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COMMENSURABILITY AND ELEMENTARY EQUIVALENCE OF POLYCYCLIC GROUPS

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It is shown that two polycyclic-by-finite groups G and H, satisfying the same sentences with one alternation of quantifiers, are commensurable. In fact we show something stronger: given $n > 1$ there is a subgroup H_n of H and a subgroup G_n of G such that $H_n \simeq G$, $G_n \simeq H$ and the indices $[G : G_n]$ and $[H : H_n]$ are finite and prime to n.

1. INTRODUCTION

This work is motivated by a problem proposed by G. Sabbagh: to find an algebraic characterisation of elementary equivalence in the class of polycyclic-by-finite groups. F.Oger solved this problem for finitely generated finite-by-nilpotent groups; in [7], he proves that two finitely generated finite-by-nilpotent groups G and H satisfy the same sentences if and only if $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$. This result can not be generalised to polycyclic-by-finite groups: there exist polycyclic-by-finite groups G and H such that $G \equiv H$ and $G \times \mathbb{Z} \not\simeq H \times \mathbb{Z}$. Here we give a necessary condition for any two polycyclic-by-finite groups to satisfy the same sentences with one alternation of quantifiers.

THEOREM. *Let G and H be polycyclic-by-finite groups satisfying the same sentences with one alternation of quantifiers. Given an integer $r > 1$, there is a subgroup H_r of H and a subgroup G_r of G such that $H_r \simeq G$, $G_r \simeq H$ and the indices $[G : G_r]$ and $[H : H_r]$ are finite and prime to r.*

This Theorem is a generalisation of [5], where Oger proves the same result in the case G and H are finitely generated finite-by-nilpotent groups. The techniques used in [5] do not work for polycyclic-by-finite groups and we do not suppose this result known in the proof of the Theorem.

The aim of Sections 2, 3 and 4 is the proof of the Theorem. In Section 2, we prove a few lemmas concerning subgroups defined by formulas. Section 3 deals only with polycyclic-by-finite groups: we give a sufficient condition for a subgroup to have finite index prime to a given integer and we show how to arrive at this condition applying

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the lemmas of Section 2. All we need in the proof of the Theorem is given in this section; the proof itself is given in Section 4. In Section 5 we discuss the connections between some equivalence relations in the class of polycyclic-by-finite groups (the two equivalence relations appearing in the Theorem, commensurability, $\equiv$, "having the same finite images"); in particular, we give an example showing that the converse of the Theorem is not true.

[**12**] is our basic reference for polycyclic-by-finite groups. If G is a group, $H < G$ means that H is a subgroup of G; if $n \geqslant 1$ is an integer, G^n is the subgroup $\langle g^n \mid g \in G \rangle$; G' is the derived subgroup $[G,G]$; $\zeta(G)$ denotes the centre of G; $\mathcal{F}(G)$ is the set of isomorphism classes of the finite images of G; if G is polycyclic-by-finite, $h(G)$ is the Hirsch number of G.

The definitions of formula (existential, universal, quantifier free), sentence and language can be found in [**2**]. The formulas considered here are in the first-order language of groups, $L = \{\cdot, *^{-1}, 1\}$, where $\cdot$ is a binary function for the group operation, $*^{-1}$ is an unary fuction for the inverse and 1 is a constant for the identity element.

2. Definability

A formula ϕ with one alternation of quantifiers can be written either in the form $(\exists X_1)\dots(\exists X_n)(\forall\, Y_1)\dots(\forall\, Y_m)\theta$, or in the form $(\forall\, Y_1)\dots(\forall\, Y_m)(\exists X_1)\dots(\exists X_n)\theta$, where θ is a formula without quantifiers. The first one is called an $\exists\,\forall$ formula and the last one a $\forall\exists$ formula. Given a formula $\phi(X_1, \dots, X_n)$ with n free variables and a group G with $g_1, \dots, g_n$ in G, we say that $g_1, \dots, g_n$ satisfy ϕ in G, and we write $G \vDash \phi(g_1, \dots, g_n)$, if $\phi(g_1, \dots, g_n)$ is true in G. Given groups G and H we write $G \equiv H$ if G and H are elementary equivalent; we write $G \equiv_1 H$ if G and H satisfy the same sentences with one alternation of quantifiers.

Definition: Let ϕ be a formula with one free variable and let G be a group. We denote by G_ϕ the set $\{g \in G \mid G \vDash \phi(g)\}$. Clearly, G_ϕ is a subgroup of G if and only if $\phi(1)$ and $(\forall X)(\forall Y)\big((\phi(X) \wedge \phi(Y)) \to \phi(XY^{-1})\big)$ are true in G. In this case, we say that ϕ defines the subgroup G_ϕ in G.

From now on we shall often make use of formulas with one free variable defining subgroups in a given group G. When it is clear from the context that the formula must have one free variable we do not state this hypothesis explicitly.

Lemma 2.1. *Let ϕ and ψ be formulas. The following four statements hold.*

(i) *For any group G, if G_ϕ is a subgroup of G then G_ϕ is a characteristic subgroup of G (in particular, $G_\phi \lhd G$).*

(ii) *For any group G such that G_ϕ and G_ψ are subgroups of G, we have $G_{(\phi \wedge \psi)} = G_\phi \cap G_\psi$.*

(iii) *There is a formula* γ *such that* $G_\gamma = G_\phi G_\psi$, *for any group* G *in which* G_ϕ *and* G_ψ *are subgroups. If* ϕ *and* ψ *are existential, so is* γ.

(iv) *If* ϕ *and* ψ *are existential, then there is an existential formula* λ *such that* $G_\lambda = (G_\phi)_\psi$, *for any group* G *with* G_ϕ *subgroup of* G *and* $(G_\phi)_\psi$ *subgroup of* G_ϕ.

PROOF: In order to prove (i), it suffices to observe that formulas are preserved by automorphisms (and in particular by the inner automorphisms). The proof of (ii) is straightforward. For (iii), set $\gamma(Z)$ equal to $(\exists X)(\exists Y)(\phi(X) \wedge \psi(Y) \wedge (Z = XY))$.

To prove (iv), we write ψ in the form $(\exists Y_1)\ldots(\exists Y_n)\psi'(Y_1, \ldots, Y_n, Y)$, where ψ' is a formula without quantifiers. The statement follows easily if we set $\lambda(Z)$ equal to

$$(\exists Z_1)\ldots(\exists Z_n)\Big(\phi(Z) \wedge \bigwedge_{i=1}^{n} \phi(Z_i) \wedge \psi'(Z_1, \ldots, Z_n, Z)\Big). \qquad \square$$

The fact that subgroups defined by formulas are normal (Lemma 2.1 (i)) will often be used without any further comment.

