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A TRIPLE IN CAT

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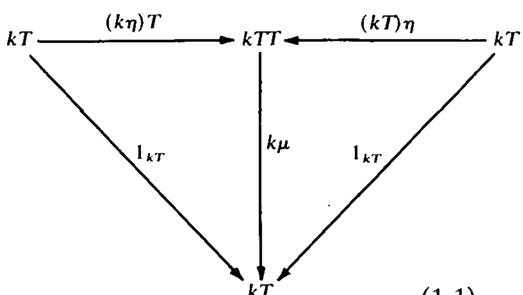
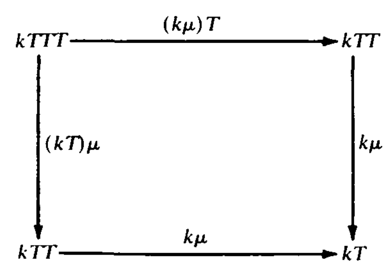
1. Introduction

A *triple* (or *monad*) in a category $\boldsymbol{K}$ is a triple $\mathcal{T}=(T, \mu, \eta)$ where $T: \boldsymbol{K}\to\boldsymbol{K}$ is a functor and $\mu: TT\to T$, $\eta: 1_{\boldsymbol{K}}\to T$ are natural transformations for which (1.1) and (1.2) commute:



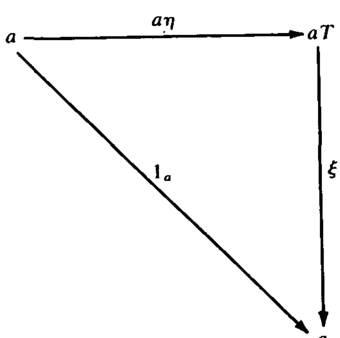
(1.1)



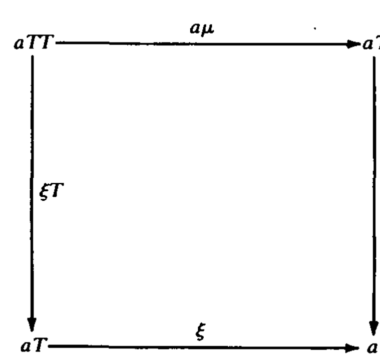
(1.2)

In these diagrams the component of a natural transformation α at an object x is denoted $x\alpha$. Thus for example $(k\eta)T$ is the value of the functor T applied to the component of η at k, whereas $(kT)\eta$ is the component of η at the object kT. I write functions and functors on the right and composition from left to right.

A pair (a, ξ) is a $\mathcal{T}$-*algebra* if a is an object of $\boldsymbol{K}$, $\xi: aT\to a$, and (1.3) and (1.4) commute:

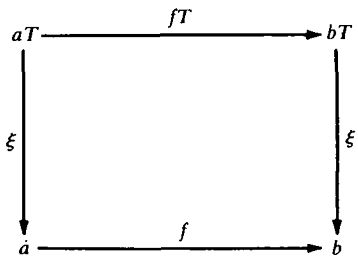


(1.3)



(1.4)

If (a, ξ), (b, ζ) are $\mathcal{T}$-algebras, an arrow $f: a \to b$ is a *$\mathcal{T}$-algebra homomorphism* if (1.5) commutes:



(1.5)

The $\mathcal{T}$-algebras and $\mathcal{T}$-algebra homomorphisms form a category $\boldsymbol{K}^{\mathcal{T}}$. Furthermore, the functor $U: \boldsymbol{K}^{\mathcal{T}} \to \boldsymbol{K}$ taking (a, ξ) to a and (f, fT) to f has a left adjoint $F: \boldsymbol{K} \to \boldsymbol{K}^{\mathcal{T}}$ such that $T = FU$. Details may be found in MacLane (**4**, Ch. VI) or Manes (**5**).

