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Closed Left Ideal Decompositions of $U(G)$

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Abstract. Let G be an infinite discrete group and let βG be the Stone–Čech compactification of G. We take the points of βG to be the ultrafilters on G, identifying the principal ultrafilters with the points of G. The set $U(G)$ of uniform ultrafilters on G is a closed two-sided ideal of βG. For every $p \in U(G)$, define $I_p \subseteq \beta G$ by $I_p = \bigcap_{A \in p} \mathrm{cl}(GU(A))$, where $U(A) = \{p \in U(G) : A \in p\}$. We show that if $|G|$ is a regular cardinal, then $\{I_p : p \in U(G)\}$ is the finest decomposition of $U(G)$ into closed left ideals of βG such that the corresponding quotient space of $U(G)$ is Hausdorff.

Let G be an infinite discrete group of cardinality κ and let βG be the Stone–Čech compactification of G. We take the points of βG to be the ultrafilters on G, identifying the principal ultrafilters with the points of G. The topology of βG is generated by taking as a base the subsets of the form $\overline{A} = \{p \in \beta G : A \in p\}$, where $A \subseteq G$. For $p, q \in \beta G$, the ultrafilter pq has a base consisting of subsets of the form $\bigcup_{x \in A} xB_x$, where $A \in p$ and $B_x \in q$. Under this operation, all right translations of βG and the left translations by elements of G are continuous. See [3] for an elementary introduction to the semigroup βG.

The set $U(G)$ of uniform ultrafilters on G is a closed two-sided ideal of βG. It has long been known that $U(G)$ can be decomposed (*i.e.,* partitioned) into 2^{2^κ} left ideals of βG [1] (see also [3, Theorem 6.53]). Relatively recently, this theorem was strengthened by showing that $U(G)$ can be decomposed into 2^{2^κ} closed left ideals of βG such that the corresponding quotient space of $U(G)$ is Hausdorff. This was first done in the case where κ is a regular cardinal in [5] and then for all κ in [4]. The proof was based on slowly oscillating functions.

Considering the diagonal of the quotient mappings justifies the following definition.

Definition 1 Let $\mathcal{I}(G)$ denote the finest decomposition of $U(G)$ into closed left ideals of βG with the property that the corresponding quotient space of $U(G)$ is Hausdorff.

Note that if $\mathcal{I}$ is a decomposition of $U(G)$ into left ideals of $U(G)$, then every member of $\mathcal{I}$ is also a left ideal of βG. To see this, assume the contrary. Then there are distinct $I, J \in \mathcal{I}$, $p \in I$ and $x \in \beta G$ such that $xp \in J$. From this we obtain that $p(xp) \in J$ and $(px)p \in I$, since $px \in U(G)$, a contradiction.

In this paper we present an intrinsic characterization of $\mathcal{I}(G)$ in the case where κ is a regular cardinal. In the case $\kappa > \omega$ we construct a decomposition of $U(G)$

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into closed left ideals of βG such that the corresponding quotient space of $U(G)$ is homeomorphic to $U(\kappa)$. Also as a consequence we obtain the result from [4].

Definition 2 For every $p \in U(G)$, define $I_p \subseteq \beta G$ by

$$I_p = \bigcap_{A \in p} \operatorname{cl}(GU(A)),$$

where $U(A) = \overline{A} \cap U(G)$.

The next theorem is the main result of this paper.

Theorem 3 *Let κ be a regular cardinal. Then $\mathfrak{I}(G) = \{I_p : p \in U(G)\}$. Furthermore, for any $p, q \in U(G)$, $I_p \cap I_q = \varnothing$ if and only if there are $A \in p$ and $B \in q$ such that $|(xA) \cap B| < \kappa$ for all $x \in G$.*

Before proving Theorem 3 we establish several auxiliary statements.