LEMMA 2.2. *Let* ϕ *and* ψ *be existential formulas and let* G *and* H *be groups such that* $G \equiv_1 H$. *The following statements hold:*

(i) $G_\phi \lhd G$ *if and only if* $H_\phi \lhd H$.

(ii) $G_\psi \subset G_\phi$ *if and only if* $H_\psi \subset H_\phi$.

PROOF: Since ϕ is existential, $\phi(1)$ and $(\forall X)(\forall Y)((\phi(X) \wedge \phi(Y)) \to \phi(XY^{-1}))$ are $\forall\exists$ formulas. This proves (i). For any group K, the sentence $(\forall X)(\psi(X) \to \phi(X))$ is satisfied in K if an only if $K_\psi \subset K_\phi$. Statement (ii) follows from the fact that this sentence is $\forall\exists$ if ψ and ϕ are existential. $\square$

In Chapter 3, we deal with several situations in which we have to describe finite groups by formulas. In order not to repeat slighty variations of the same argument, we state Lemma 2.3 bellow.

LEMMA 2.3. *Let* ϕ *and* ψ *be formulas and let* G *be a group such that* G_ψ *and* G_ϕ *are both subgroups of* G *with* $G_\psi \subset G_\phi$ *and* $|G_\phi / G_\psi| = n$ *finite. Then, there is a formula* $\beta(X_1, \ldots, X_n)$ *satisfying the following two conditions.*

(i) *For every group* H *with subgroups* H_ϕ *and* H_ψ *such that* $H_\psi \subset H_\phi$, *we have* $H \models \beta(b_1, \ldots, b_n)$ *if and only if* $\{b_1 H_\psi, \ldots, b_n H_\psi\}$ *is a subgroup of* H_ϕ / H_ψ *isomorphic to* G_ϕ / G_ψ.

(ii) *If* ϕ *and* ψ *are existential then* β *is* $\exists\forall$; *if* ϕ *and* ψ *are quantifier free formulas then* β *is quantifier free.*

Moreover, if ϕ *and* ψ *are existential and* H *is a group with* $G \equiv_1 H$ *we have also:*

(iii) $G_\phi / G_\psi \simeq H_\phi / H_\psi$;

(iv) $H \models \beta(b_1, \ldots, b_n)$ *if and only if* $\{b_1 H_\psi, \ldots, b_n H_\psi\} = H_\phi / H_\psi$.

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PROOF: Set $G_\phi/G_\psi = \{a_1G_\psi, \dots, a_nG_\psi\}$ and let $\sigma : \{1,\dots,n\}\times\{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ be such that $a_ia_ja^{-1}_{\sigma(i,j)} \in G_\psi$ for $1 \leqslant i,j \leqslant n$. Denote by S_n the permutation group of $\{1,\dots,n\}$ and define $\beta(X_1, \dots, X_n)$ as:

$$\overbrace{\Big(\bigwedge_{1\leqslant i\leqslant n}\phi(X_i)\Big)\wedge\Big(\bigwedge_{1\leqslant i<j\leqslant n}\neg\,\psi(X_iX_j^{-1})\Big)}^{\beta_1(X_1,\dots,X_n)}\wedge\overbrace{\Big(\bigvee_{\pi\in S_n}\bigwedge_{1\leqslant i,j\leqslant n}\psi(X_{\pi(i)}X_{\pi(j)}X^{-1}_{\pi\circ\sigma(i,j)})\Big)}^{\beta_2(X_1,\dots,X_n)}$$

Proof of (i). Suppose H is a group in which $H_\phi \subset H_\psi$ are both subgroups of H and let $b_1, \dots, b_n$ be in H. By interpretating $\beta_1(b_1, \dots, b_n)$ and $\beta_2(b_1, \dots, b_n)$ in H, we conclude that $H \vDash \beta(b_1, \dots, b_n)$ if and only if the two following conditions hold:

(a) $\{b_1H_\psi, \dots, b_nH_\psi\}$ is a subset of H_ϕ/H_ψ which has n elements.

(b) There is $\pi \in S_n$ such that $b_ib_jH_\psi$ is equal to $b_{\pi\circ\sigma(i,j)}H_\psi$, for $1 \leqslant i,j \leqslant n$.

So, if $H \vDash \beta(b_1, \dots, b_n)$, by (a) we can define the function $f_\pi : G_\phi/G_\psi \to H_\phi/H_\psi$ given by the equalities $f_\pi(a_iG_\psi) = b_{\pi(i)}H_\psi$, for $1 \leqslant i \leqslant n$; moreover, this function is injective and $\Im(f)$ is $\{b_1H_\psi, \dots, b_nH_\psi\}$. By (b) we know that $f_\pi \in \operatorname{Hom}(G_\phi/G_\psi, H_\phi/H_\psi)$. Therefore, $\{b_1H_\psi, \dots, b_nH_\psi\}$ is a subgroup of H_ϕ/H_ψ isomorphic to G_ϕ/G_ψ.

Conversely, suppose $\{b_1H_\psi, \dots, b_nH_\psi\}$ is a subgroup of H_ϕ/H_ψ isomorphic to G_ϕ/G_ψ. It is clear that (a) holds. So, we only have to prove that (b) holds. Let $f : G_\phi/G_\psi \hookrightarrow H_\phi/H_\psi$ be a monomorphism whose image is $\{b_1H_\psi, \dots, b_nH_\psi\}$ and take $\pi \in S_n$ such that $f(a_iG_\psi) = b_{\pi(i)}H_\psi$ for $1 \leqslant i \leqslant n$. Since $a_ia_jG_\psi$ is equal to $a_{\sigma(i,j)}G_\psi$, f sends $a_ia_jG_\psi$ to $b_{\pi\circ\sigma(i,j)}H_\psi$. As f is a group homomorphism we conclude that $b_ib_jH_\psi$ is equal to $b_{\pi\circ\sigma(i,j)}H_\psi$ for $1 \leqslant i,j \leqslant n$. Then (b) is true and (i) is proved.

Proof of (ii). This is a direct consequence of the definition of β.

