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SUPPLEMENTS OF HÖLDER'S INEQUALITY

DAVID C. BARNES

1. Introduction. Given vectors $\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$ (or functions $f(x)$ and $g(x)$) we define the Hölder Quotient H_{pq} by

$$(1) \qquad H_{pq}(\vec{a}, \vec{b}) = \frac{(\vec{a}, \vec{b})}{||a||_p||p||_q}$$

or in case of functions by

$$(2) \qquad H_{pq}(f, g) = \frac{\int_0^\ell f(x)g(x)dx}{||f||_p||g||_q}.$$

Here $||\cdot||_p$ and $||\cdot||_q$ are the usual L_p and L_q norms. We assume throughout that

$$1/p + 1/q = 1 \text{ or } pq = p + q$$

$$a_i \geqq 0, b_i \geqq 0$$

$$f(x) \geqq 0, g(x) \geqq 0.$$

If p and q are both greater than one then they are positive but if we allow p and q to be less than one then one of them must be positive and the other one must be negative. This may cause a problem if for example, some value a_i is zero and p is negative. In this case we use the convention that $||\vec{a}||_p = 0$ and $H_{pq}(\vec{a}, \vec{b}) = \infty$. The classical Hölder Inequality may then be stated in the form (see [**1**] page 27)

$$(3) \qquad \text{if } p, q > 1 \text{ then } H_{pq}(\vec{a}, \vec{b}) \leqq 1$$

$$(4) \qquad \text{if } p, q < 1 \text{ and } a_i > 0, b_i > 0 \text{ then } H_{pq}(\vec{a}, \vec{b}) \geqq 1.$$

In either case equality holds if and only if $(a_i)^p$ is proportional to $(b_i)^q$.

There are several results in the literature (see [**2**]) which give "complements of" or "inverse" Hölder inequalities. These results take the form of bounds on $H_{pq}(\vec{a}, \vec{b})$ of the form

$$(5) \qquad \text{if } p, q > 1 \text{ then } H_{pq}(\vec{a}, \vec{b}) \geqq C_1$$

$$(6) \qquad \text{if } p, q < 1 \text{ then } H_{pq}(\vec{a}, \vec{b}) \leqq C_2$$

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where C_1 and C_2 are constants with $C_1 < 1$ and $C_2 > 1$.

A supplement of Hölder's inequality gives, for $p, q > 1$ a smaller upper bound on $H_{pq}(\vec{a}, \vec{b})$ than given by (3) or for $p, q < 1$ a larger lower bound than (4). These results are of the form

(7) if $p, q > 1$ then $H_{pq}(\vec{a}, \vec{b}) \leqq C_1 < 1$

(8) if $p, q < 1$ then $H_{pq}(\vec{a}, \vec{b}) \geqq C_2 > 1$.

In order to obtain these improved estimates for $H_{pq}(\vec{a}, \vec{b})$ we shall, of course, need to impose on the vectors $\vec{a}, \vec{b}$ some additional constraint which prevents $(a_i)^p$ from being proportional to $(b_i)^q$. This will prevent the case of equality in (3) and (4) which can lead to improvements (7) and (8). A more common notation for inequalities (7) (8) is

(7*) $$\sum_{j=1}^{n} a_j b_j \leqq C_1 \left[\sum_{j=1}^{n} (a_j)^p\right]^{1/p} \left[\sum_{j=1}^{n} (b_j)^q\right]^{1/q} \text{ with } C_1 < 1.$$

(8*) If $p, q < 1$ then

$$\sum_{j=1}^{n} a_j b_j \geqq C_2 \left[\sum_{j=1}^{n} (a_j)^p\right]^{1/p} \left[\sum_{j=1}^{n} (b_j)^q\right]^{1/q} \text{ with } C_2 > 1.$$

In this paper we give a general method for finding supplements of Hölder's inequality of the form (7*) or (8*).

2. The general method. Suppose first of all that $p, q > 1$.The other case will be treated in Theorem 4 below. Along with the vectors $\vec{a}, \vec{b}$ we consider their normalized forms $\vec{\alpha}, \vec{\beta}$ defined by

(9) $$\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \vec{\alpha}/||\vec{\alpha}||_p$$

(1) $$\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) = \vec{b}/||\vec{b}||_q.$$

We introduce functions $h(u, v)$ and $S(\vec{a}, \vec{b})$ defined by

(11) $$h(u, v) = (1/p)u + (1/q)v - u^{1/p} v^{1/q}$$

(12) $$S(\vec{\alpha}, \vec{\beta}) = \sum_{j=1}^{n} h(\alpha_j^p, \beta_j^q).$$

It follows that

(13) $$H_{pq}(\vec{a}, \vec{b}) = (\vec{\alpha}, \vec{\beta}) = 1 - S(\vec{\alpha}, \vec{\beta}) = 1 - \sum_{j=1}^{n} h(\alpha_j^p, \beta_j^q).$$

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One of the standard proofs of Hölder's inequality proceeds by showing that $h(u, v) \geqq 0$ and $h(u, v) = 0$ if and only if $u = v$. The relation (13) then gives $H_{pq}(\vec{a}, \vec{b}) \leqq 1$ and the associated conditions for equality. Our method of finding supplementary inequalities entails finding positive lower bounds on $h(u, v)$.

