)$, where S_{f_u} is obtained by the metaplectic representation of $\mathrm{Sp}(n,\mathbb{R})^\sim$ via the homomorphism $L \to \mathrm{Sp}(n,\mathbb{R})^\sim$ obtained by the action of L on $\mathfrak{u}/\mathfrak{u}_{f_u} = T_{f_u}(\mathcal{O}_{f_u})$.

This proves in particular that the decomposition of T_f is compatible with the Satake decomposition of a representation of $U \rtimes L$ as in [**21**, Theorem III.2]. □

THEOREM 4.11 *Let G be a simply connected Lie group, Δ^+ a $\mathfrak{k}$-adapted positive system of roots, and $f \in \mathrm{int}\, C^\star_{\min} \subseteq \mathfrak{t}^*$ regular such that $\lambda := if$ is dominant integral with respect to Δ^+_k. Then T^G_f is the highest weight representation with highest weight $\lambda - \rho$ with respect to Δ^+.*

Proof. For the case where G is reductive, the existence of a $\mathfrak{k}$-adapted positive system implies that $\mathfrak{g}$ is quasihermitian, i.e., its commutator algebra is a direct sum of compact and hermitian simple ideals (cf. Definition 3.1(d)). In this case the assertion is exactly Proposition 4.9 because for $X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}$ the element $[X_\alpha, \overline{X_\alpha}]$ is a positive multiple of $\check{\alpha}$, so that $f \in \mathrm{int}\, C^\star_{\min}$ means that $if(\check{\alpha}) < 0$ for all $\alpha \in \Delta^+_p$.

In general $G \cong U \rtimes L$, where L is a simply connected reductive quasihermitian Lie group and U, the nilradical of G, is a central extension of a Heisenberg algebra. Let $T := T^G_f$. We may assume that U is not central. Otherwise G is reductive and Proposition 4.9 applies.

Then the largest ideal contained in $\ker f$ acts trivially on $\mathcal{O}_f$ and therefore, since f is regular, is contained in $\mathfrak{t}$. Hence it is central. It follows in particular that $f([\mathfrak{u},\mathfrak{u}]) \neq \{0\}$ because otherwise $\ker f$ contains the ideal $[\mathfrak{g},\mathfrak{u}]$. From $[\mathfrak{u},\mathfrak{u}] \subseteq \mathfrak{z} := \mathfrak{z}(\mathfrak{g})$ we infer that f does not vanish on $\mathfrak{z}$.

Now let $\mathfrak{a} := \ker dT \cap \mathfrak{u}$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. We claim that $\mathfrak{a} \subseteq \mathfrak{z}$. To see this, we note that $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{z}) \oplus [\mathfrak{t},\mathfrak{a}]$ and that $\mathfrak{a} \cap \mathfrak{z} \subseteq \ker f$ as well as $[\mathfrak{t},\mathfrak{a}] \subseteq [\mathfrak{t},\mathfrak{g}] \subseteq \ker f$. Therefore $\mathfrak{a}$ is an ideal contained in $\ker f$ and therefore central as we have seen above.

Let $\mathfrak{m} = [\mathfrak{t},\mathfrak{u}]$ and consider the skew-symmetric bilinear form $q(X,Y) := f([X,Y])$ on $\mathfrak{m} \times \mathfrak{m}$. We claim that q is non-degenerate. Suppose that $q(X,\mathfrak{m}) = \{0\}$. Then $[X,\mathfrak{m}] \subseteq \ker f$ and therefore $[X,\mathfrak{u}] \subseteq \ker f$. On the other hand $[X,\mathfrak{l}] \subseteq \mathfrak{m} \subseteq \ker f$ and consequently $X \in \mathfrak{g}_f = \{Y \in \mathfrak{g} : f \circ \mathrm{ad}\, Y = 0\} = \mathfrak{t}$, contradicting the regularity of f. This proves that the image $\mathfrak{u}/\mathfrak{a}$ is isomorphic to a Heisenberg algebra $\mathfrak{h}_n$ and the homomorphism $\mathfrak{u} \mapsto \mathfrak{u}/\mathfrak{a}$ can be written as

$$\beta : \mathfrak{n} = \mathfrak{m} \oplus \mathfrak{z} \to \mathfrak{h}_n, \qquad (X,Y) \mapsto (X, f(Y))$$

because the kernel of this homomorphism is $\mathfrak{a}$.

