Canad. Math. Bull. Vol. **30** (2), 1987

HOW INTRICATE ARE $(2s + 1)$-FACTORIZATIONS?

BY

A. J. W. HILTON

ABSTRACT. The intricacy of the problem of $(2s + 1)$-factorizating K_n is determined. Some generalizations are also given.

The subject of "The intricacy of combinatorial construction problems" was introduced by W. E. Opencomb in [6], where a formal definition was given, together with a wide variety of examples. Many problems have intricacy 1; for example, the problem of filling up a latin square row by row [an $r \times n$ $(r < n)$ latin rectangle on n symbols can always be extended to an $(r + 1) \times n$ latin rectangle]. When the intricacy of a problem is not 1, then usually all that is known are bounds on the value of the intricacy. For example, the intricacy of the problem of completing partial latin squares (without the stipulation that the completion be done row-by-row) is between 2 and 4: what is meant by this is that the elements of any partial $n \times n$ latin square L on n symbols can be shared between 4 $n \times n$ matrices of cells to form 4 partial $n \times n$ latin squares (the sharing being performed in such a way that if a cell (i,j) of L is occupied by a symbol σ, then in exactly one of L_1, L_2, L_3, L_4 is the cell (i, j) occupied by σ, and in the other partial latin squares, cell (i, j) is unfilled), and the sharing can be done so that each of L_1, L_2, L_3 and L_4 can be completed; thus the intricacy is at most 4. Furthermore, there are some partial latin squares which cannot be completed, so the intricacy is at least 2. It is usually thought that the intricacy of this problem and of a number of similar problems is 2. But apart from the problem of filling an $n \times n$ chess board with dominos, and a problem involving Cayley tables, for no problem where the intricacy is not 1 is the exact value of the intricacy known. Here we introduce another problem for which the intricacy is not 1 but can be determined.

Let n be even, let $1 \leqq 2s + 1 < n$ and let $n - 1 = q(2s + 1) + r$, where $0 \leqq r < 2s + 1$. Let $Q(2s + 1, n)$ be the problem of finding q edge-disjoint $(2s + 1)$-factors in K_n (leaving an r-factor). Suppose that a student, acting without foresight, attempted to find q edge-disjoint $(2s + 1)$-factors as follows. First he removed a $(2s + 1)$-factor F_1, then he removed a $(2s + 1)$-factor F_2 from $K_n \backslash F_1$, then he removed a $(2s + 1)$-factor F_3 from $K_n \backslash (F_1 \cup F_2)$, and so on; but after a while he gets stuck: the graph remaining, say $K_n \backslash (F_1 \cup \ldots \cup F_t)$, has no $(2s + 1)$-factor! We show that the student need not be unduly perturbed, for the t $(2s + 1)$-factors need not be wasted. The set $\{F_1, \ldots, F_t\}$

Received by the editors November 14, 1985, and, in revised form, July 8, 1986.
AMS Subject Classification (1980): 05C99

https://doi.org/10.4153/CMB-1987-034-2 Published online by Cambridge University Press

can be partitioned into two, $\{F_{\sigma(1)}, \ldots, F_{\sigma(w)}\}$ and $\{F_{\sigma(w+1)}, \ldots, F_{\sigma(t)}\}$, for some permutation σ of $\{1, \ldots, t\}$, in such a way that the graphs $K_n \backslash (F_{\sigma(1)} \cup \ldots \cup F_{\sigma(w)})$ and $K_n \backslash (F_{\sigma(w+1)} \cup \ldots \cup F_{\sigma(t)})$ contain $q - w$ and $q - (t - w)$ edge-disjoint $(2s + 1)$-factors, respectively. In other words, $\kappa(Q(2s + 1, n))$, the intricacy of the problem $Q(2s + 1, n)$, is in this case given by $\kappa(Q(2s + 1, n)) = 2$.

More precisely we have the following theorem.

THEOREM 1. *Let n be even, let $3 \leqq 2s + 1 < n$ and let $n - 1 = q(2s + 1) + r$, where $0 \leqq r < 2s + 1$. Then*

$$\kappa(Q(2s + 1, n)) = \begin{cases} 2 & \text{if} \quad q \geqq 4, \\ 1 & \text{if} \quad q \leqq 3. \end{cases}$$

It is shown by Chetwynd and Hilton in [1] that $\kappa(Q(1, n)) \leqq 7$, and it would follow from a conjecture in [1] that $\kappa(Q(1, n)) = 2$; this would be a deep result of considerable interest, so it is a little surprising that Theorem 1 is provable without a great deal of difficulty.