Proof of (iii). Suppose that $G \equiv_1 H$. Recall that, by Lemma 2.2, $H_\psi \subset H_\phi$ and they are both normal subgroups of H. From (i) and (ii) we know that G satisfies the $\exists\,\forall$ sentence $(\exists X_1)\dots(\exists X_n)\beta$. Then, there are $b_1, \dots, b_n$ such that $H \vDash \beta(b_1, \dots, b_n)$. Again by (i), there is a group monomorphism from G_ϕ/G_ψ to H_ϕ/H_ψ. So, to prove that these two finite quotient groups are isomorphic, it is enough to show that $|H_\phi/H_\psi| \leqslant n$. Consider the sentence θ given by:

$$(\forall X_1)\dots(\forall X_{n+1})\Big(\bigwedge_{i=1}^{n+1}\phi(X_i)\longrightarrow\Big(\bigvee_{1\leqslant i<j\leqslant n+1}\psi(X_iX_j^{-1})\Big)\Big).$$

When interpreted in a group K such that $K_\psi \subset K_\phi$ are both subgroups of K, θ says "K_ϕ/K_ψ has $\leqslant n$ elements". The sentence θ is $\forall\exists$ and $G \vDash \theta$. Therefore, if $H \equiv_1 G$ then H satisfies θ, and so, $|H_\phi/H_\psi| \leqslant n$.

Proof of (iv). Since $|G_\phi/G_\psi| = n$, (iv) is a direct consequence of (i) and (iii). $\square$

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COROLLARY 2.4. *Let L be a finite group and suppose $|L| = n$. Then, there exists a quantifier free formula $\beta_L(X_1, \ldots, X_n)$ and an universal formula $\rho_L(X_1, \ldots, X_n)$ such that, for every group H, the following statements hold:*

(1) *$H \models \beta_L(h_1, \ldots, h_n)$ if and only if $\{h_1, \ldots, b_n\}$ is a subgroup of H isomorphic to L.*

(2) *$H \models \rho_L(h_1, \ldots, h_n)$ if and only if $\{h_1, \ldots, b_n\}$ is a normal subgroup of H isomorphic to L.*

PROOF: (i) In Lemma 2.3, take $X = X$ and $X = 1$ as being $\phi(X)$ and $\psi(X)$ respectively, and take G equals to L. Now, for every group H, $H_\phi = H$ and $H_\psi = 1$. By Lemma 2.3-(iii), the formula $\beta(X_1, \ldots, X_n)$ is quantifier-free. Taking $\beta_L(X_1, \ldots, X_n)$ equal to the this formula, Lemma 2.3-(ii) proves (i).

(ii) Take $\rho_L(\overline{X})$ as $(\forall X)\Big(\Big(\bigwedge\limits_{1 \leqslant i \leqslant n} \bigvee\limits_{1 \leqslant j \leqslant n} XX_iX^{-1} = X_j\Big) \wedge \beta_L(\overline{X})\Big)$, where $\overline{X}$ is $(X_1, \ldots, X_n)$. We have added to the formula β_L a formula saying "the conjugate of X_i by any element is among $X_1, \ldots, X_n$". Now (ii) follows from (i). ∎

COROLLARY 2.5. *Let ϕ and ψ be existential formulas and let G be a group such that G_ψ and G_ϕ are both subgroups of G. Suppose that $G_\psi \subset G_\phi$ and that $|G_\phi/G_\psi|$ is finite. Let $y_1, \ldots, y_k$ be elements of G such that $G_\phi \subset \langle y_1, \ldots, y_k \rangle G_\psi$. Then, there exists an $\exists\forall$ formula $\gamma(Y_1, \ldots, Y_k)$ satisfying the following two conditions:*

(i) *$G \models \gamma(y_1, \ldots, y_k)$;*

(ii) *$H \models \gamma(z_1, \ldots, y_k)$ implies $H_\phi \subset \langle z_1, \ldots, z_k \rangle H_\psi$, for every group H with $G \equiv_1 H$.*

PROOF: Put $n = |G_\phi/G_\psi|$ and consider the $\exists\forall$ formula $\beta(X_1, \ldots, X_n)$ given in Lemma 2.3. Item (iv) of the same lemma implies that, for every group H with $G \equiv_1 H$, we have: $\beta(b_1, \ldots, b_n)$ is true in H if and only $\{b_1 H_\psi, \ldots, b_n H_\psi\}$ is equal to H_ϕ/H_ψ.

Let $a_1, \ldots, a_n$ in G be such that $G_\phi/G_\psi = \{a_1 G_\psi, \ldots, a_n G_\psi\}$, and set $\overline{y} = (y_1, \ldots, y_k)$ and $\overline{Y} = (Y_1, \ldots, Y_k)$. Since $G_\phi \subset \langle y_1, \ldots, y_k \rangle G_\psi$, there are n words, $u_1(\overline{Y}), \ldots, u_n(\overline{Y})$, such that $u_1(\overline{y})G_\psi = a_1 G_\psi, \ldots, u_n(\overline{y})G_\psi = a_n G_\psi$. Define $\gamma(\overline{Y})$ as being $\beta(u_1(\overline{Y}), \ldots, u_n(\overline{Y}))$. This formula is $\exists\forall$ because β is $\exists\forall$. As $\beta(a_1, \ldots, z_n)$ is true in G, we know that $G \models \gamma(\overline{y})$. This proves (i).

Now, let H be a group for which $G \equiv_1 H$ and let $\overline{z} = (z_1, \ldots, z_k)$ be such that $H \models \gamma(\overline{z})$. Then, $H \models \beta(u_1(\overline{z}), \ldots, u_n(\overline{z}))$ and we have that H_ϕ/H_ψ is equal to $\{u_1(\overline{z})H_\psi, \ldots, u_n(\overline{z})H_\psi\}$. Consequently, H_ϕ is contained in $\langle z_1, \ldots, z_k \rangle H_\psi$. ∎

3. POLYCYCLIC-BY-FINITE GROUPS

We begin this section with a result on polycyclic-by-finite groups that gives a sufficient condition for a subgroup to have finite index prime to a given integer.

Proposition 3.1. *Consider a polycyclic-by-finite group G, an integer $c \geqslant 1$ and a chain of normal subgroups of G, $1 = G_{c+1} \lhd G_c \lhd \cdots \lhd G_1 \lhd G$, such that G/G_1 is finite and G_i/G_{i+1} is Abelian for $1 \leqslant i \leqslant c$. Suppose K is a subgroup of G and $r \geqslant 1$ is an integer. If $G = KG_1$ and $G_i \subset KG_i^r$, for $1 \leqslant i \leqslant c$, then $[G : K]$ is finite and prime to r.*

Proof: We first treat the Abelian case.

(i) *Given finitely generated Abelian groups $B < A$ and an integer $r > 1$, if $A = BA^r$, then $[A : B]$ is finite and prime to r.*
Given a finitely generated Abelian group H, write H as the direct sum of a free Abelian group and a finite group. Now it is easy to show that, if $H^r = H$, then $|H|$ is finite and prime to r. Setting H equal to A/B, (i) follows.