We introduce the points R_i in the u, v plane defined by

$$(14) \qquad R_i = (\alpha_i^p, \beta_i^q)$$

and we abbreviate notation so that $h(\alpha_i^p, \beta_i^q) = h(R_i)$. The size of $H_{pq}(\vec{a}, \vec{b})$ is thus determined by the location of the points R_i in the u, v plane. If all the points R_i lie on the straight line defined by $u = v$ then

$$h(R_i) = 0 \text{ and } H_{pq}(\vec{a}, \vec{b}) = 1.$$

If however some constraint forces the points R_i off the line then $h(R_i) > 0$ and so

$$H_{pq}(\vec{a}, \vec{b}) < 1.$$

To apply the method we will minimize $S(\vec{\alpha}, \vec{\beta})$ (and thus maximize $H_{pq}(\vec{a}, \vec{b})$) by moving the points R_i around in a suitable way. We may move the points R_i in the (u, v) plane in any way which is appropriate as long as we maintain at all times the relations

$$(15) \qquad \alpha_i \geqq 0, \beta_i \geqq 0, \sum_{j=1}^{n} (\alpha_j)^p = 1, \sum_{j=1}^{n} (\beta_j)^q = 1.$$

The difficult part is to move the points around maintaining (15) while at the same time decreasing $S(\vec{\alpha}, \vec{\beta})$ to its minimum value.

Let A denote the subset of $2n$ dimensional space $(\vec{\alpha}, \vec{\beta})$ defined by (15). Now $S(\vec{\alpha}, \vec{\beta})$ is a continuous function defined on the compact set A. If B is any closed subset of A it follows that there exists a point $(\vec{\alpha}_0, \vec{\beta}_0) \in B$ for which

$$H_{pq}(\vec{\alpha}, \vec{\beta}) \leqq H_{pq}(\vec{\alpha}_0, \vec{\beta}_0)$$

for all $(\vec{\alpha}, \vec{\beta}) \in B$.

Thus we may take $C = H_{pq}(\vec{\alpha}_0, \vec{\beta}_0)$ and we have a supplementary inequality

$$H_{pq}(\vec{\alpha}, \vec{\beta}) \leqq C \quad \text{for all } (\vec{\alpha}, \vec{\beta}) \in B$$

or in other notation

$$\sum_j a_j b_j \leqq C\left[\sum_j a_j^p\right]^{1/p}\left[\sum b_j^p\right]^{1/q}.$$

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In order to find the vectors $\vec{\alpha}_0$, $\vec{\beta}_0$ we need to move the points R_i around. Now if we were to move only the one point R_i then one of the norm conditions (15) would be violated. It turns out however that we can move points R_i, R_j in pairs while maintaining (15). To simplify the notation we let

$$R_i = (\alpha_i^p, \beta_i^q) = (x_1, y_2) \quad \text{and} \quad R_j = (\alpha_j^p, \beta_j^q) = (x_2, y_1).$$

In order to move pairs of points R_i, R_j we will find the following theorems very useful. See also figure 1.

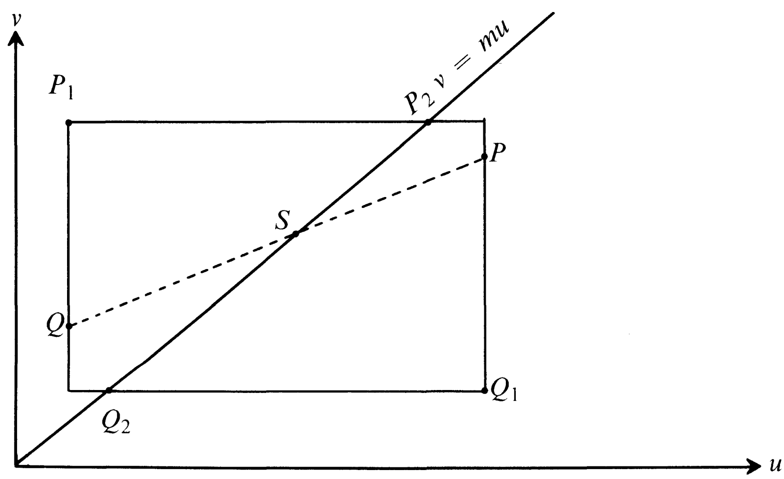

Figure 1 The Geometry of Theorems 1, 2, 3, 4

Consider any rectangle having sides parallel to the u, v axes. Let P and Q be any pair of antipodal points on the boundary of the rectangle. Then the maximum value of $h(P) + h(Q)$ occurs when $P = P_1$ and $Q = Q_1$, the northwest and southeast corners. The minimum value of $h(P) + h(Q)$ occurs when the points P, Q and the origin are all collinear, that is when P, Q are both on the same line $v = mu$. As drawn in Figure 1 the line $v = mu$ intersects the top and bottom of the rectangle (at P_2 and Q_2) but depending on the location of the rectangle the line may intersect the left and right ends. In any case if the line joining P and Q rotates about the center S then $h(P) + h(Q)$ is a strictly monotonic function between its maximum and minimum values.

THEOREM 1. *Let* x_1, y_1, x_2, y_2 *be nonnegative numbers satisfying*

$$x_1 \leqq x_2, y_1 \leqq y_2, x_1^2 + x_2^2 > 0, y_1^2 + y_2^2 > 0.$$

Then

(16) $\quad h(x_1, y_2) + h(x_2, y_1) > h(x_1, y_2 - t) + h(x_2, y_1 + t)$

for all t *satisfying* $0 < t < y_2 - y_1$.

Furthermore

(17) $\quad h(x_1, y_2) + h(x_2, y_1) > h(x_1 + s, y_2) + h(x_2 - s, y_1)$

for all s satisfying $0 < s < x_2 - x_1$.

