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Actions of lattices in Sp (1, *n*)

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Abstract. We study questions concerning the ergodic theory, von Neumann algebras, geometry, and topology of actions of lattices in Sp $(1, n)$.

1. *Introduction*

In this paper we study actions of lattice subgroups of the Lie groups Sp $(1, n)$. There are four main results. The first two are of an ergodic theoretic nature, concerning the measurable orbit equivalence of actions of such groups, as well as the von Neumann algebra associated with the action by the Murray–von Neumann group-measure space construction. The last two results are of a geometric nature, concerning the actions of such lattices preserving a geometric structure on a manifold. An interesting feature of the proof of these is the new role played by von Neumann algebras in helping (along with geometric and ergodic theoretic arguments) to establish purely geometric results. We now describe the results in more detail.

Let Γ (respectively Γ') be a discrete group acting essentially freely, properly ergodically, and with finite invariant measure on a (standard) measure space (S, μ) (respectively (S', μ')). We recall that these actions are called orbit equivalent if there is a measurable bijection (modulo null sets) $\theta: S \to S'$ that is measure-class preserving and such that for (almost) all $s \in S$, $\theta(s\Gamma) = \theta(s)\Gamma'$. The groups Γ and Γ' are called weakly equivalent, and we write $\Gamma \approx \Gamma'$, if such orbit equivalent S and S' exist. If Γ, Γ' are both amenable then $\Gamma \approx \Gamma'$ by [**1, 5**]. Suppose now that G, G' are connected non-compact simple Lie groups with finite center, and $\Gamma \subset G$, $\Gamma' \subset G'$ are lattices. Then the main result of [**12**] (see also [**14**]) implies that if $\mathbb{R}$-rank $(G) \geq 2$, then $\Gamma \approx \Gamma'$ implies G and G' are locally isomorphic. A basic open problem in this direction is to clarify the extent to which this result holds if both G and G' are of $\mathbb{R}$-rank 1. From [**13**], we deduce that if $\Gamma \approx \Gamma'$, G has Kazhdan's property if and only if G' does as well. Our first main theorem is to extend the result quoted above for $\mathbb{R}$-rank $(G) \geq 2$ to the case Sp $(1, n)$.

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THEOREM 1.1(a). *Let* $\Gamma \subset \mathrm{Sp}(1, n)$, $\Gamma' \subset \mathrm{Sp}(1, m)$ *be lattices* $(n, m \geq 1)$. *Assume* $\Gamma \approx \Gamma'$. *Then* $n = m$.

(b) *More generally, let I, J be finite subsets of* $\{n \in \mathbb{Z} \mid n \geq 2\}$. *For* $i \in I$ (*respectively* $j \in J$) *let* $\Gamma_i \subset \mathrm{Sp}(1, i)$ (*respectively* $\Gamma_j \subset \mathrm{Sp}(1, j)$) *be a lattice. Let D and D′ be discrete groups each of which is a* (*finite*) *product of lattices in groups of the form* $\mathrm{SO}(1, k)$ *or* $\mathrm{SU}(1, p)$. *If*

$$D \times \prod_{i \in I} \Gamma_i \approx D' \times \prod_{j \in J} \Gamma_j,$$

then $\prod_{i \in I}(2i - 1) = \prod_{j \in J}(2j - 1)$ (*where we take an empty product to equal* 1.)

It would of course be interesting to determine if we must have $I = J$ in (b). We remark that although $\prod_{i \in I} \Gamma_i$ is a lattice in a semisimple group of $\mathbb{R}$-rank ≥ 2 if the cardinality of I is at least 2, the results of [**12, 14**] are not directly applicable since these lattices will not be irreducible.

The technique of proof of Theorem 1.1 is related to (and was inspired by) the work of Cowling and Haagerup [**4**] showing that the von Neumann algebras of lattices in $\mathrm{Sp}(1, n)$ vary as n varies. Making precise the ideas of Haagerup [**8**], Cowling and Haagerup define for any discrete group Γ, a number $\Lambda(\Gamma)$ which depends only on $VN(\Gamma)$, the von Neumann algebra generated by the regular representation of Γ, and which is $2n - 1$ for a lattice in $\mathrm{Sp}(1, n)$. In the present paper we construct a number $\mathrm{C}(S, \Gamma)$, where S is a measure space on which Γ acts in a measure class preserving way, so that $\mathrm{C}(S, \Gamma) = \Lambda(\Gamma)$ if Γ acts essentially freely, ergodically and with finite invariant measure. We then show that for such actions, $\mathrm{C}(S, \Gamma)$ is an invariant of orbit equivalence, and this will yield Theorem 1.1 (using, of course, the computations of $\Lambda(\Gamma)$ for various lattices, which appear in [**3, 4, 8**]). The number $\mathrm{C}(S, \Gamma)$ is constructed using the group measure space von Neumann algebra, but it is not clear whether or not this depends only on this von Neumann algebra. (I.e., a priori it depends on a Cartan subalgebra as well.)