We prove the proposition by induction on c. If $c = 1$, then G_1 is Abelian and $G_1 = KG_1^r \cap G_1 = (K \cap G_1)G_1^r$; from (i), it follows that $[G_1 : K \cap G_1]$ is finite and prime to r. Since $G = KG_1$ and G_1 is normal in G, then $[G : K]$ is $[G_1 : K \cap G_1]$. Now suppose $c > 1$. Set $\overline{K} = KG_c/G_c$, $\overline{G} = G/G_c$ and $\overline{G_i} = G_i/G_c$, for $1 \leqslant i \leqslant c$. By the induction hypothesis, we know that $[\overline{G} : \overline{K}]$ is finite and prime to r. As $[\overline{G} : \overline{K}]$ is equal to $[G : KG_c]$, it is enough to prove that $[KG_c : K]$ is finite and prime to r. The subgroup G_c is finitely generated Abelian and $G_c = KG_c^r \cap G_c = (K \cap G_c)G_c^r$. Therefore, by (i), the index $[G_c : K \cap G_c]$ is finite and prime to r. As $[KG_c : K]$ is equal to $[G_c : K \cap G_c]$, the proposition is proved. $\square$

In the proof of the Theorem, one of the things we need is: a $\exists\forall$ formula γ with n free variables and such that if $H \vDash \gamma(h_1, \ldots, h_k)$ then $H_r = \langle h_1, \ldots, h_k \rangle$ is a subgroup of H whose index is finite and prime to r. We shall use the proposition above to achieve this. So, the next results clear the way to build a formula implying the proposition's hypothesis. The formula itself is given in Corollary 3.6, at the end of this section. We recall two results that allow us to describe certain subgroups by existential formulas:

(i) *Let G be a polycyclic group. There exists a positive integer k such that every g belonging to G' can be written as a product $[g_1, g_2] \cdots [g_{2k-1}, g_{2k}]$, where $g_1, \ldots, g_{2k}$ are in G.*

(ii) *Let G be a polycyclic-by-finite group and let $m \geqslant 1$ be an integer. There is an integer r such that every g in G^m can be written as $g = g_1^m \cdots g_r^m$, with $g_1, \ldots, g_r$ in G.*

The first result is a consequence of a theorem proved by Rhemtulla in [**10**]. A proof of (ii) can be found in [**6**]. See also [**11**], where Romankov generalises these results to any verbal subgroup provided the group G is polycyclic. A direct consequence of these results is the following lemma.

Lemma 3.2. (i) *If K and H are polycyclic groups, there is an existential formula λ such that $K_\lambda = K'$ and $H_\lambda = H'$.*

(ii) *If K and H are polycyclic-by-finite groups and $m > 1$ is an integer, there is an existential formula γ such that $K_\gamma = K^m$ and $H_\gamma = H^m$.*

Proof: (i) Define $\lambda(X)$ as $(\exists Y_1 \ldots \exists Y_{2l})(X = [Y_1, Y_2] \cdots [Y_{2l-1}, Y_{2l}])$, where each element of H' and each element of K' is a product of l commutators.

(ii) Similarly, define $\gamma(X)$ as $(\exists Y_1 \ldots \exists Y_s)(X = Y_1^m \cdots Y_s^m)$, where each element of H^m and each element of K^m is a product of s m-th powers.

Now we apply the results of the preceding section to polycyclic-by-finite groups. In order to make notation lighter we have chosen to write G_ϕ^m instead of $(G_\phi)^m$ (where G is a group, m is an integer and ϕ is a formula).

Lemma 3.3. *Let G and H be polycyclic-by-finite groups such that $G \equiv_1 H$. Let ψ and ϕ be existential formulas such that G_ψ and G_ϕ are subgroups of G with $G_\psi \subset G_\phi$. If G_ϕ / G_ψ is Abelian then H_ϕ / H_ψ is isomorphic to G_ϕ / G_ψ.*

Proof: Before beginning the proof, we remark that H_ϕ / H_ψ makes sense since, by Lemma 2.2, $H_\psi \subset H_\phi$ and these two sets are normal subgroups of H. The sentence $(\forall X)(\forall Y)\big((\phi(X) \wedge \phi(Y)) \rightarrow \psi(XYX^{-1}Y^{-1})\big)$ is $\forall\exists$; this sentence means "K_ϕ / K_ψ is Abelian" in any group K for which $K_\psi \lhd K$, $K_\psi \lhd K$ and $K_\psi \subset K_\phi$. Thus, G_ϕ / G_ψ Abelian implies H_ϕ / H_ψ Abelian.

Two finitely generated Abelian groups A and B are isomorphic if and only if, for each integer $m > 1$, the finite groups A/A^m and B/B^m are isomorphic. Moreover, we have:

$$(G_\phi/G_\psi)/(G_\phi/G_\psi)^m \simeq G_\phi/\big(G_\psi G_\phi^m\big) \quad \text{and} \quad (H_\phi/H_\psi)/(H_\phi/H_\psi)^m \simeq H_\phi/\big(H_\psi H_\phi^m\big).$$

Thus, it is enough to show that $G_\phi/\Big(G_\psi G_\phi^m\Big) \simeq H_\phi/\Big(H_\psi H_\phi^m\Big)$ for each $m > 1$.

It follows from Lemma 3.2 and Lemma 2.1-(iii) and (iv) that, for each $m > 1$, there is an existential formula θ such that $G_\theta = G_\psi G_\phi^m$ and $H_\theta = H_\psi H_\phi^m$. As G_ϕ / G_θ and H_ϕ / H_θ are finite, Lemma 2.3 yields $G_\phi / G_\theta \simeq H_\phi / H_\theta$. $\square$

Proposition 3.4. *Let G and H be polycyclic-by-finite groups such that $G \equiv_1 H$. Then, there are two positive integers m and c and there are existential formulas $\phi^1, \ldots, \phi^{c+1}$ such that $1 = G_{\phi^{c+1}} \lhd G_{\phi^c} \lhd \cdots \lhd G_{\phi^1} = G^m$ and $1 = H_{\phi^{c+1}} \lhd H_{\phi^c} \lhd \cdots \lhd H_{\phi^1} = H^m$ are the derived series of G^m and H^m. Moreover, $G_{\phi^i}/G_{\phi^{i+1}} \simeq H_{\phi^i}/H_{\phi^{i+1}}$, for $1 \leqslant i \leqslant c$.*

PROOF: There is an integer $m > 1$ such that G^m and H^m are polycyclic and therefore soluble. Let $c \geqslant 1$ be the minimal integer such that G^m and H^m are soluble of class $\leqslant c$. Consider $1 = G_{c+1} \lhd G_c \lhd \cdots \lhd G_1 = G^m$ and $1 = H_{c+1} \lhd H_c \lhd \cdots \lhd H_1 = H^m$, where $G_{i+1} = [G_i, G_i]$ and $H_{i+1} = [H_i, H_i]$, for $1 \leqslant i \leqslant c$. By Lemma 3.2, there is an existential formula that defines G^m in G and H^m in H. Also, for each integer $i > 1$, there is an existential formula that defines G_{i+1} in G_i and H_{i+1} in H_i. Using induction and Lemma 2.1-(iv), we see that, for each $i \in \{1, \dots, c+1\}$, there is an existential formula ϕ^i such that $G_i = G_{\phi^i}$ and $H_i = H_{\phi^i}$.