Proof. We will first prove (16). Define a function $g(t)$ for $0 < t < y_2 - y_1$ by

(18) $\quad g(t) = h(x_1, y_2 - t) + h(x_2, y_1 + t).$

We must show $g(t) < g(0)$. Now $g''(t) > 0$ so the result will follow if we show $g(0) \geqq g(y_2 - y_1)$. Using (18) and (11) we find, after some manipulation, that this inequality is equivalent to

(19) $\quad (x_1^{1/p} - x_2^{1/p})(y_2^{1/q} - y_1^{1/q}) \leqq 0.$

This inequality is valid since $p, q > 0$ and $x_1 \leqq x_2, y_1 \leqq y_2$. The proof of (17) is similar.

Theorem 1 shows how to move pairs of points around in the plane while maintaining (15) and at the same time decreasing $S(\vec{\alpha}, \vec{\beta})$. Other kinds of movements are sometimes useful, for example:

THEOREM 2. *Let* x_1, y_1, x_2, y_2 *be nonnegative numbers satisfying*

$$x_1 \leqq x_2, y_1 \leqq y_2 \text{ and } y_1 = mx_1, y_2 = mx_2$$

for some positive constant m.

For $x_2 > t > -x_1$ *define a function* $g(t)$ *by*

$$g(t) = h(x_1 + t, y_1) + h(x_2 - t, y_2).$$

Then $g(t)$ *is convex and has a minimum at* $t = 0$. *It is strictly decreasing as* $t \to 0$ *from the left or the right.*

For $y_2 > s > -y_1$ *define a function* $f(s)$ *by*

$$f(s) = h(x_1, y_1 + s) + h(x_2, y_2 - s).$$

Then $f(s)$ *is convex and has a minimum at* $s = 0$. *It is strictly decreasing as* $s \to 0$ *from the left or right.*

Proof. To prove the first part suppose that $y_1 > 0$. It is easy to see that $g''(t) > 0$ and $g'(0) = 0$ which gives the result. If $y_1 = 0$ then $x_1 = 0$ and we must show $g(t)$ is convex and increasing for $x_2 > t > 0$. We see that

$$g'(t) = (1/p)m^{1/q}x_2^{1/q}(x_2 - t)^{1/p-1} > 0$$

and similarly $g''(t) > 0$. This proves the first part of the theorem. The proof of the second part is similar and will not be given.

These two theorems can be combined to get other interesting motions of the points R_i.

THEOREM 3. *Let x_1, y_1, x_2, y_2 be nonnegative numbers with*

$$x_1 < x_2, y_1 < y_2.$$

Define the points P and Q by

$$P = (x_1, y_2) \quad Q = (x_2, y_1).$$

For $0 < \gamma < \frac{1}{2}$ define points P^*, Q^* by

$$P^* = P + \gamma(Q - P), Q^* = Q + \gamma(P - Q).$$

Then

$$h(P) + h(Q) > h(P^*) + h(Q^*).$$

Proof. We first use (17) to move points P, Q horizontally until they are lined up vertically with P^* and Q^* respectively. This decreases $h(P) + h(Q)$. We then use (16) to move the points P, Q vertically until they coincide with P^* and Q^*. This also decreases $h(P) + h(Q)$ which proves the theorem.

The preceding analysis assumed $p, q > 1$ but most of it remains intact if $p, q < 1$. Assume now that $p, q < 1$ or equivalently $pq < 0$ and define a new function $h(u, v)$ by

$$(11^*) \quad h(u, v) = u^{1/p}v^{1/q} - (1/p)u - (1/q)v.$$

With this new choice of $h(u, v)$ we see that (12) and (13) become

$$(12^*) \quad S(\vec{\alpha}, \vec{\beta}) = \sum_{j=1}^{n} h(\alpha_j^p, \beta_j^q)$$

$$(13^*) \quad H_{pq}(\vec{a}, \vec{\mathrm{b}}) = 1 + S(\vec{\alpha}, \vec{\beta}) = 1 + \sum_{j=1}^{n} h(\alpha_j^p, \beta_j^q).$$

Thus minimizing $S(\vec{\alpha}, \vec{\beta})$ also minimizes $H_{pq}(\vec{a}, \vec{b})$ when $pq < 0$ but this is exactly what we want to do in this case. Analogues of Theorems 1, 2 and 3 hold with $h(u, v)$ defined by (11*) and $pq < 0$. We must however avoid the division by zero which may occur in the form u^p with $p < 0$, $u = 0$ or in the form v^q with $q < 0$, $v = 0$.

THEOREM 4. *Let x_1, y_1, x_2, y_2 be nonnegative numbers satisfying*

$$x_1 \leqq x_2, y_1 \leqq y_2.$$

Assume also that $pq < 0$ and that $x_1 > 0$ if $p < 0$ but that $y_1 > 0$ if $q < 0$. Then Theorem 1, 2 *and* 3 *hold with $h(u, v)$ defined by* (11*).

The proofs in this case are essentially the same with some minor modifications. To prove (16) we define $g(t)$ using (18) and $h(u, v)$ using (11*). Inequality (19) then becomes

$$(19^*)\qquad (x_1^{1/p} - x_2^{1/p})(y_2^{1/q} - y_1^{1/q}) \geqq 0.$$

Now however $pq < 0$ so either $p < 0$ or $q < 0$. If $p < 0$ then $q > 0$ and we see that both factors in (19*) are $\geqq 0$. If however $p > 0$ and $q < 0$ then we see that both factors in (19*) are $\leqq 0$. In either case (19*) follows. The remaining parts of Theorem 4 can be proved in a similar manner.