It is well known (**4**, VI.4) that the functor $*$: **Sets**→**Sets** which takes a set X to the set X^* or words (of finite length $\geqq 0$) in X, and a function $\phi: X \to Y$ to the obvious induced function $\phi^*: X^* \to Y^*$, is the functor part of a triple whose category of algebras is isomorphic to the category of monoids. (If the empty word is excluded one gets the category of semigroups.)

In this note I shall show how monoids "are" the algebras of a triple $\mathcal{D}$ in **Cat** (the category of small categories and functors). I put "are" in quotes because one gets only an equivalence, not an isomorphism, between **Mon** and the category of $\mathcal{D}$-algebras.

The part of the triple that corresponds to the underlying set functor U for $*$ is Leech's functor D: **Mon**→**Cat** (**2**), (**3**). Given a monoid M, the objects of the small category MD are the elements of M, and the arrows are 3-tuples (k, m, n) of elements of M, with dom $(k, m, n) = m$, cod $(k, m, n) = kmn$, and composition satisfying

$$(k, m, n) \circ (k', kmn, n') = (k'k, m, nn'). \tag{1.6}$$

If $f: M \to M'$ is a homomorphism, then the corresponding functor $fD: MD \to M'D$ is defined by

$$(k, m, n)fD = (kf, mf, nf). \tag{1.7}$$

I shall construct a functor Δ : **Cat**→**Mon** which is left adjoint to D; then by the general theory of triples it will follow that ΔD is the functor part of a triple $\mathcal{D}$ in **Cat**. I shall then show directly that the category of $\mathcal{D}$-algebras is equivalent to **Mon**.

2. The functor Δ

Given a category $\boldsymbol{C}$, let $\boldsymbol{C}^L$ and $\boldsymbol{C}^R$ be two disjoint copies of the set of arrows of $\boldsymbol{C}$. If $f: b \to c$ in $\boldsymbol{C}$ I shall write f^L for f in its role as an element of $\boldsymbol{C}^L$ and call it a "left arrow". and f^R for f as an element of $\boldsymbol{C}^R$, a "right arrow". Let $\boldsymbol{C}^o$ denote the set of objects of $\boldsymbol{C}$, where $\boldsymbol{C}^o$ is taken to be disjoint from each of $\boldsymbol{C}^L$ and $\boldsymbol{C}^R$.

Let us write elements of the free monoid on $\boldsymbol{C}^L \cup \boldsymbol{C}^R \cup \boldsymbol{C}^o$ in triangular brackets; thus $\langle f^L, g^L, c, f^R \rangle$ is a typical element if f, g are arrows of $\boldsymbol{C}$ and c an object of $\boldsymbol{C}$.

Let $\sim$ be the congruence relation on this free monoid generated by requiring, for arrows $f: b \to c$, $g: c \to d$, $h: d \to e$ of $\boldsymbol{C}$,

$$\langle h^L, g^L\rangle = \langle (g \circ h)^L\rangle \tag{2.1}$$

$$\langle f^R, g^R\rangle = \langle (f \circ g)^R\rangle \tag{2.2}$$

$$\langle f^L, b, f^R\rangle = \langle c\rangle. \tag{2.3}$$

Finally, let $\boldsymbol{C}\Delta$ be the quotient monoid of the free monoid on $\boldsymbol{C}^L \cup \boldsymbol{C}^R \cup \boldsymbol{C}^0$ by the congruence $\sim$. Write equivalence classes in square brackets; thus for f, g as above, $[f^L, b, f^R, g^R] = [c, g^R] = [f^L, b, (f \circ g)^R] \in \boldsymbol{C}\Delta$. Multiplication is then by concatenation:

$$[f^L, b].[c, b, g^k] = [f^L, b, c, b, g^k]. \tag{2.4}$$

If $F: \boldsymbol{C} \to \boldsymbol{D}$ is a functor, let $\bar{F}: \boldsymbol{C}^L \cup \boldsymbol{C}^R \cup \boldsymbol{C}^o \to \boldsymbol{D}^L \cup \boldsymbol{D}^R \cup \boldsymbol{D}^0$ be defined for an object c and an arrow f by