Lemma 4 *For every $p \in U(G)$, I_p is a closed left ideal of βG contained in $U(G)$.*

Proof Clearly, I_p is a closed subset of $U(G)$. To see that I_p is a left ideal of βG, let $r \in \beta G$ and $q \in I_p$. Since the right translation of βG by q is continuous,

$$rq = \lim_{G \ni x \to r} xq.$$

Consequently, in order to show that $rq \in I_p$, it suffices to show that for every $x \in G$, $xq \in I_p$. We show that $xI_p = I_p$. Clearly,

$$xI_p = \bigcap_{A \in p} x \operatorname{cl}(GU(A)).$$

Since the left translation of βG by x is continuous,

$$x \operatorname{cl}(GU(A)) = \operatorname{cl}(xGU(A)) = \operatorname{cl}(GU(A)).$$

Hence, $xI_p = I_p$. ■

Recall that if $\mathcal{E}$ is a decomposition of a compact Hausdorff space X into closed subsets and E is the equivalence relation on X corresponding to $\mathcal{E}$, then the following statements are equivalent:

(a) the quotient space X/E is Hausdorff;
(b) $\mathcal{E}$ is upper semicontinuous, that is, for every $Y \in \mathcal{E}$ and for every neighborhood U of $Y \subseteq X$, there is a neighborhood V of $Y \subseteq X$ such that if $Z \in \mathcal{E}$ and $Z \cap V \neq \varnothing$, then $Z \subseteq U$.

(See [2, Theorem 3.2.11 and Problem 1.7.17].)

Lemma 5 *Let $\mathcal{J}$ be a decomposition of $U(G)$ into closed left ideals such that the corresponding quotient space of $U(G)$ is Hausdorff. Then for every $J \in \mathcal{J}$ and $p \in J$, $I_p \subseteq J$.*

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Proof It suffices to show that for every neighborhood V of $J \subseteq U(G)$, $I_p \subseteq \mathrm{cl}(V)$. Since $\mathcal{J}$ is upper semicontinuous, one may suppose that for every $I \in \mathcal{J}$, if $I \cap V \neq \varnothing$, then $I \subseteq V$. It follows that $GV \subseteq V$. Since V is a neighborhood of $p \in U(G)$, there is $A \in p$ such that $U(A) \subseteq V$. Consequently, $GU(A) \subseteq V$, so $\mathrm{cl}(GU(A)) \subseteq \mathrm{cl}(V)$. Hence, $I_p \subseteq \mathrm{cl}(V)$. ■

As usual, given a set X and a cardinal λ,

$$[X]^{\lambda} = \{A \subseteq X : |A| = \lambda\} \quad \text{and} \quad [X]^{<\lambda} = \{A \subseteq X : |A| < \lambda\}.$$

Definition 6 For every $p \in U(G)$, let $\mathcal{F}_p$ denote the filter on G with a base consisting of subsets of the form $\bigcup_{x\in G} x(A \setminus F_x)$, where $A \in p$ and $F_x \in [G]^{<\kappa}$ for each $x \in G$.

Lemma 7 *For every $p \in U(G)$, $I_p = \bigcap_{C\in\mathcal{F}_p} \overline{C}$.*

Proof To see that $I_p \subseteq \bigcap_{C\in\mathcal{F}_p} \overline{C}$, let $A \in [G]^{\kappa}$ and $F_x \in [G]^{<\kappa}$ for each $x \in G$ and let $C = \bigcup_{x\in G} x(A \setminus F_x)$. For every $x \in G$,

$$xU(A) \subseteq x\overline{(A \setminus F_x)} = \overline{x(A \setminus F_x)} \subseteq \overline{C}.$$

Consequently, $\mathrm{cl}(GU(A)) \subseteq \overline{C}$.