By Lemma 3.3, we know that $G_{\phi^i}/G_{\phi^{i+1}} \simeq H_{\phi^i}/H_{\phi^{i+1}}$, for $1 \leqslant i \leqslant c$. The existence of these isomorphisms implies that $G_{\phi^i} = 1$ if and only if $H_{\phi^i} = 1$ and so, H^m and G^m are both soluble of class c. ∎

COROLLARY 3.5. *Let G and H be polycyclic-by-finite groups such that $G \equiv_1 H$. Then G and H have the same Hirsch number.*

PROOF: We follow the same notation of the preceding proposition. The groups G/G^m and H/H^m are finite and so $h(G/G^m) = h(H/H^m) = 0$. Proposition 3.4 implies that

$$h(G) = h(G^m) = \sum_{i=1}^{c} h\big(G_{\phi^i}/G_{\phi^{i+1}}\big) = \sum_{i=1}^{c} h\big(H_{\phi^i}/H_{\phi^{i+1}}\big) = h(H^m) = h(H). \qquad ∎$$

Now, we can finally show that the hypothesis of Proposition 3.1 can be expressed by a formula.

COROLLARY 3.6. *Let G and H be polycyclic-by-finite groups with $G \equiv_1 H$ and let $r \geqslant 1$ be an integer. Suppose $\{g_1, \dots, g_k\}$ is a set of generators of G. Then, there is an $\exists\forall$ formula $\gamma(X_1, \dots, X_k)$ satisfying the following conditions:*

(i) *$\gamma(g_1, \dots, g_k)$ is true in G;*
(ii) *if $H \vDash \gamma(h_1, \dots, h_k)$ then $\langle h_1, \dots, h_k \rangle$ is a subgroup of H whose index is finite and prime to r.*

PROOF: By Proposition 3.4, there exist integers $m \geqslant 1$, $c \geqslant 1$ and existencial formulas $\phi^1, \dots, \phi^{c+1}$, such that the derived series of G^m and H^m are

$$1 = G_{\phi^{c+1}} \lhd G_{\phi^c} \lhd \cdots \lhd G_{\phi^1} = G^m \quad \text{and} \quad 1 = H_{\phi^{c+1}} \lhd H_{\phi^c} \lhd \cdots \lhd H_{\phi^1} = H^m.$$

We write $\overline{X} = (X_1, \dots, X_k)$. The group $G/G_{\phi^1} = G/G^m$ is finite and $G = \langle g_1, \dots, g_k \rangle G_{\phi^1}$. By Corollary 2.5, there exists an $\exists\forall$ formula, $\gamma_0\big(\overline{X}\big)$, such that $G \vDash \gamma_0(g_1, \dots, g_k)$ and, if $H \vDash \gamma_0(h_1, \dots, h_k)$, then $H = \langle h_1, \dots, h_k \rangle H_{\phi^1}$.

By Lemma 2.1-(iv) and Lemma 3.2, for each $i \in \{1, \dots, c\}$, there is an existential formula ψ^i such that $\big(G_{\phi^i}\big)^r = G_{\psi^i}$ and $\big(H_{\phi^i}\big)^r = H_{\psi^i}$. The group G_{ϕ^i}/G_{ψ^i} is finite

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and G_{ϕ^i} is contained in $\langle g_1, \ldots, k\rangle G_{\psi^i}$. Therefore, Corollary 2.5 implies that there is an $\exists\forall$ formula $\gamma_i(\overline{X})$, such that: $G \vDash \gamma_i(g_1, \ldots, g_k)$ and, if $H \vDash \gamma_i(h_1, \ldots, h_k)$, then $H_{\phi^i} \subset \langle h_1, \ldots, h_k\rangle H_{\psi^i}$.

Denote by $\gamma(\overline{X})$ the $\exists\forall$ formula $\bigwedge_{i=0}^{c} \gamma_i(\overline{X})$. Clearly G satisfies $\gamma(g_1, \ldots, g_k)$. Moreover, $H \vDash \gamma(h_1, \ldots, h_k)$ implies that $H = \langle h_1, \ldots, h_k\rangle H^m$ and that H_{ϕ^i} is contained in $\langle h_1, \ldots, h_k\rangle (H_{\phi^i})^r$ for $1 \leqslant i \leqslant c$. So, (i) is proved and (ii) is a direct consequence of Proposition 3.1. $\square$

Following the notation of the Corollary above, we would like to have $H_r = \langle h_1, \ldots, h_k\rangle$ isomorphic to G. Using the above results, it is not dificult to obtain H_r isomorphic to a quotient of G by a finite normal subgroup (this is detailed in the theorem's demonstration). So, what we need is a formula that forces H_r to be isomorphic to G, knowing that $H_r \simeq G/S$ with $|S| < \infty$. Finite subgroups can be described by formulas; consequently, we can oblige H_r to contain a subgroup isomorphic to a certain finite subgroup of G. So, we are done if we have a finite subgroup T_G of G such that $T_G \hookrightarrow G/S$ with S finite obliges $S = 1$. Then, it is enough to take T_G as the maximal finite normal subgroup of G.

DEFINITION: Given a polycyclic-by-finite group G, we shall note by T_G the unique maximal finite normal subgroup of G. (The existence of such a group is guaranteed because G satisfies the maximal condition on subgroups; uniqueness follows from the fact that the product of two finite normal subgroups is a finite normal subgroup).

4. PROOF OF THE THEOREM

Let G and H be polycyclic-by-finite groups and let $r \geqslant 1$ be an integer. As the hypothesis of the Theorem are symmetrical on G and H, it is enough to show that there exists H_r subgroup of H such that $G \simeq H_r$ and $[H : H_r]$ is finite and prime to r. So, the Theorem is a consequence of the following two facts.

FACT 1. There is a subgroup H_r of H such that:

(i) H_r is an homomorphic image of G;
(ii) $[H : H_r]$ is finite and prime to r;
(iii) there exists $T \lhd H$ such that $T \subset H_r$ and $T \simeq T_G$.