Our second main result concerns precisely this point, i.e. to clarify in which circumstances one can expect non-isomorphism of the group measure space von Neumann algebras given non-orbit equivalent actions. It is known that this does not hold in complete generality [**2**], but it is widely expected that some such phenomenon exists for actions of lattices in (possibly higher rank) semisimple groups. If Γ acts on a measure space S, we let $\mathrm{VN}(S, \Gamma)$ be the group measure space von Neumann algebra. (We recall the definition in § 2.)

THEOREM 1.2. *Let* $\Gamma \subset \mathrm{Sp}(1, n)$, $\Gamma' \subset \mathrm{Sp}(1, m)$ *be lattices* $(n, m \geq 2)$. *Let* $\Gamma \to X$, $\Gamma' \to X'$ *be embeddings where X and X′ are* (*separable*) *pro-finite groups. Let* Γ *act on X* (*and* Γ' *and X′*) *by translations. If* $\mathrm{VN}(X, \Gamma) \cong \mathrm{VN}(X', \Gamma')$, *then* $n = m$.

We remark that any such lattice is residually finite and hence admits a pro-finite embedding. Thus, Theorem 1.2 provides a natural infinite set of mutually non-isomorphic II_1 group measure space factors. It would of course be of considerable interest to determine if the conclusion of Theorem 1.2 remains valid in a more general context, e.g. for lattices in higher rank groups. Another natural question is the sensitivity of $\mathrm{VN}(X, \Gamma)$ to changing the pro-finite embedding for a fixed Γ. For example, if p is a prime and $\Gamma \to X_p$ is an embedding in a pro-p group, does VN (X_p, Γ) vary as p changes?

Our third main result concerns connection preserving actions on manifolds. We first recall the main result of [17] (see also [16]) for actions of lattices in higher rank groups. Suppose H is a connected simple Lie group with finite center and that $\mathbb{R}$-rank $(H) \geq 2$. Let $\Gamma \subset H$ be a lattice. Suppose M^m is a compact manifold and that M is endowed with both a connection and a G-structure, where $G \subset \mathrm{SL}(m, \mathbb{R})$ is an algebraic group. If Γ acts on M so as to preserve the connection and the G-structure, then the main result of [17] implies that either $\mathfrak{h} \to \mathfrak{g}$ (i.e. H embeds in G locally), or Γ preserves a smooth Riemannian metric on M. For example, $\mathrm{SL}(n, \mathbb{Z})$ acts on $\mathbb{R}^n/\mathbb{Z}^n$ preserving the standard connection and a volume form (i.e. a $\mathrm{SL}(n, \mathbb{R})$-structure). It follows from the theorem quoted above that any volume preserving connection preserving action of $\mathrm{SL}(n, \mathbb{Z})$ $(n \geq 3)$ on a compact manifold M with dim $M < n$ must preserve a Riemannian metric. From the fact that every homomorphism of $\mathrm{SL}(n, \mathbb{Z})$ $(n \geq 3)$ into a compact Lie group has finite image, we deduce that every such action is finite (i.e. factors through a finite quotient.) In this paper, we establish a result of the same nature for lattices $\Gamma \subset \mathrm{Sp}(1, n)$. Using the fact that Γ has Kazhdan's property (T), it follows from the results of [18] that if Γ preserves a G-structure on a compact manifold M^m, $G \subset \mathrm{SL}(m, \mathbb{R})$ algebraic, and a connection on M, where G is locally isomorphic to a group of the form SO $(1, p)$ or SU (1, p), then Γ preserves a smooth Riemannian metric on M. Here we prove:

THEOREM 1.3. *Let* $\Gamma \subset \mathrm{Sp}(1, n)$ *be a lattice. Suppose* Γ *acts on a compact manifold* M *preserving a connection and an* $\mathrm{Sp}(1, m)$*-structure for any linear representation of* $\mathrm{Sp}(1, m)$. *If* Γ *acts properly on the frame bundle of* M, *then* $n \leq m$.

THEOREM 1.4. *Let* $\Gamma \subset \mathrm{Sp}(1, n)$ *be a lattice. Suppose* Γ *acts on a compact Riemannian manifold preserving the Levi-Civita connection and a* $\mathrm{Sp}(1, m)$*-structure for any linear representation of* $\mathrm{Sp}(1, m)$. *Let* Iso (M) *be the isometry group of* M.