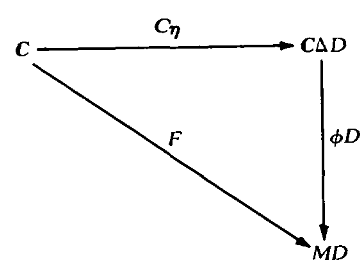
$$c\bar{F} = cF,\ f^L\bar{F} = (fF)^L,\ f^R\bar{F} = (fF)^R, \tag{2.5}$$

and set

$$[x_1, x_2, \ldots, x_n]F\Delta = [x_1\bar{F}, x_2\bar{F}, \ldots, x_n\bar{F}] \tag{2.6}$$

for $x_i \in \boldsymbol{C}^L \cup \boldsymbol{C}^R \cup \boldsymbol{C}^o$. This is well-defined and makes Δ: **Cat**$\to$**Mon** a functor.

To show that Δ is left adjoint to D, it suffices (**4**, Theorem IV.2) to construct a natural transformation $\eta: 1_{\mathbf{Cat}} \to \Delta D$ with the property that if M is any monoid and $F: M \to \boldsymbol{C}D$ a functor, then there is a unique homomorphism $\phi: \boldsymbol{C}\Delta \to M$ such that



(2.7)

commutes. (This η will be the η of the triple.)

For a category $\boldsymbol{C}$, define $\boldsymbol{C}\eta: \boldsymbol{C} \to \boldsymbol{C}\Delta D$ by

$$c.\ \boldsymbol{C}\eta = [c] \tag{2.8}$$

$$f.\ \boldsymbol{C}\eta = ([f^L], [\mathrm{dom}\, f], [f^R]). \tag{2.9}$$

It is straightforward to verify that $\boldsymbol{C}\eta$ is a functor and η is a natural transformation.

Given $F: \boldsymbol{C} \to MD$, the requisite ϕ making (2.7) commute is defined this way: Suppose $f: b \to c$ in $\boldsymbol{C}$ and $fF = (k, m, n)$ in MD. Then $[b]\phi = m$, $[f^L]\phi = k$, and $[f^R]\phi = n$.

3. The triple

By the general theory of triples the adjunction $\Delta \vdash D$ gives rise to a triple $\mathcal{D} = (\Delta D, \eta, \mu)$ where ΔD and η have already been defined. Following (**4**, p. 134), μ must be defined in terms of the co-unit $\varepsilon: D\Delta \to 1_{\mathbf{Mon}}$ of the adjunction, which is defined for each monoid M on the generators of $MD\Delta$ by

$$[m]M\varepsilon = m, \tag{3.1}$$

$$[(k, m, n)^L]M\varepsilon = k, \text{ and} \tag{3.2}$$

$$[(k, m, n)^R]M\varepsilon = n. \tag{3.3}$$

The value of $M\varepsilon$ on the equivalence class of a string is obtained by multiplying (in M) the values at each entry.

The natural transformation μ is by definition $\Delta\varepsilon D: \Delta D\Delta D \to \Delta D$; that is, for a category $\boldsymbol{C}$, $\boldsymbol{C}\mu: \boldsymbol{C}\Delta D\Delta D \to \boldsymbol{C}\Delta D$ is the result of applying D to the component at $\boldsymbol{C}\Delta$ of the natural transformation ε: thus $\boldsymbol{C}\mu = (\boldsymbol{C}\Delta)\varepsilon D$.