We shall need the following lemmas. The first is due to the author [4].

LEMMA 1. *Let $s \geqq 1$, let n be even, let $d = x(2s + 1) + y$, where $0 \leqq y < 2s + 1$, and let $n/2 \leqq d \leqq n - 1$. A regular simple graph of degree d and order n can be expressed as the union of x $(2s + 1)$-factors and a y-factor.*

The next is an old and very well known result of Petersen [7].

LEMMA 2. *A regular graph of even degree is the union of edge-disjoint 2-factors.*

The next is a nice result of Bill Jackson [5]; it, and Petersen's theorem, are the tools by which Lemma 1 was proved.

LEMMA 3. *Let G be a regular 2-connected graph of order $n \geqq 9$ and degree $d(G)$. Then G contains a set of*

$$\left[\frac{1}{6}(3d(G) - n + 4)\right]$$

edge-disjoint Hamiltonian circuits.

We also need the following result of Chvatal [2].

LEMMA 4. *Let G be a simple graph of order n with degrees $d_1 \leqq d_2 \leqq \ldots \leqq d_n$. If, for every k with $0 < k < n/2$, the following condition holds:*

$$d_k \leqq k \Rightarrow d_{n-k} \geqq n - k,$$

then G contains a Hamiltonian circuit.

Finally we shall use the following special case, due to Dirac [3], of Lemma 4.

LEMMA 5. *Let G be a simple graph of order $n \geqq 3$ and minimum degree $\delta(G)$. If $\delta(G) \geqq n/2$ then G contains a Hamiltonian circuit.*

PROOF OF THEOREM 1. The conditions imply that $q \geqq 1$. Suppose $q = 1$. By Lemma 1, K_n contains a $(2s + 1)$-factor. Therefore $\kappa(Q(2s + 1, n)) = 1$.

Now suppose $q = 2$. By Lemma 1, K_n contains two edge-disjoint $(2s + 1)$-factors. Furthermore, suppose a $(2s + 1)$-factor F_1 is given. If $r = 0$ then $n - 1 - (2s + 1) = 2s + 1$, so $K_n \backslash F_1$, is another $(2s + 1)$-factor. If $r > 0$ then the degree of $K_n \backslash F_1$ is $\geqq n/2$, so, by Lemma 1, $K_n \backslash F_1$ contains a $(2s + 1)$-factor. Thus $\kappa(Q(2s + 1, n)) = 1$.

Now suppose that $q = 3$. Then, since n is even and $n - 1 = 3(2s + 1) + r$, it follows that r is even. By Lemma 1, K_n contains 3 edge-disjoint $(2s + 1)$-factors. If one $(2s + 1)$-factor, say F_1, is given, then the degree of $K_n \backslash F_1$ is $\geqq n/2$, so, by Lemma 1, $K_n \backslash F_1$ contains two edge-disjoint $(2s + 1)$-factors. Now suppose that two edge-disjoint $(2s + 1)$-factors, say F_1 and F_2, are given. If $r = 0$ then $K_n \backslash (F_1 \cup F_2)$ is a $(2s + 1)$-factor. Therefore suppose that $r > 0$. Then $K_n \backslash (F_1 \cup F_2)$ is a regular graph of degree $(2s + 1) + r$ and order $n = 3(2s + 1) + r + 1$. We now proceed to show that $K_n \backslash (F_1 \cup F_2)$ has a 1-factor. Since the degree of $K_n \backslash (F_1 \cup F_2)$ is more than $(1/3)n$, $K_n \backslash (F_1 \cup F_2)$ can have at most two components. A regular graph of odd degree must have even order, so if there are two components, say G_1 and G_2, then they both have even order. Then $\min(|V(G_1)|, |V(G_2)|) \geqq (2s + 1) + r + 1$, so $\max(|V(G_1)|, |V(G_2)|) \leqq n - (2s + 1) - r - 1 = 2(2s + 1)$. Therefore, by Lemma 5, both G_1 and G_2 have Hamiltonian circuits, and therefore both have 1-factors. Therefore $K_n \backslash (F_1 \cup F_2)$ has a 1-factor. Now suppose $K_n \backslash (F_1 \cup F_2)$ has a cut vertex, w. Let $K_n \backslash (F_1 \cup F_2) = G_1^* \cup G_2^*$, where $V(G_1^*) \cap V(G_2^*) = \{w\}$ and $\min(|V(G_1^*)|, |V(G_2^*)|) \geqq (2s + 1) + r + 1$. Then one of G_1^* and G_2^* has even order, say G_1^*, and one, G_2^*, has odd order. Then $\max(|V(G_1^*)|, |V(G_2^*)|) \leqq 2(2s + 1) + 1$. Then, by Lemma 5, both G_1^* and G_2^* have Hamiltonian circuits, and so G_1^* has a 1-factor, and G_2^* has a near 1-factor which excludes w. Therefore $K_n \backslash (F_1 \cup F_2)$ has a 1-factor. Finally suppose that $K_n \backslash (F_1 \cup F_2)$ is 2-connected. By Lemma 3, $K_n \backslash (F_1 \cup F_2)$ contains