(i) If $\Gamma \cap \mathrm{Iso}(M)$ is finite, then $n \leq m$.

(ii) If $m = 2$ and the structural representation of $\mathrm{Sp}(1, m)$ is irreducible, then either:

   (a) $n = 2$; or

   (b) Γ preserves a smooth Riemannian metric on M.

To illustrate, we remark that under the standard embedding $\mathrm{Sp}(1, n) \subset \mathrm{GL}(4(n+1), \mathbb{R})$ the subgroup $\mathrm{Sp}(1, n)$ is the set of real points of an algebraic $\mathbb{Q}$-group, and hence the group of integer points $\Gamma = \mathrm{Sp}(1, n)_{\mathbb{Z}}$ is a lattice. This group acts naturally on the torus $M = \mathbb{R}^{4(n+1)}/\mathbb{Z}^{4(n+1)}$, preserving a Riemannian connection and a $\mathrm{Sp}(1, n)$-structure, and $\Gamma \cap \mathrm{Iso}(M)$ is finite. Theorem 1.4 implies that there is no such action on a smaller dimensional manifold.

The proof of Theorems 1.3, 1.4 is in two parts. We consider the action of Γ on the frame bundle $P(M)$ of M. If the action is proper, then we use von Neumann algebra techniques to show $n \leq m$. If the action is not proper, then we use arguments of geometry and the notion of algebraic hull of an action (reflecting the relationship between algebraic structures and the ergodic theory of the action) to deduce the existence of a smooth invariant metric.

The technical tools used to prove Theorems 1.3 and 1.4 when the Γ-action on $P(M)$ is proper also establish our last result, which extends the work of [**19**].

THEOREM 1.5. *Suppose the discrete group* Γ *acts properly on the universal covering space* $\tilde{M}$ *of a compact manifold M, that this action commutes with the action of the fundamental group* $\pi_1(M)$ *and that the projected action of* Γ *on M admits a finite invariant measure. Then* $\Lambda(\pi_1(M)) \geq \Lambda(\Gamma)$. *In particular, if* Γ *is a lattice in* $\mathrm{Sp}(1, n)$, *then* $\pi_1(M)$ *cannot be embedded as a discrete subgroup of* $\mathrm{Sp}(1, m)$ (*with* $m < n$) *or in* $\mathrm{SO}(1, m)$ *or in* $\mathrm{SU}(1, m)$ (*with* $m \geq 1$).

Using a result of M. Gromov [7], the following Corollary is immediate.

COROLLARY 1.6. *Suppose that* $\mathrm{Sp}(1, n)$ *acts* (*non-trivially and*) *real analytically on a compact manifold M and preserves a connection and a finite measure. Then* $\Lambda(\pi_1(M)) \geq 2n - 1$.

2. *Approximations of the identity on von Neumann algebras*

Let M be a von Neumann algebra. We recall that an operator $T: M \to M$ is called completely bounded if $T \otimes \mathrm{Id}$ is a bounded map on the spatial tensor product algebra $M \bar{\otimes} N$ for any von Neumann algebra N, and the completely bounded norm is given by $\|T\|_{CB} = \|T \otimes \mathrm{Id}\|$ for $N = B(H)$, H a separable Hilbert space. (See [3] for details and discussion.) We also recall that M is the dual space of a Banach space, and hence M has a weak-*-topology. As in [**3**, **4**, **8**], we have the following invariant of M. Consider nets $\{T_i\}_{i \in I}$ where $T_i: M \to M$ is a weak-*-continuous operator, $\dim T_i(M) < \infty$, and for all $x \in M$, $T_i x \to x$ in weak-*. Let $\Lambda(M)$ be the infimum of those numbers C for which there exists such a net with $\|T_i\|_{CB} \leq C$ for all i. Otherwise, set $\Lambda(M) = \infty$.

If Γ is a discrete group, we let VN (Γ) be the von Neumann algebra generated by the regular representation of Γ. Thus, if we let $\pi: \Gamma \to \mathrm{U}(L^2(\Gamma))$ be the regular representation, then VN (Γ) is the closure in the weak operator topology of operators of the form $\sum_{\gamma \in \Gamma} a_\gamma \pi(\gamma)$ where $a_\gamma \in C$ and $a_\gamma = 0$ except on a finite set of Γ. If $T: \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$, and $F \subset \Gamma$ is a finite set, we say that T is supported on F if $T(\mathrm{VN}(\Gamma)) \subset \{\sum_{\gamma \in F} a_\gamma \pi(\gamma)\}$. (In particular, $\dim T(\mathrm{VN}(\Gamma)) < \infty$.) From [**8**] we have:

LEMMA 2.1. [**8**]. *If* Γ *is a discrete group, then* $\Lambda(\mathrm{VN}(\Gamma))$ *is the infimum* (*if it exists*) *of those numbers C for which there is a net of weak-*-continuous* $T_i: \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$ *such that* (a) *each* T_i *is supported on a finite set* (*possibly depending on i*); (b) *for all* $x \in \mathrm{VN}(\Gamma)$, $T_i x \to x$ *in weak-**; *and* (c) $\|T_i\|_{CB} \leq C$ *for all i.*

We shall also need the dual description of $\Lambda(\mathrm{VN}(\Gamma))$ given in [**4**, **8**]. Let G be a locally compact group. Let $B(G)$ be the space of matrix coefficients of unitary representations of G. I.e. a function f on G is in $B(G)$ if and only if f is of the form $f(g) = \langle \pi(g)v, w \rangle$ where π is a unitary representation of G. Then $B(G)$ is a Banach algebra with the norm

$$\|f\|_B = \inf\{\|v\| \, \|w\| \mid f(g) = \langle \pi(g)v, w \rangle\}.$$

We let $A(G) \subset B(G)$ be the subalgebra of matrix coefficients of the regular representation of G. Then $A(G)$ is the closed ideal generated by $B_c(G) = B(G) \cap C_c(G)$. Furthermore, we have a natural identification $A(G)^* = \text{VN}(G)$, the von Neumann algebra generated by the regular representation. Now let

$$M(G) = \{m \in C(G) \mid mA(G) \subset A(G)\},$$

and

$$M_0(G) = \{m \in M(G) \mid \text{the adjoint operator on VN}(G) \text{ is completely bounded}\}.$$

It is known that $m \in M_0(G)$ if and only if there are bounded continuous functions $P, Q: G \to H$ where H is a Hilbert space, such that $m(\gamma^{-1}\lambda) = \langle P(\lambda), Q(\gamma)\rangle$ for $\gamma, \lambda \in G$. Furthermore, if we let $\|m\|_{CB}$ be the completely bounded norm of the adjoint operator on VN (G), then

$$\|m\|_{CB} = \|m\|_{M_0(G)} \stackrel{\text{def}}{=} \inf\{\|P\|_\infty \|Q\|_\infty\}$$

where P, Q satisfy the above equation. With this norm, $M_0(G)$ is a Banach space, and we have $A(G) \subset B(G) \subset M_0(G) \subset M(G)$. We also remark that if Γ is discrete, $l^1(\Gamma) \subset l^2(\Gamma) \subset A(\Gamma)$.

LEMMA 2.2. [**4**, **8**]. *For any locally compact G, let $\Lambda(G)$ be the infimum of those numbers C for which there is a net $\mu_j \in A_c(G)$ such that $\mu_j \to 1$ uniformly on compact sets and $\|\mu_j\|_{M_0(G)} \le C$. Then*
(i) *If $\Gamma \subset G$ is a lattice, $\Lambda(\Gamma) = \Lambda(G)$.*
(ii) *If Γ is discrete, then $\Lambda(\Gamma) = \Lambda(\text{VN}(\Gamma))$.*
(iii) *$\Lambda(G)$ is the infimum of those numbers C for which there is a net $\phi_j \in A(G)$ such that $\phi_j \to 1$ uniformly on compact sets and $\|\phi_j\|_{M_0(G)} \le C$.*

We summarize some of the basic results of [**3**, **4**].

THEOREM 2.3. [**3**, **4**].
(a) *If $\Gamma \subset \text{SO}(1, n)$ or $\Gamma \subset \text{SU}(1, n)$ is a lattice, $n \ge 2$, then $\Lambda(\text{VN}(\Gamma)) = 1$.*
(b) *If $\Gamma \subset \text{Sp}(1, n)$ is a lattice, $n \ge 2$, then $\Lambda(\text{VN}(\Gamma)) = 2n - 1$.*
(c) $\Lambda(\text{VN}(\Gamma_1 \times \Gamma_2)) = \Lambda(\text{VN}(\Gamma_1))\Lambda(\text{VN}(\Gamma_2))$.