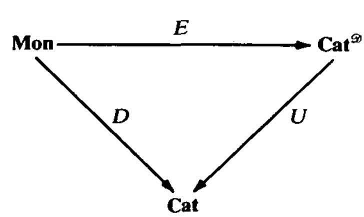
The way $\boldsymbol{C}\mu$ acts is best illustrated by an example. A typical object of the category $\boldsymbol{C}\Delta D\Delta D$ is a string of equivalence classes of strings and "left" and "right" triples of equivalence classes of strings like

$$\Gamma = [[f^L, c], ([c, b, f^R], [f^L], [g^R, f^L, c])^R, ([g^L], [c, f^R], [b, g^R])^L],$$

where b, c are objects of $\boldsymbol{C}$ and f and g are arrows. Then $\Gamma\boldsymbol{C}\mu = [f^L, c, g^R, f^L, c, g^L]$, an object of $\boldsymbol{C}\Delta D$. An arrow of $\boldsymbol{C}\Delta D\Delta D$ is an ordered triple of such strings, on which because of (1.7) $\boldsymbol{C}\mu$ acts coordinatewise by the same rule.

4. The main theorem

Theorem. *The category* $\mathbf{CAT}^{\mathscr{D}}$ *of* $\mathscr{D}$*-algebras is equivalent to the category of monoids and monoid homomorphisms by a functor E making*



commute.

Proof. The functor E is the well known comparison functor (**4**, VI.3), (**5**, 2.2.21) which takes a monoid M to the algebra $M\varepsilon D: MD\Delta D \to MD$ and a homomorphism $f: M \to M'$ to $(fD, fD\Delta D)$. One could presumably deduce that E is an equivalence by using one of the criteria developed by Beck (**1**), (**4**, VI.7, Exercise 6), (**6**, 21.5.7), but that involves coequalizers in **Cat**, which I hate, so I shall prove directly that E is an equivalence by showing that it is full and faithful and that every $\mathscr{D}$-algebra is isomorphic to an algebra of the form $(MD, M\varepsilon D)$.

That E is faithful follows from the fact that D is faithful.

Suppose $H: MD \to M'D$ is a functor such that $(H, H\Delta D)$ is a morphism of $\mathscr{D}$-algebras from ME to $M'E$, so that

$$\begin{array}{ccc} MD\Delta D & \xrightarrow{H\Delta D} & M'D\Delta D \\ \Big\downarrow{\scriptstyle M\varepsilon D} & & \Big\downarrow{\scriptstyle M'\varepsilon D} \\ MD & \xrightarrow{H} & M'D \end{array} \tag{4.1}$$

commutes. To show that E is full, it is sufficient to show (for every such H) that $H = hD$ for some monoid homomorphism $h: M \to M'$.

I shall first show that for all $k, m, n \in M$,

$$(k, m, n)H = (kH, mH, nH). \tag{4.2}$$

Let

$$(k, m, n)H = (k', mH, n') \tag{4.3}$$

for some $k', n' \in M'$ (we know the domain of $(k, m, n)H$ is mH). Then

$$\begin{aligned} nH &= [(k, m, n)^R]M\varepsilon D . H \\ &= [(k, m, n)^R]H\Delta D . M'D && \text{by (4.1)} \\ &= [(k', mH, n')^R]M'\varepsilon D && \text{by (2.5) and (4.3)} \\ &= n' && \text{by (3.3).} \end{aligned}$$

Similarly $kH = k'$, so (4.2) is proved.

Applying this to $(1_M, 1_M, 1_M)$ it follows that $1_M H$ is the unity of M'. Also

$$\begin{aligned} (mn)H &= [m, n]M\varepsilon D . H && \text{(definition of } \varepsilon) \\ &= [m, n]H\Delta D . M'\varepsilon D && (4.1) \\ &= [mh, nh]M'\varepsilon D && (2.6) \\ &= mHnH && \text{(definition of } \varepsilon), \end{aligned}$$

so that H restricted to the objects of MD, i.e. to M, is a monoid homorphism from M to M', which I shall denote h. It is then immediate from (1.7) and (4.2) that $hD = H$.

Finally, given a $\mathcal{D}$-algebra ξ: $\boldsymbol{C}\Delta D \to \boldsymbol{C}$, it is necessary to construct a monoid M such that the algebra $(MD, M\varepsilon D)$ is isomorphic in $\mathbf{Cat}^{\mathcal{D}}$ to $(\boldsymbol{C}, \xi)$. For this purpose, I shall repeatedly need formulas (4.4) through (4.8) below.