$$\left[\frac{1}{6}(3d(K_n \backslash (F_1 \cup F_2)) - \mathrm{n} + 4)\right]$$
$$= \left[\frac{1}{6}(3(2s + 1) + 3r - 3(2s + 1) - r - 1 + 4)\right] = \left[\frac{1}{6}(2r + 3)\right] \geqq 1$$

Hamiltonian circuits, since $n \geqq 9$. Therefore $K_n \backslash (F_1 \cup F_2)$ contains a 1-factor in this final case also. Let F^* be a 1-factor of $K_n \backslash (F_1 \cup F_2)$. Then $K_n \backslash (F_1 \cup F_2 \cup F^*)$ is a regular graph of even degree, and so, by Lemma 2, it is the union of edge-disjoint 2-factors, and so, in particular, it contains a $(2s)$-factor. But the union of this with F^* yields the $(2s + 1)$-factor we wished for. Therefore, in this case also, $\kappa(Q(2s + 1, n)) = 1$.

Now suppose that $q \geqq 4$. We first show that $\kappa(Q(2s + 1, n)) \leqq 2$. Let $\{F_1, \ldots, F_t\}$ be a set of t edge-disjoint $(2s + 1)$-factors of K_n. We may assume that $t \leqq q - 1$, for if $t = q$ there is nothing to show. Then the graphs $K_n \backslash (F_1 \cup \ldots \cup F_{[t/2]})$ and $K_n \backslash (F_{[t/2]+1} \cup \ldots \cup F_t)$ both have degree $\geqq n/2$. Therefore, by Lemma 1, they contain $q - [t/2]$ and $q - t + [t/2]$ edge-disjoint $(2s + 1)$-factors respectively. Therefore the

https://doi.org/10.4153/CMB-1987-034-2 Published online by Cambridge University Press

set $\{F_1, \ldots, F_{[t/2]}\}$ can be extended to a set of q edge-disjoint $(2s + 1)$-factors of K_n, and so can the set $\{F_{[t/2]+1}, \ldots, F_t\}$. Therefore $\kappa(Q(2s + 1, n)) \leqq 2$ in this case.

Next we show that $\kappa(Q(2s + 1, n)) \geqq 2$. Suppose first that r is odd. Let H be a graph consisting of two components, one a K_{2s+2+r} and the other a graph formed by uniting $\frac{1}{2}(r + 2s + 1)$ edge-disjoint Hamiltonian circuits from a $K_{n-(2s+2+r)}$. Then H is a regular of order n and degree $2s + 1 + r$, and, since both components are of odd order, it contains no $(2s + 1)$-factor. However $\bar{H}$ (the complement of H) is a regular graph of order n and degree $n - (2s + 2) - r = (q - 1)(2s + 1) \geqq \frac{1}{2}(q(2s + 1) + r + 1) = n/2$, so, by Lemma 1, it is the union of $(q - 1)$ $(2s + 1)$-factors, say $F_1, \ldots, F_{q-1}$. Thus the set $\{F_1, \ldots, F_{q-1}\}$ cannot be extended to a set of q edge-disjoint $(2s + 1)$-factors of K_n. Therefore, in this case, $\kappa(Q(2s + 1, n)) \geqq 2$.