To see the converse inclusion, let $B \subseteq G$ and $I_p \subseteq \overline{B}$. It then follows that there is $A \in p$ such that $\mathrm{cl}(GU(A)) \subseteq \overline{B}$. (Indeed, $G \setminus \overline{B} = \overline{G \setminus B}$ is compact and for every $y \in \overline{G \setminus B}$, there is $A_y \in p$ such that $y \notin \mathrm{cl}(GU(A_y))$.) For every $x \in G$, one has $xU(A) \subseteq \overline{B}$; consequently, there is $F_x \in [G]^{<\kappa}$ such that $x(A \setminus F_x) \subseteq B$. Let $C = \bigcup_{x\in G} x(A \setminus F_x)$. Then $C \in \mathcal{F}_p$ and $C \subseteq B$. ■

Lemma 8 *Suppose that κ is a regular cardinal. Let $A \in [G]^{\kappa}$ and $F_x \in [G]^{<\kappa}$ for every $x \in G$ and let $B = G \setminus \bigcup_{x\in G} x(A \setminus F_x)$. Then there are $H_x, K_x \in [G]^{<\kappa}$ for every $x \in G$ such that*

$$\Big(\bigcup_{x\in G} x(A \setminus H_x)\Big) \cap \Big(\bigcup_{x\in G} x(B \setminus K_x)\Big) = \varnothing.$$

Proof Enumerate G as $\{x_\alpha : \alpha < \kappa\}$. For every $\alpha < \kappa$, define $H_{x_\alpha}, K_{x_\alpha} \in [G]^{<\kappa}$ by

$$H_{x_\alpha} = \bigcup_{\beta\leq\alpha} F_{x_\beta^{-1}x_\alpha} \quad \text{and} \quad K_{x_\alpha} = \bigcup_{\beta\leq\alpha} x_\alpha^{-1}x_\beta F_{x_\alpha^{-1}x_\beta}.$$

Then for every $\alpha < \kappa$ and $\beta \leq \alpha$,

$$x_\beta^{-1}x_\alpha(A \setminus H_{x_\alpha}) \cap B = \varnothing \quad \text{and} \quad x_\alpha^{-1}x_\beta A \cap (B \setminus K_{x_\alpha}) = \varnothing,$$

and so

$$x_\alpha(A \setminus H_{x_\alpha}) \cap x_\beta B = \varnothing \quad \text{and} \quad x_\beta A \cap x_\alpha(B \setminus K_{x_\alpha}) = \varnothing.$$

Consequently, for every $\alpha, \beta < \kappa$,

$$x_\alpha(A \setminus H_{x_\alpha}) \cap x_\beta(B \setminus K_{x_\beta}) = \varnothing.$$ ■

Now we are in a position to prove Theorem 3.

Proof of Theorem 3 Let $\mathcal{I} = \{I_p : p \in U(G)\}$. By Lemma 4, all members of $\mathcal{I}$ are closed left ideals of βG contained in $U(G)$. To show that $\mathcal{I}$ is an upper semicontinuous decomposition of $U(G)$, let $p \in U(G)$, $A \in p$, and $F_x \in [G]^{<\kappa}$ for every $x \in G$, and let $B = \bigcup_{x\in G} x(A \setminus F_x)$. By Lemma 8, there are $H_x, K_x \in [G]^{<\kappa}$ for every $x \in G$ such that $Q \cap R = \varnothing$, where

$$Q = \bigcup_{x\in G} x(A \setminus H_x) \quad \text{and} \quad R = \bigcup_{x\in G} x(B \setminus K_x).$$

By Lemma 7, $I_p \subseteq \overline{Q}$ and for every $r \in U(B)$, $I_r \subseteq \overline{R}$; consequently, $I_r \subseteq \overline{G \setminus Q}$. This shows that $\mathcal{I}$ is a decomposition. It follows from this also that for every $q \in U(Q)$, $I_q \subseteq \overline{G \setminus B}$, which shows that $\mathcal{I}$ is upper semicontinuous. Thus, $\mathcal{I}$ is a decomposition of $U(G)$ into closed left ideals such that the corresponding quotient space of $U(G)$ is Hausdorff. That $\mathcal{I}$ is the finest decomposition of this kind follows from Lemma 5.