When $pq < 0$ there can be some difficulty with the existence of a minimizing point for $H_{pq}(\vec{a}, \vec{b})$. Consider the case $p < 0$ and $q > 0$. Now $\sum_j (\alpha_j)^p = 1$ implies that $(\alpha_j)^p \leqq 1$ so $\alpha_j \geqq 1$. The set A defined by (15) is no longer compact. However $\alpha_j \geqq 1$ implies that $S(\vec{\alpha}, \vec{\beta})$ defined by (12*) is continuous on A. In order to insure the existence of a minimizing point we impose a boundedness condition. Let $D > 1$ be a constant and define a set $A(D)$ as follows:

If $p < 0$, $q > 0$ let

$$A(D) = \{ (\vec{\alpha}, \vec{\beta}) \mid \sum_j (\alpha_j)^p = \sum_j (\beta_j)^q = 1,$$
$$0 \leqq \beta_j,\ 1 \leqq \alpha_j \leqq D \}.$$

If however $p > 0$ and $q < 0$ let

$$A(D) = \{ (\vec{\alpha}, \vec{\beta}) \mid \sum_j (\alpha_j)^p = \sum_j (\beta_j)^q = 1,$$
$$0 \leqq \alpha_j,\ 1 \leqq \beta_j \leqq D \}.$$

Now $A(D)$ is compact and if B is any closed subset of $A(D)$ then there is some point $(\vec{\alpha}_0, \vec{\beta}_0) \in B$ which satisfies

$$H_{pq}(\vec{\alpha}, \vec{\beta}) \geqq H_{pq}(\vec{\alpha}_0, \vec{\beta}_0)$$

for all $(\vec{\alpha}, \vec{\beta}) \in B$.

3. Applications of the method. In this section we show some of the results which may be obtained using the preceding ideas.

THEOREM 5. *Let $p + q = pq$ and $p, q > 1$. Let $\vec{a} = (a_i)$ be an increasing sequence*

$$a_i < a_{i+1},\ i = 1, 2, \dots, n - 1.$$

Then there is a constant $C < 1$ such that for any sequence $\vec{b} = (b_i)$ satisfying

$$(20)\qquad b_i \geqq b_{i+1},\ b_i \geqq 0 \quad i = 1, 2, \dots, n - 1$$

it follows that

$$(21) \quad \sum_j a_j b_j \leqq C\left(\sum_j (a_j)^p\right)^{1/p}\left(\sum_j (b_j)^q\right)^{1/q}.$$

Equality holds if and only if (b_i) *is a constant sequence. For a given sequence* $\vec{a}$ *the constant* C *may be calculated using*

$$C = H_{pq}(\vec{a}, 1) = (1/n)^{1/q} \sum_j a_j\left(\sum_j a_j^p\right)^{-p}.$$

Proof. Under these conditions it is not possible for (a_i^p) to be proportional to (b_i^q). Normalizing the vectors $\vec{a}$ and $\vec{b}$ we see that

$$\alpha_i < \alpha_{i+1}, \ \beta_i \geqq \beta_{i+1}, \quad i = 1, 2, \dots, n - 1.$$

Let $\vec{\beta} = (\beta_i^*)$ be that sequence which minimizes $S(\vec{a}, \vec{b})$ subject to (15) and (20) for the fixed vector $\vec{\alpha}$. Suppose that (β_i^*) is not a constant sequence. Then there is an index j for which $\beta_j^* > \beta_{j+1}^*$. To use Theorem 1 we take

$$(x_1, y_2) = (\alpha_j^p, \beta_j^{*q}) = R_j, (x_2, y_1) = (\alpha_{j+1}^p, \beta_{j+1}^{*q}) = R_{j+1}.$$

We see $x_1 < x_2$ and $y_1 < y_2$. Thus (16) shows how to decrease the value of $S(\vec{a}, \vec{\beta}^*)$ subject to the constraints (15), (20). This contradicts the definition of $\vec{\beta}^*$ so the sequence (β_i^*) must be constant. Using (1) it follows that any vector $\vec{b}$ which maximizes $H_{pq}(\vec{a}, \vec{b})$ must be a constant and furthermore any constant vector $\vec{b}$ will in fact maximize $H_{pq}(\vec{a}, \vec{b})$ subject to (15), (20). We may take $\vec{b} = (1, 1, 1, \dots, 1)$ and we see

$$H_{pq}(\vec{a}, \vec{b}) \leqq H_{pq}(\vec{a}, 1) = C.$$

Since (a_i) is strictly decreasing it follows from the classical Hölder inequality that $C < 1$.

There is no essential difficulty in applying the method to multiple sums. We consider sums of the form $\sum_n \sum_m a_{nm}b_{nm}$. In this case the points R_{ij} in the u, v plane have a double index and are defined by

$$R_{ij} = \left(\frac{a_{ij}^p}{\sum_n \sum_m a_{nm}^p}, \frac{b_{ij}^q}{\sum_n \sum_m b_{nm}^q} \right)$$

In this case we have

THEOREM 6. *Let* $p + q = pq$ *and* $p, q > 1$. *Let* (a_{ij}) *be a given strictly increasing sequence,* $a_{i,j} < a_{i+1,j}$ *and* $a_{i,j} < a_{i,j+1}$ *for all* i, j. *Then there is a constant* $C < 1$ *which depends on* (a_{ij}) *such that for any decreasing sequence* (b_{ij}) *satisfying* $b_{i,j} \geqq b_{i+1,j}$ *and* $b_{i,j} \geqq b_{i,j+1}$ *for all* i, j *it follows that*