FACT2. If H_r is a subgroup of finite index in H, satisfying conditions (i) and (iii) of Fact 1, then $H_r \simeq G$.

PROOF OF FACT 1: Consider a finite presentation of G:

$$G = \langle g_1, \ldots, g_k : R_1(g_1, \ldots, g_k) = \cdots = R_l(g_1, \ldots, g_k) = 1\rangle.$$

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From now on, we write $\overline{X} = (X_1, \ldots, X_k)$ and $\overline{g} = (g_1, \ldots, g_k)$. Let $\gamma(\overline{X})$ be the $\exists\,\forall$ formula given by Corollary 3.6 for the generators $g_1, \ldots, g_k$ of G and for the integer r. We know that G satisfies $\gamma(\overline{g})$. Suppose $\{x_1, \ldots, x_n\} = T_G$ and consider the universal formula $\rho_{T_G}(X_1, \ldots, X_n)$ given by Corollary 2.4. Let $u_1(\overline{X}), \ldots, u_n(\overline{X})$ be n words such that $x_i = u_i(\overline{g})$, for $1 \leqslant i \leqslant n$. Then G satisfies $\rho_{T_G}(\mu_1(\overline{g}), \ldots, \mu_n(\overline{g}))$.

Define the formula $\theta(\overline{X})$ as

$$\left(\bigwedge_{i=1}^{l} R_i(\overline{X}) = 1\right) \wedge \gamma(\overline{X}) \wedge \rho_{T_G}(u_1(\overline{X}), \ldots, u_n(\overline{X})).$$

$G \vDash \theta(\overline{g})$ and $\theta(\overline{X})$ is an $\exists\,\forall$ formula. Since $G \equiv_1 H$, the $\exists\,\forall$ sentence $(\exists \overline{X})\theta(\overline{X})$ is satisfied in H. Let $h_1, \ldots, h_k$ be such that $H \vDash \theta(h_1, \ldots, h_k)$ and set $H_r = \langle h_1, \ldots, h_k \rangle$. As $H \vDash \left(\bigwedge_{i=1}^{l} R_i(h_1, \ldots, h_k) = 1\right)$, the map $g_i \to h_i$ induces a group homomorphism from G to H. Since $H \vDash \gamma(h_1, \ldots, h_k)$, Corollary 3.6 implies that $[H : H_r]$ is finite and prime to r. As $H \vDash \rho_{T_G}(u_1(h_1, \ldots, h_k), \ldots, u_n(h_1, \ldots, h_k))$, Corollary 2.4 implies that $T = \{u_1(h_1, \ldots, h_k), \ldots, u_n(h_1, \ldots, h_k)\}$ is a normal subgroup of H isomorphic to T_G. ∎

Proof of Fact 2: By (i), we can take $S \lhd G$ such that $G/S \simeq H_r$. By Corollary 3.5, G and H have the same Hirsch number. Since $[H : H_r]$ is finite, we have $h(G) = h(H_r)$ and so, $h(S) = 0$, that is, S is finite.

By (iii), there is $T \lhd H$ such that $T \simeq T_G$ and $T \subset H_r$. The isomorphism between G/S and H_r, implies the existence of $T^* \lhd G$ such that $S \subset T^*$ and $T^*/S \simeq T_G$. Since S and T_G are finite, so is T^*; hence, $T^* \subset T_G$, by the maximality of T_G. Now we have $T_G \simeq T^*/S \subset T_G/S$ and so, $S = 1$. ∎

5. Some equivalence relations

We begin by giving the definition of commensurability and giving a name for the equivalence relation that appears in the Theorem.

Definition: Two groups G and H are said to be commensurable if there exist G_* and H_*, subgroups of G and H respectively, such that $G_* \simeq H_*$ and the indices $[G : G_*]$ and $[H : H_*]$ are finite. This notion has been introduced by Baumslag [**1**, p.9].

Definition: Two groups G and H are said to be strongly commensurable if, for each $n > 1$, there are G_n and H_n, subgroups of G and H respectively, such that $G_n \simeq H$, $H_n \simeq G$ and the indices $[H : H_n]$ and $[G : G_n]$ are finite and prime to n.

Let G and H be polycyclic-by-finite groups and consider the following six equivalence relations:

(1) $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$;

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(2) $G \equiv H$;
(3) $G \equiv_1 H$;
(4) G and H strongly commensurable;
(5) G and H commensurable;
(6) $\mathcal{F}(G) = \mathcal{F}(H)$.

(1) $\Longrightarrow$ (2) is a result of Oger and holds in the class of groups (see [**8**]). An example showing that
(2) $\not\Longleftarrow$ (1) can be found in [**9**, Proposition].
(2) implies (3) is evident. To the best of the author's knowledge it is not known if (2) and (3) are equivalent.
(3) $\Longrightarrow$ (4) is the Theorem we have proved here and (4) $\not\Longrightarrow$ (3) is a consequence of the example at the end of this section.
(4) $\Longrightarrow$ (5) is clear from the definitions and (4) $\Longrightarrow$ (6) can be easily proved (a proof is given in next lemma).
Neither (5) implies (6) nor (6) implies (5) (see [**1**, p.9]). Consequently (4) is not implied by either (5) or (6).

LEMMA 5.1. *Suppose G and H are polycyclic-by-finite groups strongly commensurable. Then $\mathcal{F}(G) = \mathcal{F}(H)$.*

PROOF: It is enough to show that $\mathcal{F}(G) \subset \mathcal{F}(H)$. Moreover, we remark that $\mathcal{F}(G) \subset \mathcal{F}(H)$ if and only if, for each integer $m > 1$, we have $G/G^m \in \mathcal{F}(H)$. The proof of this is straightforward once we use that G/G^m and H/H^m are finite.

For each integer $m > 1$, there is $K < G$ such that $K \simeq H$ and $[G : K]$ is finite and prime to m. By the first paragraph, it is enough to show that G/G^m is isomorphic to a quotient of K. If p is a prime that divides $|G/G^m|$, there exists an element of order p in G/G^m and so, p divides m. We conclude that $|G/G^m|$ is prime to $[G : K]$. As the index $[G : KG^m]$ divides $[G : G^m]$ and also divides $[G : K]$, we have G equals to KG^m. Then G/G^m is isomorphic to $K/(G^m \cap K)$. ∎

EXAMPLE. Following [**3**], given a commutative ring R with 1 and a faithfull R-module M, we define

$$\Gamma(R, M) = \left\{ \begin{pmatrix} 1 & r & n \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \Big/ r \in R, \; m, n \in M \right\}.$$

$\Gamma(R, M)$ is a group under the usual matrix product. We identify $\Gamma(R, M)$ to the set $R \times M \times M$ with multiplication defined by:

$$(r, m, n)(r', m', n') = (r + r', m + m', rm' + n + n').$$

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We have $1_{\Gamma(R,M)} = (0,0,0)$ and $[(r,m,n),(r',m',n')] = (0,0,rm'-r'm)$. The centre $\zeta(\Gamma(R,M))$ and the derived subgroup $\Gamma(R,M)'$ are both equal to $(0,0,M)$. So, $\Gamma(R,M)$ is nilpotent of class 2.