Now suppose that (S, μ) is a (standard) measure space and that the discrete group Γ acts in a measure class preserving way on S. For $\gamma \in \Gamma$, let $\pi(\gamma) \in U(L^2(S \times \Gamma))$ be given by

$$(\pi(\gamma)f)(s, g) = f(s\gamma, g\gamma)r(s, \gamma)^{1/2}$$

where $r(s, \gamma) = (d(\gamma_* \mu)/d\mu)(s)$. For $a \in L^\infty(S)$, we have the multiplication operator (which we still denote by a) on $L^2(S \times \Gamma)$, i.e. $(a \cdot f)(s, \gamma) = a(s)f(s, \gamma)$. We let VN (S, Γ) be the von Neumann algebra generated by $\{a \in L^\infty(S)\} \cup \{\pi(\gamma) \mid \gamma \in \Gamma\}$. This is the group measure space von Neumann algebra. It is the closure in the weak operator topology of operators of the form $\{\sum_{\gamma \in \Gamma} a_\gamma \pi(\gamma)\}$ where $a_\gamma \in L^\infty(S)$ and $a_\gamma(s) = 0$ for all s except for γ in a finite subset of Γ. Obviously, if $S = \{pt\}$, then VN $(S, \Gamma) = \text{VN}(\Gamma)$. On the other hand, for essentially free actions, VN (S, Γ)

depends only on the equivalence relation defined by the Γ-action on S, or in other words, depends on the action up to orbit equivalence. (See [6] for discussion.) We shall need one explicit feature of this isomorphism, and hence we recall this with a little detail. Let Γ act on (S, μ), Γ' act on (S', μ'), and suppose $\theta: S \to S'$ is an orbit equivalence. We suppose both actions are essentially free, and (for simplicity) finite measure preserving. Define $\alpha: S \times \Gamma \to \Gamma'$ by $\theta(s)\alpha(s, \gamma) = \theta(s\gamma)$. Then the map $\tilde{\theta}: S \times \Gamma \to S' \times \Gamma'$ given by $\tilde{\theta}(s, \gamma) = (\theta(s), \alpha(s, \gamma))$ is a measure preserving bijection. Let $U: L^2(S' \times \Gamma') \to L^2(S \times \Gamma)$ be the associated unitary operator. I.e., for $h \in L^2(S' \times \Gamma')$, $Uh = h \circ \tilde{\theta}$. Then one easily checks that $U^{-1}\,\mathrm{VN}\,(S, \Gamma) U = \mathrm{VN}\,(S', \Gamma')$. Namely, it is clear that for $a \in L^\infty(S)$, $U^{-1} a U = a \circ \theta^{-1}$. Furthermore, for $\gamma \in \Gamma$, $\gamma' \in \Gamma'$, let $A_{\gamma,\gamma'} = \{s' \in S' \mid \alpha(\theta^{-1}s, \gamma) = \gamma'\}$. Then one verifies that

$$U^{-1}\pi(\gamma)U = \sum_{\gamma' \in \Gamma'} \chi_{A_{\gamma,\gamma'}}(s')\pi(\gamma') \in \mathrm{VN}\,(S', \Gamma').$$

Now let $\mathscr{F}$ be the set of finite subsets of Γ. Suppose $s \to F_s$ is a measurable map $S \to \mathscr{F}$, (i.e. $\{(s, \gamma) \mid \gamma \in F_s\}$ is measurable). We call $\{F_s\}$ a field of finite subsets of Γ (on S).

DEFINITION 2.4. If $T: \mathrm{VN}\,(S, \Gamma) \to \mathrm{VN}\,(S, \Gamma)$ is an operator, and $\{F_s\}$ is a field of finite subsets of Γ, we say that T is supported on $\{F_s\}$ if $T(\mathrm{VN}\,(S, \Gamma))$ is contained in the closure (in the weak operator topology) of operators of the form $\sum_{\gamma \in \Gamma} a_\gamma(s)\pi(\gamma)$ where $a_\gamma \in L^\infty(S)$ and $a_\gamma(s) = 0$ if $\gamma \notin F_s$.

DEFINITION 2.5. Let $\mathrm{C}\,(S, \Gamma)$ be the infimum (if it exists) of those numbers C for which there exists a net of weak-* continuous operators $T_i: \mathrm{VN}\,(S, \Gamma) \to \mathrm{VN}\,(S, \Gamma)$ such that

(i) for each i, there is a field subsets $\{F_s^i\}$ such that T_i is supported on $\{F_s^i\}$;
(ii) for all $x \in \mathrm{VN}\,(S, \Gamma)$, $T_i x \to x$ in weak-*;
(iii) $\|T_i\|_{CB} \le C$ for all i.

If no such C exists we set $\mathrm{C}\,(S, \Gamma) = \infty$.