Finally, applying Lemma 7 gives us that $q \notin I_p$ if and only if there are $A \in p$ and $B \in q$ such that $|(xA) \cap B| < \kappa$ for all $x \in G$. ∎

The decomposition constructed in [4] had an additional property that for every member I of the decomposition, $IG \subseteq I$.

Definition 9 Let $\mathcal{I}'(G)$ denote the finest decomposition of $U(G)$ into closed left ideals of βG with the property that the corresponding quotient space of $U(G)$ is Hausdorff and for every member I of the decomposition, $IG \subseteq I$.

Definition 10 For every $p \in U(G)$, define $I'_p \subseteq \beta G$ by

$$I'_p = \bigcap_{A\in p} \operatorname{cl}(GU(A)G).$$

As in the proof of Lemma 7, one shows that $I'_p = \bigcap_{C\in \mathcal{F}'_p} \overline{C}$, where $\mathcal{F}'_p$ denotes the filter on G with a base consisting of subsets of the form

$$\bigcup_{x,y\in G} x(A \setminus F_{x,y})y,$$

where $A \in p$ and $F_{x,y} \in [G]^{<\kappa}$ for every $x, y \in G$. The next lemma is the corresponding version of Lemma 8.

Lemma 11 *Suppose that κ is a regular cardinal. Let $A \in [G]^{\kappa}$ and $F_{x,y} \in [G]^{<\kappa}$ for every $x, y \in G$ and let $B = G \setminus \bigcup_{x,y\in G} x(A \setminus F_{x,y})y$. Then there are $H_{x,y}, K_{x,y} \in [G]^{<\kappa}$ for every $x, y \in G$ such that*

$$\Big(\bigcup_{x,y\in G} x(A \setminus H_{x,y})y\Big) \cap \Big(\bigcup_{x,y\in G} x(B \setminus K_{x,y})y\Big) = \varnothing.$$

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Proof Enumerate $G \times G$ as $\{(x_\alpha, y_\alpha) : \alpha < \kappa\}$, and for every $\alpha < \kappa$, define $H_{x_\alpha,y_\alpha}, K_{x_\alpha,y_\alpha} \in [G]^{<\kappa}$ by

$$H_{x_\alpha,y_\alpha} = \bigcup_{\beta\leq\alpha} F_{x_\beta^{-1}x_\alpha,y_\alpha y_\beta^{-1}} \quad \text{and} \quad K_{x_\alpha,y_\alpha} = \bigcup_{\beta\leq\alpha} x_\alpha^{-1}x_\beta F_{x_\alpha^{-1}x_\beta,y_\beta y_\alpha^{-1}} y_\beta y_\alpha^{-1}.$$

Then for every $\alpha < \kappa$ and $\beta \leq \alpha$,

$$x_\beta^{-1}x_\alpha(A \setminus H_{x_\alpha,y_\alpha})y_\alpha y_\beta^{-1} \cap B = \varnothing \quad \text{and} \quad x_\alpha^{-1}x_\beta A y_\beta y_\alpha^{-1} \cap (B \setminus K_{x_\alpha,y_\alpha}) = \varnothing,$$

so

$$x_\alpha(A \setminus H_{x_\alpha,y_\alpha})y_\alpha \cap x_\beta B y_\beta = \varnothing \quad \text{and} \quad x_\beta A y_\beta \cap x_\alpha(B \setminus K_{x_\alpha,y_\alpha})y_\alpha = \varnothing,$$

and consequently, for every $\alpha, \beta < \kappa$,

$$x_\alpha(A \setminus H_{x_\alpha,y_\alpha})y_\alpha \cap x_\beta(B \setminus K_{x_\beta,y_\beta})y_\beta = \varnothing. \qquad \blacksquare$$

It is easy to see that the corresponding versions of Lemmas 4 and 5 also hold. Hence, we obtain the following analogue of Theorem 3.