Let R be $\mathbb{Z}[\sqrt{5}i]$, the ring of algebraic integers in $\mathbb{Q}(\sqrt{5}i)$, and let M be the R ideal generated by 2 and $1+\sqrt{5}i$. Then M is a non principal ideal of R. The group $\Gamma(R,R)$ is generated by $(1,0,0)$, $(\sqrt{5}i,0,0)$, $(0,1,0)$ and $(0,\sqrt{5}i,0)$. The group $\Gamma(R,M)$ is generated by $(1,0,0)$, $(\sqrt{5}i,0,0)$, $(0,2,0)$ and $(0,1+\sqrt{5}i,0)$. So, both groups are finitely generated and nilpotent, therefore polycyclic. Proposition B of [**3**] shows that $\Gamma(R,M)$ and $\Gamma(R,R)$ have the same finite images and are not isomorphic. Here we prove that $\Gamma(R,M)$ and $\Gamma(R,R)$ are strongly commensurable but they do not satisfy the same sentences with one alternation of quantifiers.

Lemma 5.2. *Let R be $\mathbb{Z}[\sqrt{5}i]$, the ring of algebraic integers of $\mathbb{Q}(\sqrt{5}i)$, and let M be the ideal of R generated by 2 and $1+\sqrt{5}i$. Then $\Gamma(R,M)$ and $\Gamma(R,R)$ do not satisfy the same sentences with one alternation of quantifiers.*

Proof: Let $\sigma(X)$ be the universal formula $(\forall Y)([X,Y]=1)$. The formula $\sigma(x)$ is true in $\Gamma(R,R)$ (in $\Gamma(R,M)$) if and only if x is in $\Gamma(R,R)'$ (in $\Gamma(R,M)'$). Now consider the $\forall\exists$ sentence $(\forall X_1 \forall X_2)(\exists Y_1 \dots \exists Y_4)\phi(X_1,X_2,Y_1,\dots,Y_4)$ where ϕ is given by:

$$\bigwedge_{i=1,2} \left(\neg\sigma(Y_iY_{i+2}) \wedge \neg\sigma\left(Y_iY_{i+2}^{-1}\right)\right)$$
$$\wedge\ [Y_1,Y_3]=1 \ \wedge\ [Y_2,Y_4]=1 \ \wedge\ [Y_1,Y_2]=[X_1,X_2][Y_3,Y_4]=[X_1,X_2]^3.$$

We shall show that this sentence is true in $\Gamma(R,M)$ but is not true in $\Gamma(R,R)$.

We first work in $\Gamma(R,M)$. Clearly M is equal to $\{x+y\sqrt{5}i \in R \mid x+y \text{ is even }\}$. Moreover, for all $r\in R$, each of the complex numbers $r(1\pm\sqrt{5}i)/2$ belongs to R if and only if r belongs to M. We shall use these facts without stating. Given x_1 and x_2 in $\Gamma(R,M)$, the commutator $[x_1,x_2]$ is equal to $(0,0,a)$ with a in M. So, $a(1\pm\sqrt{5}i)/2$ belongs to R. Consider the following elements of $\Gamma(R,M)$:

$$y_1=(1,0,0)y_2=(0,a,0) \quad y_3=\left(a\left(1+\sqrt{5}i\right)/2,0,0\right) \quad y_4=\left(0,1-\sqrt{5}i,0\right).$$

We have $y_3^{-1}=(-a(1+\sqrt{5}i)/2,0,0)$ and $y_4^{-1}=(0,-1+\sqrt{5}i,0)$. Therefore, we know that

$$y_1y_3=\left(1+a\left(1+\sqrt{5}i\right)/2,0,0\right) \qquad y_1y_3^{-1}=\left(1-a\left(1+\sqrt{5}i\right)/2,0,0\right)$$
$$y_2y_4=\left(0,a+1-\sqrt{5}i,0\right) \qquad y_2y_4^{-1}=\left(0,a-1+\sqrt{5}i,0\right).$$

As a belongs to R, either $1+a(1+\sqrt{5}i)/2$ and $1-a(1+\sqrt{5}i)/2$ are different from zero and so, $y_1y_3 \notin \Gamma[R,M]'$ and $y_1y_3^{-1} \notin \Gamma[R,M]'$. Also if we have $a \neq -1+\sqrt{5}i$ and $a \neq 1-\sqrt{5}i$, we know that y_2y_4 and $y_2y_4^{-1}$ are not in $\Gamma[R,M]'$. Direct computations show that

$$[y_1,y_2]=(0,0,a),\quad [y_1,y_3]=(0,0,0),\quad [y_2,y_4]=(0,0,0) \text{ and}$$
$$[y_3,y_4]=\left(0,0,a\left(1+\sqrt{5}i\right)\left(1-\sqrt{5}i\right)/2\right)=(0,0,3a)=(0,0,a)^3.$$

So, we have proved that if $[x_1,x_2] \neq -1+\sqrt{5}i$ and $[x_1,x_2] \neq 1-\sqrt{5}i$, then $\phi(x_1,x_2,y_1,\dots,y_4)$ is true in $\Gamma(R,M)$. In the case $[x_1,x_2]=-1+\sqrt{5}i$ or $[x_1,x_2]=1-\sqrt{5}i$, we take

$$z_1=(1,0,0)\quad z_2=(0,a,0)\quad z_3=\left(a\left(1-\sqrt{5}i\right)/2,0,0\right)\quad z_4=\left(0,1+\sqrt{5}i,0\right),$$

and we have that $\phi(x_1,x_2,z_1,\dots,z_4)$ is true in $\Gamma(R,M)$. Consequently $\Gamma(R,M)$ satisfies the sentence $(\forall X_1\,\forall X_2)(\exists Y_1\dots\exists Y_4)\phi(X_1,X_2,Y_1,\dots,Y_4)$.