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$$\sum_i \sum_j a_{ij} b_{ij} \leqq C\Big[\sum_i \sum_j a_{ij}{}^p\Big]^{1/p} \Big[\sum_i \sum_j b_{ij}{}^q\Big]^{1/q}.$$

Equality holds if and only if (b_{ij}) *is a constant sequence. The constant C may be calculated using*

$$C = H_{pq}((a_{ij}), 1).$$

Proof. Normalizing the sequence (a_{ij}) and (b_{ij}) we obtain sequences (α_{ij}) and (β_{ij}) satisfying

(22) $\quad \alpha_{i,j} < \alpha_{i+1,j}, \alpha_{i,j} < \alpha_{i,j+1}, \beta_{i,j} \geqq \beta_{i+1,j}, \beta_{i,j} \geqq \beta_{i,j+1}.$

Let $(\beta^*_{i\,j})$ be that sequence which, for the fixed sequence (α_{ij}), minimizes $S((\alpha_{ij}), (\beta_{ij}))$ over all sequences $\beta_{i,j}$ which satisfy (22). Let $R^*_{i\,j}$ be the points in the u, v plane defined by

$$R^*_{i\,j} = ((\alpha_{ij})^p, (\beta^*_{i\,j})^q).$$

We must show $\beta^*_{i\,j}$ is constant. Suppose not. Then there is some pair i, j for which $\beta^*_{i,j} > \beta^*_{i+1,j}$ or else $\beta_{ij} > \beta^*_{i,j+1}$. There are two different ways to order the corresponding $\alpha_{i,j+1}$ and $\alpha_{i+1,j}$. This gives a total of four cases to consider:

case I $\quad \beta^*_{i,j} > \beta^*_{i+1,j}, \quad \alpha_{i+1,j} > \alpha_{i,j+1};$

case II $\quad \beta^*_{i,j} > \beta^*_{i+1,j}, \quad \alpha_{i+1,j} < \alpha_{i,j+1};$

case III $\quad \beta^*_{i,j} > \beta^*_{i,j+1}, \quad \alpha_{i+1,j} > \alpha_{i,j+1};$

case IV $\quad \beta^*_{i,j} > \beta^*_{i,j+1}, \quad \alpha_{i+1,j} < \alpha_{i,j+1}.$

Consider case I where we have

$$\beta^*_{i,j} > \beta^*_{i+1,j} \text{ and } \alpha_{i+1,j} > \alpha_{i,j+1}.$$

Since $\beta^*_{i,j}$ is decreasing we may have either $\beta^*_{i,j+1} < \beta^*_{i\,j}$ or else $\beta^*_{i,j+1} = \beta^*_{i,j}$. If $\beta^*_{i,j+1} < \beta^*_{i\,j}$ we apply (16) using $R^*_{i,j} = (x_1, y_2)$ and $R^*_{i+1,j} = (x_2, y_1)$. If however $\beta^*_{i,j+1} = \beta^*_{i\,j}$ we may not use (16) in the same way without violating the decreasing condition $\beta_{i,j} \geqq \beta_{i,j+1}$. However in the event that $\beta^*_{i,j+1} = \beta^*_{i\,j}$ we may apply (16) to the points

$$R^*_{i,j+1} = (x_1, y_2) \quad \text{and} \quad R^*_{i+1,j} = (x_2, y_1).$$

Thus (16) shows how to decrease $S((\alpha_{ij}), (\beta^*_{i\,j}))$ which contradicts the definition of $\beta^*_{i\,j}$. This eliminates case I as a possibility.

Case IV can be treated in a similar way. In this case we apply (16) to the pair of points $R^*_{i,j} = (x_1, y_2)$ and $R^*_{i,j+1} = (x_2, y_1)$ or else to the pair $R^*_{i+1,j} = (x_1, y_2)$ and $R^*_{i,j+1} = (x_2, y_1)$ according as

$$\beta^*_{i+1,j} < \beta^*_{i,j} \text{ or } \beta^*_{i+1,j} = \beta^*_{i,j}.$$

This shows how to decrease $S((\alpha_{ij}), (\beta^*_{ij}))$ and eliminates case IV.

Case II can also be eliminated using (16). We first apply (16) to $R^*_{i,j} = (x_1, y_2)$ and $R^*_{i,j+1} = (x_2, y_1)$. If $\beta^*_{i,j+1} < \beta^*_{i,j}$ this decreases $S((\alpha_{ij}), (\beta^*_{ij}))$. We may therefore assume $\beta^*_{i,j+1} = \beta^*_{i,j}$. We next apply (16) to $R^*_{i,j+1} = (x_1, y_2)$ and $R^*_{i+1,j+1} = (x_2, y_1)$.

This will decrease $S((\alpha_{ij}), (\beta^*_{ij}))$ unless $\beta^*_{i+1,j} = \beta^*_{i+1,j+1}$. Thus we may assume

$$\beta^*_{i,j+1} = \beta^*_{i,j} \text{ and } \beta^*_{i+1,j} = \beta^*_{i+1,j+1}.$$

We would now like to apply (16) to the points $R^*_{i,j} = (x_1, y_2)$ and $R^*_{i+1,j} = (x_2, y_1)$. However since $\beta^*_{i,j} = \beta^*_{i,j+1}$ we may not decrease $\beta^*_{i,j}$ and still maintain the decreasing condition on $\beta_{i,j}$. We will therefore apply (16) simultaneously to the pair of points $R^*_{i,j}$ and $R^*_{i+1,j}$ and to the pair $R^*_{i,j+1}$ and $R^*_{i+1,j+1}$. That is let $t > 0$ be a small number and replace $(\beta^*_{i,j})^q$ with $(\beta^*_{i,j})^q - t$ and replace $(\beta^*_{i+1,j})^q$ with $(\beta^*_{i+1,j})^q + t$ and at the same time replace $(\beta^*_{i,j+1})^q$ with $(\beta^*_{i,j+1})^q - t$ and replace $(\beta^*_{i+1,j+1})^q$ with $(\beta^*_{i+1,j+1})^q + t$. This process does not violate the decreasing condition on the β terms and two applications of (16) shows that it decreases $S((\alpha_{ij}), (\beta^*_{ij}))$. This eliminates case II.