Now suppose that r is even. By assumption, $q \geqq 4$. But since $n - 1 = q(2s + 1) + r$ and n and r are even, q must be odd, so, in fact, $q \geqq 5$. In this case, instead of H, we construct a regular graph H^* of order n and degree $4s + 2$ which does not contain a $(2s + 1)$-factor. The argument is then more or less the same as in the case above, except that $\overline{H^*}$ is the union of $(q - 2)$ $(2s + 1)$-factors. Let H^* consist of two components, one a K_{4s+3} and the other the union of $(2s + 1)$ edge-disjoint Hamiltonian circuits from a Hamiltonian decomposition of $K_{n-(4s+3)}$. Then H^* is regular of order n and degree $2(2s + 1)$, and contains no $(2s + 1)$-factor. It follows that, in this case also, $\kappa(Q(2s + 1, n)) \geqq 2$.

Thus, if $q \geqq 4$ then $\kappa(Q(2s + 1, n)) = 2$. This proves Theorem 1.

Next we consider a slightly less tractable variation on the same theme. Again, let n be even and let $n - 1 = q(2s + 1) + r$, where $0 \leqq r < 2s + 1$. Let R be a regular graph of order n and degree r. We are now concerned to determine the intricacy of the problem of finding a $(2s + 1)$-factorization of $K_n \backslash R$; we denote this problem by $P(R, 2s + 1, n)$. Thus, in the previous problem, we were left with some r-factor, and it did not matter which. In this problem, the r-factor is predetermined.

THEOREM 2. *Let n be even, let $3 \leqq 2s + 1 < n$ and let $n - 1 = q(2s + 1) + r$, where $0 \leqq r < 2s + 1$. Let R be an r-factor of K_n. Then*

$$\kappa(P(R, 2s + 1, n)) = \begin{cases} 1 & \text{if} \quad q \leqq 3, \\ 2 & \text{if} \quad q \geqq 11, \end{cases}$$

and,

$$1 \leqq \kappa(P(R, 2n + 1, n)) \leqq 2 \quad \text{if} \quad 4 \leqq q \leqq 11.$$

PROOF. The conditions imply that $q \geqq 1$. If $q = 1$ then $K_n \backslash R$ is a $(2s + 1)$-factor, so $\kappa(P(R, 2s + 1, n)) = 1$ in this case. Suppose that $q = 2$. By Lemma 1, $K_n \backslash R$ is the union of two edge-disjoint $(2s + 1)$-factors. If one $(2s + 1)$-factor F_1 of $K_n \backslash R$ is given, then $K_n \backslash (F_1 \cup R)$ is another $(2s + 1)$-factor. Therefore, $\kappa(P(R, 2s + 1, n)) = 1$ in this case also. Now suppose that $q = 3$. By Lemma 1, $K_n \backslash R$ can be $(2s + 1)$-factorized. If F_1 is a given $(2s + 1)$-factor of $K_n \backslash R$, then $K_n \backslash (R \cup F_1)$ is a regular graph of degree $(n - 1) - (2s + 1) - r = 2(2s + 1) \geqq n/2$, so, by Lemma 1, $K_n \backslash (R \cup F_1)$ is the union of two edge-disjoint $(2s + 1)$-factors. Finally, if F_1 and F_2 are two edge-disjoint $(2s + 1)$-factors of $K_n \backslash R$, then $K_n \backslash (R \cup F_1 \cup F_2)$ is another $(2s + 1)$-factor. Therefore $\kappa(P(R, 2s + 1, n)) = 1$ in this case also.