Now we work in $\Gamma(R,R)$. For $1\leqslant i\leqslant 4$, take $w_i=(w_{i1},w_{i2},w_{i3})$ in $\Gamma(R,R)$ and call $\overline{w_i}$ the element (w_{i1},w_{i2}) in $R\times R$. Then $[w_i,w_j]$ is equal to $(0,0,det(\overline{w_i},\overline{w_j}))$. Suppose that $[w_1,w_2]=(0,0,1)$, $[w_3,w_4]=(0,0,3)$ and $[w_1,w_3]=[w_2,w_4]=(0,0,0)$. We have $det(\overline{w_1},\overline{w_3})=det(\overline{w_2},\overline{w_4})=0$ and so, there are α_{13} and α_{24} in $\mathbb{Q}(\sqrt{5}i)$ such that $\overline{w_3}=\alpha_{13}\overline{w_1}$ and $\overline{w_4}=\alpha_{24}\overline{w_2}$. As $[w_3,w_2]$ is in $(0,0,R)$, we know that $det(\overline{w_3},\overline{w_2})$ is in R; moreover, $det(\overline{w_1},\overline{w_2})=1$ implies $\alpha_{13}=det(\overline{w_3},\overline{w_2})$ and so, α_{13} is in R. The same argument shows that α_{24} is in R. Since $[w_3,w_4]=(0,0,3)$, we have:

$$3=det(\overline{w_3},\overline{w_4})=\alpha_{13}\alpha_{24}det(\overline{w_1},\overline{w_2})=\alpha_{13}\alpha_{24}.$$

As α_{13} and α_{24} are in R and $\alpha_{13}\alpha_{24}=3$, we conclude that either $\alpha_{13}=\pm1$ or $\alpha_{24}=\pm1$. So, we have either $\overline{w_3}=\pm\overline{w_1}$ or $\overline{w_4}=\pm\overline{w_2}$. Since $(u_1,u_2,u_3)^{-1}=(-u_1,-u_2,u_1u_2-u_3)$ for every (u_1,u_2,u_3) in $\Gamma(R,R)$, we conclude that one of the four elements w_1w_3, $w_1w_3^{-1}$, w_2w_4 or $w_2w_4^{-1}$ must be in $(0,0,R)=\Gamma(R,R)'$. What we have proved above is that if w_1, w_2, w_3 and w_4 are elements of $\Gamma(R,R)$ such that $[w_1,w_2]=(0,0,1)$, $[w_3,w_4]=(0,0,3)$ and $[w_1,w_3]=[w_2,w_4]=(0,0,0)$, then one of the four elements w_1w_3, $w_1w_3^{-1}$, w_2w_4 or $w_2w_4^{-1}$ must be in $\Gamma(R,R)'$. This shows that if we put $x_1=(1,0,0)$ and $x_2=(0,1,0)$ then $[x_1,x_2]=(0,0,1)$ and, for any $y_1,\dots,y_4$ in $\Gamma(R,R)$, the formula $\phi(x_1,x_2,y_1,\dots,y_4)$ is not true in $\Gamma(R,R)$. $\square$

LEMMA 5.3. *Let R be the ring of algebraic integers of a finite Galois-extention of $\mathbb{Q}$. Let M and N be non zero ideals of R. Then $\Gamma(R,M)$ and $\Gamma(R,N)$ are strongly commensurable.*

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PROOF: By the hypothesis of simmetry, it is enough to show that for each integer $m > 1$, there exists a subgroup H of $\Gamma(R, N)$ such that $H \simeq \Gamma(R, M)$ and the index $[\Gamma(R, N) : H]$ is finite and prime to m. We first prove the following fact:

FACT (A). For each integer $m > 1$, there is an integer k which is prime to m and there is an ideal M' of R such that $M' \simeq M$ and $kN \subset M' \subset N$.

Consider the ring $R_m = \{r/n : r \in R$ and $n \in \mathbb{Z}$ is prime to m $\}$. Since R is a Dedekind domain, so is R_m. Moreover, R_m is a principal domain because it has a finite number of prime ideals (see [4, Corollary on page 13]). Consequently, there exists a R_m-isomorphism $f : R_m M \longrightarrow R_m N$. As M and N are finitely generated R-ideals, there exists an integer $t \geqslant 1$ which is prime to m and such that $tf(M) \subset N$ and $tf^{-1}(N) \subset M$. Therefore, $f^{-1}(t^2 N) \subset tM$ and this implies $t^2 N \subset f(tM) = tf(M) \subset N$. Setting $k = t^2$ and $M' = tf(M)$, we obtain (A).

Now, put $H = \Gamma(R, M')$. It remains to prove that the index $[\Gamma(R, N) : H]$ is prime to m. Direct computations show that if L is an R-submodule of N then the index $[\Gamma(R, N) : \Gamma(R, L)]$ is equal to $[N : L]^2$. Hence, it is enough to show that $[N : M']$ is prime to m. By (A), we know that $[N : M']$ divides $[N : kN]$; moreover, the index $[N : kN]$ is prime to m because it has the same prime divisors as k. So, $[N : M']$ is prime to m. ∎

REFERENCES

[1] G. Baumslag, *Lecture notes on nilpotent groups*, Regional Conference Series in Math., **2** (American Mathematical Society, Providence, RI, 1971).

[2] C.C. Chang and H.J. Keisler, *Model theory* (North-Holland, Amsterdam, New York, 1973).

[3] F.J. Grunewald and R. Scharlau, 'A note on finitely generated torsion-free nilpotent groups of class 2', *J. Algebra* **58** (1979), 162–175.

[4] S. Lang, *Algebraic number theory*, Graduate Texts in Math **110** (Springer Verlag, Berlin, Heidelberg, New York, 1986).

[5] F. Oger, 'Elementary equivalence and genus of finitely generated nilpotent groups', *Bull. Austral. Math. Soc.* **37** (1988), 61–68.

[6] F. Oger, 'Équivalence élémentaire entre groupes finis-par-abéliens de type fini', *Comment. Math. Helv.* **57** (1982), 469–480.

[7] F. Oger, 'Cancellation and elementary equivalence of finitely generated finite-by-nilpotent groups', *J. London Math. Soc.* **44** (1990), 173–183.

[8] F. Oger, 'Cancellation and elementary equivalence of groups', *J. Pure Appl. Algebra* **30** (1983), 293–299.

[9] F. Oger, 'Elementary equivalence and profinite completions: a characterization of finitely generated abelian-by-finite groups', *Proc. Amer. Math. Soc.* **103** (1988), 1041–1048.

[10] A.H. Rhemtulla, 'Commutators of certain finitely generated soluble groups', *Canad. J. Math.* **21** (1969), 1160–1164.

[11] V.A. Romankov, 'Width of verbal subgroups in solvable groups', *Algebra and Logic* **21** (1982), 41–49.

[12] D. Segal, *Polycyclic groups* (Cambridge University Press, Cambridge, 1983).

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