Therefore the group $T(U)$ is a Heisenberg group, and as we have shown in [**21**, Theorem III.2], Satake's decomposition theorem applies because the homomorphism $\mathfrak{u} \to \mathfrak{h}_n$ extends to the homomorphism

$$\beta : \mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l} \to \mathfrak{u} \rtimes \mathfrak{sp}(n,\mathbb{R}), \qquad (X,Y) \mapsto (\beta(X), \mathrm{ad}\, Y|_{\mathfrak{u}/\mathfrak{a}}).$$

Note that to apply Satake's decomposition ([**21**, Theorem III.2]) one does not need that the representation under considertion is irreducible.

So we obtain a tensor product decomposition $T = T_1 \otimes T_2$, where T_1 is the metaplectic representation obtained by the homomorphism $G \to H_n \rtimes \mathrm{Sp}(n,\mathbb{R})^\sim$ (cf. [**21**, Proposition III.1]) and T_2 is a representation with $U \subseteq \ker T_2$. It remains to show that $T_2 = T^L_{f_l}$, where

$f_l := f|_l$. Since $\mathfrak{g}_f = \mathfrak{t}$, the regularity of f_l follows from $\mathfrak{g}_f = \mathfrak{u}_{f_u} \rtimes \mathfrak{l}_{f_l}$ (cf. [**15**, Lemma I.17]). Now Remark 4.10 applies and completes the proof. □

THEOREM 4.12. *A unitary highest weight representtaions π_λ with kernel in K can be obtained by Duflo's orbit method as a representation T_f, $f \in \mathfrak{t}^*$ regular, if and only if it is square integrable.*

Proof. Suppose that the representation with highest weight $\lambda \in \operatorname{int} C^\star_{\min}$ can be realized by Duflo's method, i.e., $\pi_\lambda = T^G_f$ for some regular $f \in \mathfrak{t}^*$. Then $\lambda|_\mathfrak{z} = if|_\mathfrak{z}$ since the infinitesimal character of T^G_f is χ_{if} ([**4**, Theorem IV.19]). Now Satake's theorem provides a decomposition $\pi_\lambda = \pi_{\lambda_1} \otimes \pi_{\lambda_2} = T^G_f = T^L_{f_l} \otimes T'$ (cf. Remark 4.10). Thus $T^L_{f_l} \cong \pi_{\lambda_1}$ is an irreducible highest weight representation of L and since $\ker \pi_\lambda \subseteq K$, the representation π_{λ_1} is square integrable. Hence it belongs to the relative discrete series (Remark 4.7). Now $\lambda_1 = if_l - \rho_l$ satisfies the Harish-Chandra condition (Theorem 3.10) and therefore the highest weight representation π_λ is square integrable (Theorem 3.15).

If, conversely, π_λ is square integrable, then we put $f := -i(\lambda + \rho)$ and note that $f \in \operatorname{int} C^\star_{\min}$ by Theorem 3.15. We show that f satisfies the requirements of Theorem 4.11. From $f \in \operatorname{int} C^\star_{\min}$ we conclude with [**19**, Proposition III.14] that $\mathfrak{g}_f \subseteq \mathfrak{k}$. For $\alpha \in \Delta^+_k$ we have

$$if(\check{\alpha}) = (\lambda + \rho)(\check{\alpha}) = (\lambda + \rho_k)(\check{\alpha}) > \lambda(\check{\alpha}) \geq 0$$

so that we finally see that $\mathfrak{g}_f = \mathfrak{t}$, i.e., f is regular. The integrality of f which is equivalent to the integrality of λ (cf. Lemma 4.5) is also satisfied. Therefore Theorem 4.11 implies that $T^G_f \cong \pi_\lambda$. □

5. Holomorphic extensions. In this section G denotes a simply connected Lie group having square integrable highest weight representations π_λ.