From now suppose that $q \geqq 4$. We now show that $\kappa(P(R, 2s + 1, n)) \leqq 2$. Let $\{F_1, \ldots, F_t\}$ be a set of edge-disjoint $(2s + 1)$-factors of $K_n \backslash R$. If $t = q$ there is nothing to prove. If $t = q - 1$, then $K_n \backslash (R \cup F_1 \cup \ldots \cup F_t)$ is a $(2s + 1)$-factor of K_n. Therefore suppose that $t \leqq q - 2$. Then $K_n \backslash (R \cup F_1 \ldots \cup F_{[t/2]})$ is a regular graph of order n and degree at least

$$\begin{aligned} n - 1 - r - \left[\frac{t}{2}\right](2s + 1) &= \left(q - \left[\frac{t}{2}\right]\right)(2s + 1) \\ &\geqq \left(q - \left[\frac{q-2}{2}\right]\right)(2s + 1) \\ &\geqq \left(\frac{q}{2} + 1\right)(2s + 1) \\ &\geqq \frac{n}{2}, \end{aligned}$$

and $K_n \backslash (R \cup F_{[t/2]+1} \cup \ldots \cup F_t)$ is a regular graph of order n and degree at least

$$\begin{aligned} n - 1 - r - \left(t - \left[\frac{t}{2}\right]\right)(2s + 1) &= \left(q - \left(t - \left[\frac{t}{2}\right]\right)\right)(2s + 1) \\ &\geqq \left(q - \left(q - 2 - \left[\frac{q-2}{2}\right]\right)\right)(2s + 1) \\ &\geqq \left(\frac{q+1}{2}\right)(2s + 1) \\ &\geqq \frac{n}{2}. \end{aligned}$$

Therefore, by Lemma 1, both these graphs are $(2s + 1)$-factorizable. This shows that $\kappa(P(R, 2s + 1, n)) \leqq 2$.

Finally, suppose that $q \geqq 11$. We show that $\kappa(P(R, 2s + 1, n)) \geqq 2$. Let H be a regular spanning subgraph of $K_n \backslash R$ of degree $2(2s + 1)$ consisting of two components. If $n \equiv 2 (\mathrm{mod}\ 4)$ let the two components be H_1 and H_2 and both have order $n/2$. If $n \equiv 0 (\mathrm{mod}\ 4)$, let the two components be H_3 and H_4 and have orders $(n/2) - 1$ and $(n/2) + 1$ respectively. Then in both cases the orders are odd, so neither component is the union of two $(2s + 1)$-factors. If $n \equiv 2 (\mathrm{mod}\ 4)$, then H_1 and H_2 are constructed as follows. Let $\{v_1, \ldots, v_n\}$ be the vertex set of $K_n \backslash R$, and let G_1 and G_2 be the subgraphs induced by $\{v_1, \ldots, v_{n/2}\}$ and $\{v_{(n/2)+1}, \ldots, v_n\}$ respectively. Then the minimum degree $\delta(G_1)$ of G_1 satisfies $\delta(G_1) \geqq (n/2) - 1 - r$. We remove from G_1 a set of $(2s + 1)$ edge-disjoint Hamiltonian circuits, one after the other. That this can be done may be seen as follows. Suppose that $1 \leqq i \leqq 2s + 1$ and that $i - 1$ edge-disjoint Hamiltonian circuits have been removed. Then the minimum degree of the graph G^*

remaining satisfies

$$\begin{aligned}\delta(G^*) \geqq \frac{n}{2} - 1 - r - 2(i-1) &\geqq \frac{n}{2} + 1 - r - 2(2s+1)\\ &= \frac{q}{2}(2s+1) + \frac{r}{2} + \frac{3}{2} - r - 2(2s+1)\\ &= \frac{1}{2}\{(q-4)(2s+1) - r + 3\}\\ &\geqq \frac{1}{2}\{(q-5)(2s+1) + 4\}\\ &\geqq \frac{1}{2}\left\{\left(\frac{q+1}{2}\right)(2s+1) + 4\right\}, \text{ since } q \geqq 11,\\ &> \frac{1}{2}\left\{\frac{q}{2}(2s+1) + \frac{r+1}{2}\right\}\\ &= \frac{1}{2}|V(G^*)|,\end{aligned}$$

and so, by Lemma 5, G^* contains a Hamiltonian circuit. We let H_1 be the union of $2s+1$ edge-disjoint Hamiltonian circuits in G_1. A similar construction works for H_2, and, in the case when $n \equiv 0 \pmod 4$, a similar construction works for H_3 and H_4. The graph $G \backslash (R \cup H)$ is, by Lemma 1, $(2s+1)$-factorizable; let $F_1, \ldots, F_{q-2}$ be edge-disjoint $(2s+1)$-factors of $K_n \backslash (R \cup H)$. Then $K_n \backslash (R \cup F_1 \cup \ldots \cup F_{q-2}) = H$, so $\{F_1, \ldots, F_{q-2}\}$ cannot be extended to a set of q edge-disjoint $(2s+1)$-factors of $K_n \backslash R$. Therefore $\kappa(P(R, 2s+1, n)) \geqq 2$ in this case. This proves Theorem 2.