For any object k of $\boldsymbol{C}$,

$$[k]\xi = k. \tag{4.4}$$

This follows from (1.3) with $a = \boldsymbol{C}$, $T = \Delta D$.

Let $w_1, w_2, \ldots, w_s$ be elements of $\boldsymbol{C}\Delta$ and W their product in $\boldsymbol{C}\Delta$. Then

$$W\xi = [w_1\xi, w_2\xi, \ldots, w_s\xi]\xi. \tag{4.5}$$

(Note that W is an object of $\boldsymbol{C}\Delta D$ so that the left side makes sense.) This is obtained from (1.4) with $a = \boldsymbol{C}$, $T = \Delta D$ by chasing the object $[w_1, w_2, \ldots, w_s]$ around the diagram.

If $w_1, w_2, w_3 \in \boldsymbol{C}\Delta$. then

$$(w_1, w_2, w_3)\xi = ([w_1\xi], [w_2\xi], [w_3\xi])\xi, \tag{4.6}$$

similarly obtained from (1.4) by chasing the arrow $([w_1],[w_2],[w_3])$.

Finally, by using (1.4) on $[([k],[m],[n])^L]$ and $[([k],[m],[n])^R]$, one has

$$k=[([k],[m],[n])^L\xi\Delta]\xi \text{ and} \tag{4.7}$$

$$n=[([k],[m],[n])^R\xi\Delta]\xi. \tag{4.8}$$

Now let M be the set of objects of C, and for $m, n \in M$, let

$$mn=[m,n]\xi. \tag{4.9}$$

Then $(km)n=[[k,m]\xi,n]\xi=[[k,m]\xi,[n]\xi]\xi=[k,m,n]\xi$ by (4.4) and (4.5) and similarly $k(mn)=[k,m,n]\xi$, so the multiplication is associative. The unity is $\wedge\xi$, where $\wedge$ is the empty word.

Define a functor $\Phi: MD\to C$ by making Φ be the identity map on objects and for an arrow $(k,m,n): m\to kmn$,

$$(k,m,n)\Phi=([k],[m],[n])\xi. \tag{4.10}$$

It follows from (4.4) that the domain of the right side is $[m]\xi=m$ and from a remark in the preceding paragraph that the codomain is $[k,m,n]\xi=kmn$.

It follows from (4.10), (4.6) and (4.4) that

$$(k',kmn,n')\Phi=([k'],[k,m,n],[n'])\xi$$

and

$$(k'k,m,nn')\Phi=([k',k],[m],[n,n'])\xi$$

so that by (1.6) and the fact that ξ is a functor, Φ preserves composition.

I shall now construct an inverse $\Psi: C\to MD$ to Φ (so I need not show Φ preserves identity arrows). Ψ is (naturally) the identity on objects, and for $f: m\to p$ in C,

$$f\Psi=([f^L]\xi,m,[f^R]\xi). \tag{4.11}$$

The domain of f is obviously m, and the codomain is

$$\begin{aligned}[f^L]\xi\cdot m\cdot[f^R]\xi&=[[f^L]\xi,[f^R]\xi]\xi\\&=[f^L,m,f^R]\xi=[p]\xi=p,\end{aligned}$$

where the second equality comes from (4.4) and (4.5) and the third from (2.3).

Then

$$\begin{aligned}(k,m,n)\Phi\Psi&=([([k],[m],[n])\xi^L]\xi,m,[([k],[m],[n])\xi^R]\xi] && \text{(4.10) and (4.11)}\\&=(k,m,n) && \text{(2.6) (4.7), and (4.8)}\end{aligned}$$

and for $f: m\to p$ in C,

$$\begin{aligned}f\Psi\Phi&=([[f^L]\xi],[[m]\xi],[[f^R]\xi])\xi && \text{(4.11) and (4.10)}\\&=[[f^L],[m],[f^R])\xi && \text{(4.6)}\\&=f.C\eta.\xi=f && \text{(2.8) and (1.3).}\end{aligned}$$

Thus Ψ is the inverse of Φ.