Let $\eta : G \to G_\mathbb{C}$ denote the universal complexification of G. For the following facts we refer to [**17**]. For a closed convex generating invariant cone $W \subseteq \mathfrak{g}$ we write $\Gamma(\mathfrak{g}, W)$ for the semigroup covering of the subsemigroup $\langle \exp(\mathfrak{g} + iW) \rangle \subseteq G_\mathbb{C}$. Such semigroups are called *Ol'shanskiĭ semigroups*. One has a natural inclusion $G \to \Gamma(\mathfrak{g}, W)$, an exponential function $\operatorname{Exp} : \mathfrak{g} + iW \to \Gamma(\mathfrak{g}, W)$, and if, in addition, $i(W \cap \mathfrak{t}_l) \subseteq (\Delta^+_{p,s})^\star$, then $\Gamma(\mathfrak{g}, W) = G \operatorname{Exp}(iW)$, where the map

$$G \times W \to \Gamma(\mathfrak{g}, W), \qquad (g, X) \mapsto g \operatorname{Exp}(iX)$$

is a homeomorphism (cf. [**17**]).

Let $\lambda \in i \operatorname{int} C^\star_{\min}$ and $\pi := \pi_\lambda$ a corresponding unitary highest weight representation. We have shown in [**21**, Corollary IV.12] that for any generating invariant cone $W \subseteq \mathfrak{g}$ where the ideal $W \cap (-W)$ is compact and $W \cap \mathfrak{t} \subseteq C_{\max}$, the representation π extends to a *holomorphic representation* π of the semigroup $S := \Gamma(\mathfrak{g}, W)$, i.e., $\pi : S \to B(\mathcal{H})$ is weakly continuous, $\pi_\lambda : S^0 := G \operatorname{Exp}(i \operatorname{int} W) \to B(\mathcal{H})$ is holomorphic, and $\pi(s^*) = \pi(s)^*$, where

$$(g \operatorname{Exp}(iX))^* = \operatorname{Exp}(iX) g^{-1}.$$

Moreover, in this case $\pi(S^0) \subseteq B_1(\mathcal{H})$ and the character $s \mapsto \Theta_\pi := \operatorname{tr} \pi(s)$ is a holomorphic function on S^0.

Thus we have a mapping

$$F : B_2(\mathscr{H}) \to \mathrm{Hol}(S^0), \qquad A \mapsto f_A, \quad \text{with} \quad f_A(s) = \mathrm{tr}(\pi(s)A).$$

If π^c denotes the representation of $S \times S$ on $B_2(\mathscr{H})$ given by

$$\pi^c(s_1, s_2)(A) := \pi(s_1)A\pi(s_2)^*,$$

then $f_{\pi^c(s_1,s_2).A}(s) = f_A(s_2^* s s_1)$, i.e., F intertwines the natural representation of $S \times S$ on the space $\mathrm{Hol}(S^0)$ of holomorphic functions on S^0 with the representation π^c on $B_2(\mathscr{H})$.

Definition 5.1. The orbit $\mathscr{M} := G.P_v \subseteq B_2(\mathscr{H})$ is called *effective* if the mapping $B_2(\mathscr{H}) \to C(M)$, $T \mapsto \sigma_T$ is injective. Note that this is equivalent to saying that span $\mathscr{M} \subseteq B_2(\mathscr{H})$ is dense, i.e., that the vector P_v is cyclic for the representation of the group G on $B_2(\mathscr{H})$ defined by $g.A := \pi(g)A\pi(g)^{-1}$. Since this representation is not irreducible, there is no a priori evidence for $\mathscr{M}$ to be effective.

Next we suppose in addition that the highest weight representation $\pi : G \to U(\mathscr{H})$ is square integrable. Let further $M = G/T$ be as in Sections 1 and 3, where $v = v_\lambda$ is a normalized highest weight vector. Then we have for $A \in B(\mathscr{H})$ and $s \in S^0$ the formula

$$f_A(s) = \mathrm{tr}(\pi(s)A) = d(\pi) \int_M \sigma_{\pi(s)A}(m)\, d\mu_M(m). \tag{5.1}$$

As a consequence of (5.1), we have for $B = \pi(s)A \in \pi(S^0)B(\mathscr{H})$ the formula

$$f_B(\mathbf{1}) = f_A(s) = d(\pi) \int_M \sigma_{\pi(s)A}(m)\, d\mu_M(m) = d(\pi) \int_M \sigma_B(m)\, d\mu_M(m).$$

A particular case of (5.1) is the character formula

$$\Theta_\pi(s) = d(\pi) \int_M \sigma_{\pi(s)}(m)\, d\mu_M(m) \tag{5.2}$$

which follows immediately from (5.1) with $A = \mathbf{1}$.