Theorem 12 *Let κ be a regular cardinal. Then $\mathcal{I}'(G) = \{I'_p : p \in U(G)\}$. Furthermore, for any $p, q \in U(G)$, $I'_p \cap I'_q = \varnothing$ if and only if there are $A \in p$ and $B \in q$ such that $|(xAy) \cap B| < \kappa$ for all $x, y \in G$.*

The next lemma will allow us to compute the cardinality of $\mathcal{I}'(G)$.

Lemma 13 *Let $A \in [G]^\kappa$. Then there are $B \in [A]^\kappa$ and $F_{x,y} \in [G]^{<\kappa}$ for every $x, y \in G$ such that whenever $B_0, B_1 \in [B]^\kappa$ and $B_0 \cap B_1 = \varnothing$, one has*

$$\Big(\bigcup_{x,y\in G} x(B_0 \setminus F_{x,y})y\Big) \cap \Big(\bigcup_{x,y\in G} x(B_1 \setminus F_{x,y})y\Big) = \varnothing.$$

Proof Enumerate $G \times G$ as $\{(x_\alpha, y_\alpha) : \alpha < \kappa\}$. Construct inductively a κ-sequence $(a_\gamma)_{\gamma<\kappa}$ in A such that for every $\gamma < \kappa$ and $\alpha \leq \gamma$,

$$x_\alpha a_\gamma y_\alpha \notin \{x_\beta a_\delta y_\beta : \beta \leq \delta < \gamma\}.$$

Define $B \in [A]^\kappa$ and $F_{x_\alpha,y_\alpha} \in [G]^{<\kappa}$ for every $\alpha < \kappa$ by

$$B = \{a_\gamma : \gamma < \kappa\} \quad \text{and} \quad F_{x_\alpha,y_\alpha} = \{a_\beta : \beta < \alpha\}.$$

We claim that these are as required. Indeed, assume the contrary. Then $x_\alpha a_\gamma y_\alpha = x_\beta a_\delta y_\beta$ for some $\alpha, \beta < \kappa$ and some distinct $\gamma, \delta < \kappa$ such that $\alpha \leq \gamma$ and $\beta \leq \delta$. But this is a contradiction. $\blacksquare$

Corollary 14 *If κ is a regular cardinal, then $|\mathcal{I}'(G)| = 2^{2^\kappa}$, and for every $I \in \mathcal{I}'(G)$, I is nowhere dense in $U(G)$.*

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Proof Let $A \in [G]^\kappa$ and let B be a subset of A guaranteed by Lemma 13. Then $|U(B)| = 2^{2^\kappa}$ and for any distinct $p, q \in U(B)$, $I_p \cap I_q = \varnothing$.

To see that I is nowhere dense in $U(G)$, suppose that $U(A) \cap I \neq \varnothing$. If $U(B) \cap I = \varnothing$, we are done. Otherwise, $I = I_p$ for some $p \in U(B)$. Pick $C \in [B]^\kappa$ such that $C \notin p$. Then $U(C) \cap I = \varnothing$. ■

The next theorem covers in some sense the case where κ is a singular cardinal.

Theorem 15 *If $\kappa > \omega$, then there is a decomposition $\mathfrak{J}$ of $U(G)$ into closed left ideals of βG such that*

(i) *the corresponding quotient space of $U(G)$ is homeomorphic to $U(\kappa)$;*
(ii) *for every $J \in \mathfrak{J}$, $JG \subseteq J$;*
(iii) *for every $J \in \mathfrak{J}$, J is nowhere dense in $U(G)$.*

The proof of Theorem 15 is based on the following lemma.

Lemma 16 *Let $\kappa > \omega$. Then there is a surjective function $f \colon G \to \kappa$ such that*

(a) *for every $\alpha < \kappa$, $|f^{-1}(\alpha)| < \kappa$;*
(b) *whenever $x, y \in G$ and $f(x) < f(y)$, one has $f(xy) = f(yx) = f(y)$.*

Proof Construct inductively a κ-sequence $(G_\alpha)_{\alpha<\kappa}$ of subgroups of G such that

(i) for every $\alpha < \kappa$, $|G_\alpha| < \kappa$;
(ii) for every $\alpha < \kappa$, $G_\alpha \subset G_{\alpha+1}$;
(iii) for every limit ordinal $\alpha < \kappa$, $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$;
(iv) $\bigcup_{\alpha<\kappa} G_\alpha = G$.