Finally case III can be treated much like case II. We first apply (16) to $R^*_{i,j}$ and $R^*_{i+1,j}$. This shows we may assume $\beta^*_{i,j} = \beta^*_{i+1,j}$. Applying (16) to $R^*_{i,j+1}$ and $R^*_{i+1,j+1}$ shows we may assume

$$\beta^*_{i,j+1} = \beta^*_{i+1,j+1}.$$

Now, just as in case II, a simultaneous application of (16) to the pair $R^*_{i,j}$ and $R^*_{i,j+1}$ and to the pair $R^*_{i+1,j}$ and $R^*_{i+1,j+1}$ shows that $S((\alpha_{ij}), (\beta^*_{ij}))$ can be made smaller. This eliminates case III.

Since none of the cases I-IV are possible it follows that β^*_{ij} must be constant.

Since $H_{pq}((a_{ij}), (b_{ij}))$ is a homogeneous function of b_{ij} we may use any constant sequence, for example $b_{ij} = 1$, to calculate the maximum value. The fact that $C < 1$ follows from the classical Hölder inequality since a_{ij} is strictly decreasing. This proves Theorem 6.

As another example of the method we consider vectors $\vec{a}$, $\vec{b}$ which for some positive constants N, D satisfy

$$(23) \qquad \|\vec{a}\|_p \leqq N, \ \|\vec{b}\|_q \leqq N$$

$$(24) \qquad a_{i+1} - a_i \geqq D \quad b_i - b_{i+1} \geqq D.$$

Now (24) means that the sequence A_i is "rapidly increasing" and the sequence b_i is "rapidly decreasing". Therefore if N is small and D is large there will not be any vectors $\vec{a}$ and $\vec{b}$ which can satisfy both (23) and (24).

In order to insure such vectors exist we will assume the inequalities (23) are both satisfied by the specific choice of vectors $a_i = (i - 1)D$ and $b_i = (n - i)D$. Putting this choice into (23) yields two compatibility conditions which we assume the constants N and D satisfy:

$$(25) \qquad D^p \sum_{j=1}^{n} (j - 1)^p < N^p, \quad D^q \sum_{j=1}^{n} (n - j)^q < N^q.$$

Under the conditions (24) it is not possible for a_i^p to be proportional to b_i^q. Normalizing the vectors $\vec{a}$ and $\vec{b}$ gives vectors $\vec{\alpha}$ and $\vec{\beta}$ which satisfy

$$(26) \qquad \alpha_{i+1} - \alpha_i \geqq d \quad \beta_i - \beta_{i+1} \geqq d$$

where $d = D/N$. We will show that the maximum of $H_{pq}(\vec{\alpha}, \vec{\beta})$ occurs for vectors $\vec{\delta}$ and $\vec{\gamma}$ for which equality always occurs in (26). We will first define the maximizing vectors, then we will show they work. Define δ_i and γ_i by

$$(27) \qquad \delta_i = \delta_1 + (i - 1)d \quad \gamma_i = \gamma_n + (n - i)d,$$

where the selection of δ_1 and γ_n is made to insure

$$\|\vec{\delta}\|_p = \|\vec{\gamma}\|_q = 1.$$

This requires δ_1 and γ_n to be roots of the equations

$$(28) \qquad \sum_j [\delta_1 + (j - 1)d]^p = 1$$

$$(29) \qquad \sum_j [\gamma_n + (n - j)d]^q = 1.$$

We must prove that these equations do in fact have solutions for δ_1 and γ_n. Define a function of δ_1, say $g(\delta_1)$, by

$$g(\delta_1) = \|\vec{\delta}\|_p^p = \sum_j [\delta_1 + (j - 1)d]^p.$$

Now $g(\delta_1) \to \infty$ if $\delta_1 \to \infty$. Furthermore equation (25) implies $g(0) < 1$ since $d = D/N$. Now $g(\delta_1)$ is continuous so it follows that (28) has a solution for δ_1. We can prove that (29) has a solution for γ_n using (25). We now have:

THEOREM 7. *Let $p + q = pq, p, q > 1$. Suppose $\vec{a}$ and $\vec{b}$ are vectors which satisfy* (23), (24) *and N, D are constants satisfying* (25). *Then there exists a constant $C < 1$ such that for all $\vec{a}, \vec{b}$ we have*

$$\sum a_j b_j \leqq C[\sum a_j^p]^{1/p} [\sum b_j^p]^{1/q}.$$

Equality holds if $\vec{a} = N\vec{\delta}$ and $\vec{b} = N\vec{\gamma}$ where $\vec{\delta}, \vec{\gamma}$ are defined by (27), (28) *and* (29). *The constant C may be calculated using*

$$C = H_{pq}(\vec{\delta}, \vec{\gamma}).$$

Proof. To prove the theorem let $\vec{\alpha}^*$ and $\vec{\beta}^*$ be the vectors which minimize $S(\vec{\alpha}, \vec{\beta})$ over all $\vec{\alpha}, \vec{\beta}$ satisfying (26) and $||\vec{\alpha}||_p = ||\vec{\beta}||_q = 1$. We need to show $\vec{\alpha}^* = \vec{\delta}$ and $\vec{\beta}^* = \vec{\gamma}$. We will first show that $\alpha^*_{i+1} = \alpha^*_i + d$ for all i. Suppose not. Then there is some index i for which $\alpha^*_{i+1} > \alpha^*_i + d$. We now use Theorem I (17) in the form

$$(x_1, y_2) = ((\alpha^*_i)^p, (\beta^*_i)^q), (x_2, y_1) = ((\alpha^*_{i+1})^p, (\beta^*_{i+1})^q).$$