For some regular graphs R it is possible to be more precise about the value of $\kappa(P(R, 2s+1, n))$. In particular we have the following result for the case when R is bipartite.

THEOREM 3. *Let* $n \equiv 2 \pmod 4$, *let* $3 \leqq 2s+1 < n$ *and let* $n - 1 = q(2s+1) + r$, *where* $0 \leqq r < 2s+1$. *Let R be a bipartite r-factor of* K_n. *Then*

$$\kappa(P(R, 2s+1, n)) = \begin{cases} 1 & \text{if} \quad q \leqq 3,\\ 2 & \text{if} \quad q \geqq 4.\end{cases}$$

PROOF. In view of Theorem 2, it is only necessary to show that $\kappa(P(R, 2s+1, n)) \geqq 2$ for $q \geqq 4$. Let the vertex set of R be $A \cup B$, where $|A| = |B| = n/2$ and each edge of R joins a vertex of A to a vertex of B. Let H be a regular spanning subgraph of $K_n \backslash R$ of degree $2(2s+1)$ which consists of two components, $H(A)$ on the vertex set A, and $H(B)$ on B. Since the orders of $H(A)$ and $H(B)$ are odd, H does not contain a $(2s+1)$-factor. The subgraph of $K_n \backslash R$ induced by A is a $K_{n/2}$, and $H(A)$ may be formed by taking $2s+1$ edge-disjoint Hamiltonian circuits from a Hamiltonian decomposition of the $K_{n/2}$. $H(B)$ may be formed similarly. By Lemma 1, $K_n \backslash (R \cup H)$ has a $(2s+1)$-factorization if $q \geqq 5$, say $\{F_1, \ldots, F_{q-2}\}$. If $q = 4$ then $K_n \backslash (R \cup H)$ is a connected $2(2s+1)$-factor of $K_n \backslash R$. It is Eulerian therefore, so colouring the edges alternately red and blue yields two $(2s+1)$-factors, say F_1 and F_2, of $K_n \backslash (R \cup H)$. Thus, for $q \geqq 4$, the set $\{F_1, \ldots, F_{q-2}\}$ of edge-disjoint $(2s+1)$-factors of $K_n \backslash R$ cannot be extended to a $(2s+1)$-factorization of $K_n \backslash R$. Therefore $\kappa(P(R, 2s+1, n)) \geqq 2$, as required. This proves Theorem 3.

It is not clear whether the precise value of $\kappa(P(R, 2s+1, n))$ given in Theorem 3 holds for all R, whether bipartite or not. In fact when $q = 4$ then $\kappa(P(R, 2s+1, n)) = 2$ if and only if $K_n \backslash R$ can be split into two $2(2s+1)$-factors, one of which consists of two components, each of odd order. It seems unlikely that this can always be done; it seems probable that the number 11 occurring in Theorem 2 should be replaced by 5.

References

1. A. G. Chetwynd and A. J. W. Hilton, *Regular graphs of high degree are 1-factorizable*, Proc. London Math. Soc., **50** (1985), pp. 193–206.

2. V. Chvatal, *On Hamilton's ideals*, J. Comb. Theory (B), **12** (1972), pp. 163–168.

3. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3), *2* (1952), pp. 69–81.

4. A. J. W. Hilton, *Factorizations of regular graphs of high degree*, J. Graph Theory, **9** (1985), 193–196.

5. Bill Jackson, *Edge-disjoint Hamilton cycles in regular graphs of large degree*, J. London Math. Soc. (2), **19** (1979), pp. 13–16.

6. W. E. Opencomb, *On the intricacy of combinatorial construction problems*, Discrete Math. **50** (1984), pp. 71–97.

7. J. Petersen, *Die Theorie der regulären Graphen*, Acta Math., **15** (1891), pp. 193–220.

Department of Mathematics
University of Reading
Whiteknights, Reading
U.K.