We remark that although $\mathrm{C}\,(S, \Gamma)$ is defined in terms of $\mathrm{VN}\,(S, \Gamma)$, it is not a priori dependent only on $\mathrm{VN}\,(S, \Gamma)$. The dependence of the definition on fields of finite subsets makes $\mathrm{C}\,(S, \Gamma)$ *a priori* dependent on the choice of Cartan subalgebra [6] $L^\infty(S) \subset \mathrm{VN}\,(S, \Gamma)$ as well. For example, by [8] we have $\Lambda(\mathrm{SL}\,(2, \mathbb{R}) \ltimes \mathbb{R}^2) = \infty$, and hence (by [8] again) $\Lambda(\mathrm{SL}\,(2, \mathbb{Z}) \ltimes \mathbb{Z}^2) = \infty$. The von Neumann algebra of this discrete group is isomorphic (via the Fourier transform in the $\mathbb{Z}^2$ variable) to the von Neumann algebra $\mathrm{VN}\,(T^2, \mathrm{SL}\,(2, \mathbb{Z}))$, where $\mathrm{SL}\,(2, \mathbb{Z})$ has the natural action by automorphisms of T^2. Thus, $\Lambda(\mathrm{VN}\,(T^2, \mathrm{SL}\,(2, \mathbb{Z})) = \infty$, while $\mathrm{C}\,(T^2, \mathrm{SL}\,(2, \mathbb{Z})) = 1$. However, we do have:

LEMMA 2.6. *If the actions of* Γ *on* S *and* Γ' *and* S' *are essentially free, finite measure preserving, and orbit equivalent, then* $\mathrm{C}\,(S, \Gamma) = \mathrm{C}\,(S', \Gamma')$.

Proof. Let $U: L^2(S' \times \Gamma') \to L^2(S \times \Gamma)$ be the unitary operator defined above, and $W: \mathrm{VN}\,(S, \Gamma) \to \mathrm{VN}\,(S', \Gamma')$ the isomorphism $W(A) = U^{-1}AU$. If $T: \mathrm{VN}\,(S, \Gamma) \to \mathrm{VN}\,(S, \Gamma)$ is supported on a field of finite subsets $\{F_s\}_{s \in S}$, then one easily verifies from the definitions that WTW^{-1}: VN $(S', \Gamma') \to$ VN (S', Γ') is supported on $\{F'_{s'}\}_{s' \in S'}$ where

$$F'_{s'} = \{\gamma' \in \Gamma' \mid \gamma' = \alpha(\theta^{-1}(s'), \gamma) \text{ for some } \gamma \in F_s\}.$$

From this, the lemma follows by a routine argument.

Given a net as in Definition 2.5, it is convenient to have a net with somewhat sharper properties.

LEMMA 2.7. *Let* C′ (S, Γ) *be the infimum of those numbers C for which there is a net of weak-* continuous operators* T_i: VN $(S, \Gamma) \to$ VN (S, Γ) *such that* (i), (ii), (iii) *of* 2.5 *hold with the additional assumption that in* (i), *for each* i, $\{F^i_s\}$ *is a constant field,* i.e. *for some* $F^i \in \mathcal{F}$, $F^i_s = F^i$ *for all s. Then* C′ $(S, \Gamma) =$ C (S, Γ).

Proof. Clearly C′ $(S, \Gamma) \geq$ C (S, Γ). To see the converse, suppose $\{T_i\}_{i \in I}$ has support on $\{F^i_s\}$, $\|T_i\|_{CB} \leq C$ and $T_i x \to x$ in weak-* for all $x \in$ VN (S, Γ). For any $F \in \mathcal{F}$, let $S_{i,F} = \{s \in S \mid F^i_s \subset F\}$. Let $T_{i,F}(x) = \chi_{S_{i,F}} \cdot (T_i(x))$. Since $L^\infty(S) \subset$ VN (S, Γ), we clearly have $T_{i,F}$: VN $(S, \Gamma) \to$ VN (S, Γ), and $T_{i,F}$ is supported on the constant field $s \to F$. For any $a \in$ VN (S, Γ), the map $x \to ax$ is completely bounded with completely bounded norm equal to $\|a\|$. Hence $\|T_{i,F}\| \leq C$. Finally, as $F \nearrow \Gamma$, $\chi_{S_{i,F}} \to$ Id in weak-*, so $T_{i,F}x \to x$ in weak-* as well for any $x \in$ VN (S, Γ). This shows C′ $(S, \Gamma) \leq C$, which suffices. ☐

3. *Proof of Theorem* 1.1

In light of Lemma 2.6 and Theorem 2.3, it suffices to prove:

THEOREM 3.1. *Suppose a* (*countable*) *discrete group* Γ *acts in a measure class preserving way on the* (*standard*) *measure space* (S, μ). *Then*

(a) C $(S, \Gamma) \leq \Lambda$(VN (Γ)).

(b) *If the measure is finite and invariant, then* C $(S, \Gamma) = \Lambda$(VN (Γ)).