Theorem 5.2. *The character formula*

$$\Theta_\pi(s) = d(\pi) \int_M \sigma_{\pi(s)}(m)\, d\mu_M(m)$$

is valid for all $s \in S^0$ and a positive constant $d(\pi)$ if and only if the highest weight representation π is square integrable modulo the center.

Proof. If π is square integrable modulo the center, then we have seen above that formula (5.2) holds. Suppose conversely that π is a highest weight representation such that (5.2) is valid.

The fact that π is irreducible yields that $\pi(S^0) \subseteq B_1(\mathscr{H})$ so that the left hand side is well defined. Pick $s \in S^0$. Then $\pi(ss^*) = \pi(s)\pi(s^*)$ is a symmetric positive operator of trace class. Thus we find a sequence $(v_n)_{n\in\mathbb{N}}$ of mutually orthogonal vectors in $\mathscr{H}$ such that $\pi(ss^*) = \sum_{n=1}^{\infty} P_{v_n}$. Then

$$\begin{aligned}\Theta_\pi(ss^*) &= d(\pi)\int_M \sigma_{\pi(ss^*)}(m)\,d\mu_M(m)\\ &\geq d(\pi)\int_M \sigma_{P_{v_n}}(m)\,d\mu_M(m)\\ &= d(\pi)\int_{G/T} \langle g\,.\,v, v_n\rangle\langle v_n, g\,.\,v\rangle\,d\mu_{G/T}(gT)\\ &= d(\pi)\int_{G/T} |\langle g\,.\,v, v_n\rangle|^2\,d\mu_{G/T}(gT).\end{aligned}$$

Since the existence of the integral on the right hand side is equivalent to the square integrability of π, the proof is complete. □

For the following lemma we recall the definition of the *projective space*

$$\mathbb{P}(\mathscr{H}) = \{\mathbb{C}v : v \in \mathscr{H}\backslash\{0\}\}$$

of $\mathscr{H}$. We write $[v] := \mathbb{C}v$ for the elements of $\mathbb{P}(\mathscr{H})$. We note that for any injective operator $A \in B(\mathscr{H})$ the mapping $\mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H})$, $[v] \mapsto A\,.\,[v] := [A\,.\,v]$ is well defined. Therefore the semigroup $B_i(\mathscr{H})$ of injective bounded operators on $\mathscr{H}$ acts on $\mathbb{P}(\mathscr{H})$ by $A\,.\,[v] := [A\,.\,v]$.

LEMMA 5.3. *The following assertions hold for any unitary highest weight representation* π_λ *with* $\lambda \in i\,\mathrm{int}\,C_{\min}^\star$.

(i) $\pi(S) \subseteq B_i(\mathscr{H})$.

(ii) *The mapping* $S^0 \times \mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H})$, $(s, [v]) \mapsto s\,.\,[v] := [s\,.\,v]$ *defines a holomorphic action of* S^0 *on* $\mathbb{P}(\mathscr{H})$.

(iii) *If* v *is a primitive element, then* $S\,.\,[v] = G\,.\,[v]$.

Proof. (i) Since we can write $s = g\,\mathrm{Exp}(iX)$ with $X \in W$, we may w.l.o.g. assume that $s = \mathrm{Exp}(iX)$. Then $\pi(s) = e^{id\pi(X)}$ and such an operator is injective by the spectral theory of selfadjoint operators.