Note that G is a disjoint union of nonempty sets $G_{\alpha+1} \setminus G_\alpha$, where $\alpha < \kappa$, and G_0. Define $f \colon G \to \kappa$ by

$$f(x) = \begin{cases} \alpha & \text{if } x \in G_{\alpha+1} \setminus G_\alpha, \\ 0 & \text{if } x \in G_0. \end{cases}$$

Clearly, f is surjective and satisfies (a). To check (b), let $x, y \in G$ and $f(x) < f(y)$. Then $x \in G_\beta$ and $y \in G_{\alpha+1} \setminus G_\alpha$ for some $\beta \leq \alpha < \kappa$. It follows that both xy and yx also belong to $G_{\alpha+1} \setminus G_\alpha$. Hence, $f(xy) = f(yx) = f(y)$. ■

Proof of Theorem 15 Let $f \colon G \to \kappa$ be a function guaranteed by Lemma 16 and let $\overline{f} \colon \beta G \to \beta\kappa$ be the continuous extension of f. Then

(i) $\overline{f}(U(G)) = U(\kappa)$ and $\overline{f}^{-1}(U(\kappa)) = U(G)$;
(ii) $\overline{f}(qp) = \overline{f}(p)$ for all $p \in U(G)$ and $q \in \beta G$;
(iii) $\overline{f}(px) = \overline{f}(p)$ for all $p \in U(G)$ and $x \in G$;
(iv) for every $u \in U(\kappa)$, $\overline{f}^{-1}(u)$ is nowhere dense in $U(G)$.

Indeed, (i) follows from surjectivity of f and condition (a). To see (ii), let $A \in p$. For every $x \in G$, let $A_x = A \setminus \{y \in G : f(y) \leq f(x)\}$. Then $A_x \in p$, and by condition (b), $f(xy) = f(y) \in f(A)$ for all $y \in A_x$. Consequently, $B = \bigcup_{x\in G} xA_x \in qp$ and $f(B) \subseteq f(A)$. Hence, $\overline{f}(qp) = \overline{f}(p)$. Checking (iii) is similar. Finally, to see (iv), let $A \in [G]^\kappa$ and suppose that $U(A) \cap \overline{f}^{-1}(u) \neq \varnothing$. Then $E = f(A) \in u$. Pick $D \in [E]^\kappa$

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such that $D \notin u$ and let $B = f^{-1}(D) \cap A$. Then $B \subseteq A$, $U(B) \neq \varnothing$, but $f(B) \notin u$, and so $U(B) \cap \overline{f}^{-1}(u) = \varnothing$. Hence, $\overline{f}^{-1}(u)$ is nowhere dense in $U(G)$.

Now let $\mathcal{J} = \{\overline{f}^{-1}(u) : u \in U(\kappa)\}$. It then follows from (i)–(iv) that $\mathcal{J}$ is as required. ■

Applying Theorem 15 in the case $\kappa > \omega$ and Corollary 14 in the case $\kappa = \omega$, we obtain the result from [4].

Theorem 17 $|\mathcal{I}'(G)| = 2^{2^{\kappa}}$, *and for every* $I \in \mathcal{I}'(G)$, I *is nowhere dense in* $U(G)$.

Clearly, Theorem 17 remains true with $\mathcal{I}'(G)$ replaced by $\mathcal{I}(G)$.

We conclude this note with the following question.

Question Is $\mathcal{I}(G)$ the finest decomposition of $U(G)$ into closed left ideals of βG?

Of special interest is the case $G = \mathbb{Z}$.

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