We let M be the set of operators in VN (S, Γ) of the form $\sum_\gamma a_\gamma \pi(\gamma)$ where $a_\gamma \in L^\infty(S)$ and $a_\gamma(s) = 0$ for all s except for γ in a finite subset of Γ.

LEMMA 3.2. *For any* $\phi \in M_0(\Gamma)$, *define* $T_\phi: M \to M$ *by* $T_\phi(\sum a_\gamma \pi(\gamma)) = \sum_\gamma \phi(\gamma) a_\gamma \pi(\gamma)$. *Then* T_ϕ *extends uniquely to a weak-* continuous map* T_ϕ: VN $(S, \Gamma) \to$ VN (S, Γ) *with* $\|T_\phi\|_{cB} \leq \|\phi\|_{M_0(\Gamma)}$.

Proof. Let $m = \sum \pi(\gamma) a_\gamma \in M$. Let $f_n, \hat{f}_n \in L^2(S \times \Gamma)$ with $\sum_n \|\hat{f}_n\|_2 \|f_n\|_2 < \infty$.

Then

$$\sum_n \langle T_\phi(m)\hat{f}_n, f_n\rangle = \sum_n \sum_\gamma \int_{S\times\Gamma} \phi(\gamma)(\pi(\gamma)a_\gamma \hat{f}_n)(s, \gamma_1)\bar{f}_n(s, \gamma_1)\, d\gamma_1\, ds$$

$$= \sum_n \sum_\gamma \int_{S\times\Gamma} \phi(\gamma)(a_\gamma \hat{f}_n)(s\gamma, \gamma_1\gamma) r(s, \gamma)^{1/2} \bar{f}_n(s, \gamma_1)\, d\gamma_1\, ds.$$

As in the discussion preceding Lemma 2.2, let $p, q: \Gamma \to l^2(\mathbb{Z}^+)$ such that

$$\phi(\gamma^{-1}\lambda) = \langle p(\lambda), q(\gamma)\rangle = \sum_{k \in Z^+} p_k(\lambda)\bar{q}_k(\gamma).$$

Thus

$$\sum_n \langle T_\phi(m)\hat{f}_n, f_n\rangle$$

$$=\sum_n\sum_\gamma \int_{S\times\Gamma} \langle p(\gamma_1\gamma), q(\gamma_1)\rangle(a_\gamma\hat{f}_n)(s\gamma, \gamma_1\gamma)(r(s,\gamma)^{1/2})\bar{f}_n(s,\gamma_1)\, d\gamma_1\, ds$$

$$=\sum_n\sum_\gamma\sum_k \int_{S\times\Gamma} p_k(\gamma_1\gamma)\bar{q}_k(\gamma_1)(a_\gamma\hat{f}_n)(s\gamma,\gamma_1\gamma)r(s,\gamma)^{1/2}\bar{f}_n(s,\gamma_1)\, d\gamma_1\, ds$$

$$=\sum_n\sum_\gamma\sum_k \int_{S\times\Gamma} (a_\gamma p_k\hat{f}_n)(s\gamma,\gamma_1\gamma)r(s,\gamma)^{1/2}(\overline{q_k f_n})(s,\gamma_1)\, d\gamma_1\, ds$$

$$=\sum_n\sum_k \int_{S\times\Gamma} \left(\sum_\gamma (\pi(\gamma)a_\gamma p_k\hat{f}_n)(s,\gamma_1)(\overline{q_k f_n})(s,\gamma_1)\, d\gamma_1\, ds\right)$$

$$=\sum_n\sum_k \langle m(p_k\hat{f}_n), q_k f_n\rangle.$$

Notice that

$$\sum_n\sum_k |\langle m(p_k\hat{f}_n), q_k f_n\rangle| \le \sum_n\sum_k \|m\|\,\|p_k\hat{f}_n\|_2\|q_k f_n\|_2$$

$$\le \|m\| \sum_n \left(\sum_k \|p_k\hat{f}_n\|_2^2\right)^{1/2}\left(\sum_k \|q_k f_n\|_2^2\right)^{1/2}$$

$$\le \|m\|\,\|p\|_\infty\|q\|_\infty \sum_n \|\hat{f}_n\|_2\|f_n\|_2$$

since

$$\left(\sum_k \|p_k\hat{f}_n\|_2^2\right)^{1/2} = \left(\sum_k \int_{S\times\Gamma} |p_k(\gamma_1)|^2|\hat{f}_n(s,\gamma_1)|^2\, d\gamma_1\, ds\right)^{1/2}$$