(ii) Let $w \in \mathscr{H}$ be a unit vector and $V_w := \{[v] \in \mathbb{P}(\mathscr{H}) : \langle v, w\rangle \neq 0\}$. Then the mapping

$$\varphi_w : w^\perp \to V_w, \qquad x \mapsto [w + x]$$

is a holomorphic chart for the open subset V_w of $\mathbb{P}(\mathscr{H})$. Its inverse is given by the map

$$\psi_w : [v] \mapsto \frac{v}{\langle v, w\rangle} - w.$$

Now pick $s_0 \in S^0$ and $v_0 \in \mathscr{H}$ with $\langle \pi(s_0)\,.\,v_0, w\rangle \neq 0$. Then there exists a neighborhood U of s_0 in S^0 and a neighborhood V of v_0 in $\mathscr{H}$ such that $\langle \pi(s)\,.\,v, w\rangle \neq 0$ holds for all $s \in U$, $v \in V$ (recall that π is norm-continuous on S^0). Thus

$$(s, v) \mapsto \psi_v([\pi(s)\,.\,v]) = \frac{\pi(s)\,.\,v}{\langle \pi(s)\,.\,v, w\rangle} - w$$

is holomorphic on $U \times V$. This proves the assertion.

https://doi.org/10.1017/S0017089500032237 Published online by Cambridge University Press

(iii) Since the representation π is irreducible, the image $\pi(Z)$ of the center Z of G is contained in $\mathbb{C}\mathbf{1}$ so that it acts trivially on the projective space $\mathbb{P}(\mathscr{H})$. Hence the action of $S=\Gamma(\mathfrak{g}, W)$ on $\mathbb{P}(\mathscr{H})$ factors to a representation of the subsemigroup $S_1 := \eta(G)\exp(iW) \subseteq G_{\mathbb{C}}$. Let $B := \langle \exp \mathfrak{b} \rangle \subseteq G_{\mathbb{C}}$, where $\mathfrak{b} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Then it follows from [**10**, Theorem II.8], that $S \subseteq \eta(G)\exp(iW_{\max}) \subseteq \eta(G)B$, where the subset $\eta(G)B \subseteq G_{\mathbb{C}}$ is open.

Let $\chi : B \to \mathbb{C}^*$ denote the holomorphic character with $d_\chi(\mathbf{1}) = \lambda$. Then the mapping

$$\eta(G)B \to \mathbb{P}(\mathscr{H}), \qquad \eta(g)b \mapsto [\chi(b)\pi(g) . v] = [g . v]$$

is a holomorphic extension of the orbit map $\eta(G) \to \mathbb{P}(\mathscr{H}), g \mapsto g . [v]$. Since holomorphic extensions are unique (cf. [**9**, Lemma 9.17]), it follows from $S \subseteq \eta(G)B$ that $S . [v] \subseteq G . [v]$. □

Now we come to the main result of this section.

THEOREM 5.4. *If v is a primitive element, then the orbit $\mathcal{M} = G . P_v \subseteq B_2(\mathscr{H})$ is effective.*

Proof. Let $A \in B(\mathscr{H})$ and suppose that $\sigma_A = 0$.
For $s \in S$ we calculate

$$\begin{aligned} f_A(ss^*) &= \operatorname{tr}(\pi(s)\pi(s^*)A) = \operatorname{tr}(\pi(s^*)A\pi(s)) \\ &= d(\pi) \int_{G/T} \langle \pi(s^*)A\pi(sg) . v, \pi(g) . v \rangle \, d\mu_{G/T}(gT) \\ &= d(\pi) \int_{G/T} \langle A\pi(sg) . v, \pi(sg) . v \rangle \, d\mu_{G/T}(gT). \end{aligned}$$

In view of Lemma 5.3(iii), we find for each $g \in G$ another $g_s \in G$ with $s . [g . v] = [g_s . v]$. Hence $sg . v = zg_s . v$ for $a \in \mathbb{C}^*$. We conclude that

$$\langle A\pi(sg) . v, \pi(sg) . v \rangle = |z|^2 \langle A\pi(g_s) . v, \pi(g_s) . v \rangle = |z|^2 \sigma_A(g_s^{-1}T) = 0.$$

Hence $f_A(ss^*) = 0$ for all $s \in S^0$. Since $S = G\,\mathrm{Exp}(iW)$, this means in particular that $f_A(\mathrm{Exp}(iX)) = 0$ for all $X \in W$. Hence $f_A = 0$ follows from the fact that f_A is a holomorphic function (cf. [**9**, Lemma 9.17]). □

REFERENCES

1. S. T. Ali and J.-P. Antoine, Quantization and Dequantization, in *Quantization and infinite dimensional systems* (Eds. J.-P. Antoine et. al.), Plenum Press, New York, London, 1994.

2. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6 (Masson, Paris, 1981).

3. M. Duflo, *Construction de representations unitaires d'un groupe de Lie*, Cours d'été de CIME, Cortona 1980.

4. M. Duflo, Théorie de Mackey pour les groupes algébriques, *Acta Math.* **149** (1982), 153–213.

5. M. Duflo, On the Plancherel formula for almost algebraic real Lie groups, *Lect. Notes in Math.* **1077** (1984), Springer, 101–165.

6. A. Guichardet, Théorie de Mackey et méthode des orbites selon M. Duflo, *Expositio Math.* **3** (1985), 303–346.
7. Harish-Chandra, Representations of semi-simple Lie groups, V, VI, *Amer. J. Math.* **78** (1956), 1–41, 564–628.
8. J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups* (Oxford University Press, 1989).
9. J. Hilgert and K.-H. Neeb, Lie semigroups and their applications, *Lecture Notes in Math.* **1552** Springer, 1993.
10. J. Hilgert and K.-H. Neeb, Compression semigroups of open orbits on complex manifolds, *Arkiv för Math.* **33** (1995), 293–322.
11. K.-H. Neeb, Globality in Semisimple Lie Groups, *Annales de l'Institut Fourier* **40** (1990), 493–536.
12. K.-H. Neeb, Invariant subsemigroups of Lie groups, *Memoirs of the Amer. Math. Soc.* **499** (1993).
13. K.-H. Neeb, Realization of general unitary highest weight representations, *Preprint, Technische Hochschule Darmstadt* **1662** (1994).
14. K.-H. Neeb, Holomorphic representations and coherent states, in *Quantization and infinite dimensional systems* (Eds. J.-P. Antoine et.al.), Plenum Press, New York, London, 1994.
15. K.-H. Neeb, On closedness and simple connectedness of adjoint and coadjoint orbits, *Manuscripta Math.* **82** (1994), 51–65.
16. K.-H. Neeb, Holomorphic representation theory II, *Acta math.* **173** (1994), 103–133.
17. K.-H. Neeb, Holomorphic representation theory I, *Math. Ann.* **301** (1995), 155–181.
18. K.-H. Neeb, On the convexity of the moment mapping for unitary highest weight representations, *J. Funct. Anal.* **127** (1995), 301–325.
19. K.-H. Neeb, Kähler structures and convexity properties of coadjoint orbits, *Forum Math.* **7** (1995), 349–384.
20. K.-H. Neeb, A Duistermaat-Heckman formula for admissible coadjoint orbits, *Proceedings of "Workshop on Lie Theory and its Applications in Physics", Clausthal, August, 1995* (Ed. Doebner, Dobrev), to appear.
21. K.-H. Neeb, Coherent states, holomorphic extensions, and highest weight representations, *Pacific J. Math.* **174** (1996), 497–542.
22. A. M. Perelomov, *Generalized coherent states and their applications* (Springer, Berlin, 1986).
23. D. Vogan, The algebraic structure of the representations of semisimple Lie groups, *Annals of Math.* **109** (1979), 1–60.
24. N. R. Wallach, *Real reductive groups I* (Academic Press Inc., Boston, New York, Tokyo, 1988).
25. G. Warner, *Harmonic analysis on semisimple Lie groups I* (Springer, Berlin, Heidelberg, New York, 1972).
26. N. J. Wildberger, On the Fourier transform of a compact semisimple Lie group, *J. Austral. Math. Soc.*, Ser. A **56** (1994), 64–116.
27. J. Wolf, Unitary representations on partially holomorphic cohomology spaces, *Mem. of the Amer. Math. Soc.* **138** (1974).
28. J. Wolf, Classification and Fourier Inversion for parabolic subgroups with square integrable nilradical, *Mem. of the Amer. Math. Soc.* **225** (1979).

MATHEMATISCHES INSTITUT
UNIVERSITÄT ERLANGEN-NÜRNBERG
BISMARCKSTRASSE $1\frac{1}{2}$
D-90154 ERLANGEN
GERMANY

https://doi.org/10.1017/S0017089500032237 Published online by Cambridge University Press