This shows how to make $S(\vec{\alpha}^*, \vec{\beta}^*)$ even smaller which contradicts the definition of $\vec{\alpha}^*, \vec{\beta}^*$. Therefore $\alpha^*_{i+1} = \alpha^*_i + d$ for all i. From this it is a simple matter to prove $\alpha^*_i = \alpha^*_1 + (i - 1)d$. Similarly $\beta^*_i = \beta^*_n + (n - i)d$. Now the condition

$$||\vec{\alpha}^*||_p = ||\vec{\beta}^*||_q = 1$$

implies that $\vec{\alpha}^* = \vec{\delta}$ and $\vec{\beta}^* = \vec{\gamma}$. It follows that

$$H_{pq}(\vec{a}, \vec{b}) = (\vec{\alpha}, \vec{\beta}) \leqq (\vec{\delta}, \vec{\gamma})$$

which proves the theorem.

We now consider an example which uses Theorem 2 with $p, q < 1$. Suppose that $p < 0$ and $q > 0$ and that D and N are constants. Let (b_i) be a fixed strictly increasing sequence. We will consider the class of all sequences (a_i) which satisfy

$$(30) \qquad 0 < a_i \leqq D \quad ||\vec{a}||_p = (\textstyle\sum a_i^p)^{1/p} \geqq N.$$

We normalize (a_i) and (b_i) to obtain vectors $\vec{\alpha}, \vec{\beta}$ satisfying

$$(31) \qquad \alpha_i \leqq d, \alpha_i^p \geqq d^p, ||\vec{\alpha}||_p = 1, ||\vec{\beta}||_q = 1,$$

where $d = D/N$. It follows from (31) that

$$(32) \qquad 1 = \sum_j \alpha_j^p \geqq \sum_j d^p = nd^p.$$

In order to avoid the trivial case in which all $\alpha_i^p = 1/n$ we will assume that $1 > nd^p$. If $d^p \leqq \beta_1^q$ then we may take $\alpha_i^p = \beta_i^q$ *for all i*. Thus

$$H_{pq}(\vec{\alpha}, \vec{\beta}) = 1$$

and no supplementary inequality is possible. We will therefore assume for the following that the constant $d = D/M$ satisfies

$$(33) \qquad \beta_1^q < d^p < 1/n.$$

Now let (β_i) be a fixed increasing sequence,

$$\beta_i < \beta_{i+1}, \; \| (\beta_i) \|_q = 1.$$

For this sequence (β_i) we propose to calculate the minimum of $H_{pq}(\vec{\alpha}, \vec{\beta})$ over all sequences (α_i) satisfying (31). We will first define the sequence (δ_i) which minimizes $H_{pq}(\vec{\alpha}, \vec{\beta})$ then show that it works.

Let m be a parameter to be chosen later and define a sequence (δ_i) by

$$(34) \qquad \delta_i^p = \text{MAX}\, \{d^p, \beta_i^q/m\}.$$

The parameter m will be selected to insure $\| (\delta_i) \|_p = 1$. Define a function $g(m)$ by

$$g(m) = \| (\delta_i) \|_p^p = \sum_j (\delta_j)^p = \sum_j \text{MAX}\, \{d^p, \beta_j^q/m\}.$$

Thus m must be a root of the equation $g(m) = 1$. We will first prove that such an m exists. Since $d^p > \beta_1^q$ it follows that

$$g(1) = \sum_j \text{MAX}\, \{d^p, \beta_j^q\} > \sum_j \beta_j^q = 1.$$

Also

$$g(\beta_n^q/d^p) = \sum_j \text{MAX}\, \{d^p, (\beta_j/\beta_n)^q d^p\} = \sum d^p = nd^p < 1.$$

In the range of values $1 < m < \beta_n^q/d^p$ the function $g(m)$ is strictly decreasing. Therefore there is a unique solution of the equation $g(m) = 1$. Henceforth m will denote that unique root.

Incidentally it follows from (33) that $N^p \geqq nD^p$ a relationship which is implicit in (30). It is a compatability condition in the sense that if it is violated then there will not exist any sequence (a_i) which can simultaneously satisfy both parts of (30).

The sequence (δ_i) minimizes $H_{pq}(\vec{\alpha}, \vec{\beta})$ as the following theorem shows:

THEOREM 8. *Let $p + q = pq$ with $p < 0$, $q > 0$. Let (b_i) be a fixed increasing sequence $0 \leqq b_i < b_{i+1}$ and let N, D be constants satisfying $N^p > nD^p$. Let (a_i) be any sequence which satisfies* (30). *Then there is a constant C which is independent of (a_i) and satisfies*

$$(35) \qquad \sum a_i b_i \geqq C\left[\sum_j a_j^p\right]^{1/p} \left[\sum_j b_j^q\right]^{1/q}.$$

If $D^p > \beta_1^q N^p$ then $C > 1$ and equality holds in (35) *if and only if $(a_i) = N(\delta_i)$ where δ_i is defined by* (34). *The constant C may be calculated using*

$$C = H_{pq}(\vec{\delta}, \vec{\beta}).$$

If $D^p \leqq \beta_1^q N^p$ then no supplementary inequality is possible.