I shall now show that the diagram

$$\begin{array}{ccc} \mathbf{C}\Delta D & \xrightarrow{\Psi\Delta D} & M D\Delta D \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle m\varepsilon D} \\ \mathbf{C} & \xrightarrow{\Psi} & MD \end{array} \tag{4.12}$$

commutes, so that Ψ (hence also Φ) is a morphism in $\mathbf{Cat}^{\mathcal{D}}$. This will complete the proof of the Theorem.

If m is an object of $\mathbf{C}$ and f, g arrows, then $[m, f^L, g^R]$ is an object of $\mathbf{C}\Delta D$ sufficiently general to illustrate the commutativity of (4.12) without involving us in subscripts. On the one hand,

$$\begin{aligned}[m, f^L, g^R]\Psi\Delta D.\, M\varepsilon D &= [m, ([f^L]\xi, \text{dom } f, [f^R]\xi)^L, ([g^L]\xi, \text{dom } g, [g^R]\xi)^R]M\varepsilon D \\ &= m\cdot[f^L]\xi\cdot[g^R]\xi,\end{aligned}$$

whereas because Ψ is the identity on objects,

$$\begin{aligned}[m, f^L, g^R]\xi\Psi &= [[m]\xi, [f^L]\xi, [g^R]\xi]\xi && (4.5)\\ &= m\cdot[f^L]\xi\cdot[g^R]\xi && (4.4)\end{aligned}$$

Thus (4.12) commutes for objects.

If w_1, w_2, w_3 are elements of $\mathbf{C}\Delta$, then

$$\begin{aligned}&(w_1, w_2, w_3)\Psi\Delta D.\, M\varepsilon D \\ &\quad = (w_1\Psi\Delta.\, M\varepsilon,\ w_2\Psi\Delta.\, M\varepsilon,\ w_3\Psi\Delta.\, M\varepsilon) && (1.7)\\ &\quad = (w_1\xi, w_2\xi, w_3\xi)\end{aligned}$$

which follows from the commutativity of (4.12) for objects (w_i is both an element of $\mathbf{C}\Delta$ and an object of $\mathbf{C}\Delta D$).

On the other hand, by (4.6)

$$(w_1, w_2, w_3)\xi\Psi = A\xi\Psi$$

where $A = ([w_1\xi], [w_2\xi], [w_3\xi])$. Then by (4.11),

$$\begin{aligned}A\xi\Psi &= ([A\xi^L]\xi, w_2\xi, (A\xi^R]\xi) \\ &= (w_1\xi, w_2\xi, w_3\xi) && (4.7), (4.8), (2.6).\end{aligned}$$

This proves the Theorem.

5. Remarks

1. **Mon** is not isomorphic to $\mathbf{Cat}^{\mathcal{D}}$; this follows from the precise tripleability theorem, since any parallel pair in **Cat** has a coequalizer which is not D of anything.

2. There are triples $\mathcal{L}$ and $\mathcal{R}$ corresponding to Leech's functors L and R in the same way that $\mathcal{D}$ corresponds to D, and a proof very similar to the one given here shows that

Cat$^{\mathscr{L}}$ and **Cat**$^{\mathscr{R}}$ are both equivalent to **Mon**. The left adjoint to R, for example, is constructed from the free monoid on $\boldsymbol{C}^R \cup \boldsymbol{C}^o$ using a congruence satisfying (2.2) and

$$\langle b, f^R \rangle = \langle c \rangle$$

for $f: b \to c$ in $\boldsymbol{C}$.

3. The extension theory which corresponds to $\mathscr{D}$ will be developed in a later paper. The extension theory for * is discussed in (**7**).

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