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Proof. Let (b_i) be the given increasing sequence and let (a_i) be any sequence which satisfies (30). Normalizing the sequences (a_i), (b_i) we obtain (α_i), (β_i) which satisfy (31). Let (α_i^*) be that sequence which minimizes $H(\vec{\alpha}, \vec{\beta})$ over all sequences (α_i) satisfying (31). We will prove $\alpha_i^* = \delta_i$. Consider the points

$$R_i^* = ((\alpha_i^*)^p, \beta_i^q).$$

Not all these points lie on the line $u = d^p$ since otherwise we have $(\alpha_i^*)^p = d^p$ for all i. Summing on i gives $1 = nd^p$ which contradicts (33). Now let R_k^* be the first point which is not on the line $u = d^p$. Suppose the equation of the line joining (0, 0) and R_k^* is $v = m^*u$. If all points which are not on the line $u = d^p$ are on the line $v = m^*u$ then (since m is uniquely defined) it follows that $m^* = m$, $\alpha_i^* = \delta_i$ and we are done. Suppose then that at least one point, say R_i^*, for $i > k$ is not on the line $u = m^*v$. Theorem 4 shows that the minimum of $h(R_i^*) + h(R_k^*)$ occurs when R_i^*, R_k^* and (0, 0) are all collinear. Since R_i^*, R_k^* and (0, 0) are not collinear R_i^* cannot minimize $H_{pq}(\vec{\alpha}, \vec{\beta})$. Thus contradiction shows that all points R_i^* which are not on the line $u = d^p$ are collinear which proves the theorem.

We now show how to obtain integral analogues of some of these results. Given an interval $0 \leqq x \leqq \ell$ we partition it into n subintervals of length $\Delta x = \ell/n$,

$$(39) \qquad 0 = x_0 < x_1 < x_2 < \ldots < x_n = \ell, \quad x_i = i\Delta x = i(\ell/n).$$

On each interval $x_{i-1} \leqq x \leqq x_i$ we replace $f(x)$ and $g(x)$ with constants f_i and g_i. We may select f_i, g_i in various ways but one convenient way to do it is to pick some $\xi_i \in [x_{i-1}, x_i]$ and define $f_i = f(\xi_i)$ and $g_i = g(\xi_i)$. Then if f and g are continuous it follows that the piecewise constant functions F_n and G_n obtained in this way will approximate f and g. Furthermore

$$(40) \qquad \begin{aligned} &\int_0^\ell F_n G_n dx \to \int_0^\ell fg dx, \\ &\int_0^\ell F_n^p dx \to \int_0^\ell f^p dx, \\ &\int_0^\ell G_n^q dx \to \int_0^\ell g^q dx, \end{aligned}$$

as $n \to \infty$. We therefore obtain

$$(41) \qquad H_{pq}(f, g) = \lim_{n \to \infty} H_{pq}(F_n, G_n).$$

Now however $H_{pq}(F_n, G_n)$ is an expression which involves finite sums. We may apply some of our preceding results to this expression and thus obtain supplementary inequalities for integrals.

As an example of this process we will prove the following:

THEOREM 9. *Let $p + q = pq$ and $p, q > 1$. Let $f(x)$ be a given continuous and strictly increasing function on $0 \leqq x \leqq \ell$. Then there is a constant $C < 1$ such that for any decreasing function $g(x)$,*

$$\int_0^\ell fg dx \leqq C[\int_0^\ell f^p dx]^{1/p} [\int_0^\ell g^q(x) dx]^{1/q}.$$

Equality holds if $g(x)$ is constant for all x. The value of C may be calculated using

$$C = H_{pq}(f(x), 1).$$

Let f and g be functions on $0 \leqq x \leqq \ell$ with f continuous and strictly increasing and g decreasing. Partition the interval $0 < x < \ell$ as described by (39). Construct f_i and g_i using $f_i = f(x_i)$, $g_i = g(x_i)$. Let $F_n(x)$ and $G_n(x)$ be the corresponding piecewise constant functions. We see that

$$(42) \qquad H_{pq}(F_n(x), G_n(x)) = H_{pq}(\vec{a}, \vec{b})$$

where vectors $\vec{a}$ and $\vec{b}$ are defined by

$$\vec{a} = (f_1, f_2, \dots, f_n)$$

$$\vec{b} = (g_1, g_2, \dots, g_n).$$

We remark that the left hand side of (42) is defined by (2) but the right hand side of (42) is defined by (1). The factors Δx have cancelled out. We now apply Theorem 5 to obtain

$$H_{pq}(F_n(x), G_n(x)) \leqq H_{pq}(\vec{a}, 1).$$

Now however it also follows that

$$H_{pq}(F_n, 1) = H_{pq}(\vec{a}, 1).$$

This implies that

$$H_{pq}(F_n, G_n) \leqq H_{pq}(f_n, 1).$$

Letting $n \to \infty$ and using (40) yields

$$H_{pq}(f, g) \leqq H_{pq}(f, 1) = C.$$

Since f is strictly increasing, $C < 1$. This proves Theorem 9.

Although (41) can be used to obtain integral analogues of many of our results it is an awkward tool and a more direct approach would be desirable. In particular, integral analogues of Theorems 1, 2 and 3 would be nice to have.

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Washington State University,
Pullman, Washington