$$= \int_{S\times\Gamma} \|p(\gamma_1)\|^2|\hat{f}_n(s,\gamma_1)|^2\, d\gamma_1\, ds \le \|p\|_\infty\|\hat{f}_n\|_2,$$

and since we have the corresponding inequality for $(\sum_k \|q_k f_n\|_2^2)^{1/2}$. (Thus, all the above formulae converge absolutely, and we may change the order of summation and integration.) It follows that if for any $m \in \mathrm{VN}\,(S,\Gamma)$, we define $T_\phi(m)$ by

$$\langle T_\phi(m)\hat{f}, f\rangle = \sum_k \langle m(p_k\hat{f}), q_k f\rangle,$$

then T_ϕ, so defined, is the unique weak-$*$ continuous extension from M to $\mathrm{VN}\,(S,\Gamma)$, and we have

$$\|T_\phi(m)\| \le \|\phi\|_{M_0(\Gamma_0)}\|m\|.$$

To see that T_ϕ is completely bounded, one replaces Γ by $\tilde{\Gamma} = \Gamma\times K$ in the above argument where $K = \mathrm{SU}\,(2)$, acting trivially on S. The function ϕ on Γ extends naturally to a function $\tilde{\phi}$ on $\tilde{\Gamma}$ ($\tilde{\phi}(\gamma, k) = \phi(\gamma)$) and by [3], we have $\tilde{\phi} \in M_0(\tilde{\Gamma})$ and $\|\tilde{\phi}\|_{M_0(\tilde{\Gamma})} = \|\phi\|_{M_0(\Gamma)}$. The above argument shows that $T_{\tilde{\phi}}$ is a bounded map on $\mathcal{A}$, the von Neumann algebra on $L^2(S\times\Gamma\times K)$ generated by $L^\infty(S)$, Γ acting on the first two components as before, and K acting by right translations on K, with $\|T_{\tilde{\phi}}\| \le \|\tilde{\phi}\|_{M_0(\tilde{\Gamma})}$. Thus, we have $\mathcal{A} \cong \mathrm{VN}\,(S,\Gamma)\otimes\mathrm{VN}\,(K)$ with $T_{\tilde{\phi}}$ corresponding to $T_\phi\otimes\mathrm{Id}$. It follows that T_ϕ is completely bounded with $\|T_\phi\|_{\mathrm{CB}} \le \|\phi\|_{M_0(\Gamma)}$.

Proof of Theorem 3.1

(a) If $\Lambda(\mathrm{VN}(\Gamma)) = \infty$, there is nothing to prove. Otherwise, for $C > \Lambda(\mathrm{VN}(\Gamma))$, choose a net ϕ_i of finitely supported functions with $\|\phi_i\|_{M_0(\Gamma)} \le C$ and $\phi_i \to 1$ pointwise (Lemma 2.2). By Lemma 3.1, we have that T_{ϕ_i} is completely bounded with $\|T_{\phi_i}\|_{CB} \le C$. We clearly have $T_{\phi_i} m \to m$ for all $m \in M$, and hence for all $m \in \mathrm{VN}(S, \Gamma)$. Finally, since ϕ_i is compactly supported, T_{ϕ_i} is supported on a field of finite subsets on S, and hence $\mathrm{C}(S, \Gamma) \le C$, proving (a).

(b) We suppose there is a finite Γ-invariant measure on S, that $\mathrm{C}(S, \Gamma) < \infty$, and show $\Lambda(\mathrm{VN}(\Gamma)) \le \mathrm{C}(S, \Gamma)$. Choose a net T_i of operators on $\mathrm{VN}(S, \Gamma)$ such that $\|T_i\|_{CB} \le C$, $T_i m \to m$ in weak-* for all $m \in \mathrm{VN}(S, \Gamma)$, and each T_i is supported on a constant field of finite subsets, say $s \to F^i \in \mathcal{F}$ (Lemma 2.7). Define $I: \mathrm{VN}(\Gamma) \to \mathrm{VN}(S, \Gamma)$ and $P: \mathrm{VN}(S, \Gamma) \to \mathrm{VN}(\Gamma)$ by $I(\sum_{\gamma \in \Gamma} c_\gamma \rho(\gamma)) = \sum_{\gamma \in \Gamma} \pi(\gamma) c_\gamma$ (where ρ is the regular representation of Γ and we view c_γ as either a complex number or a constant function on S), and

$$P\left(\sum_{\gamma \in \Gamma} \pi(\gamma) a_\gamma\right) = \sum_{\gamma \in \Gamma} \left(\int_S a_\gamma(s)\, d\mu(s)\right) \rho(\gamma).$$

Then I and P are completely positive maps of completely bounded norm 1. I is the canonical injection of $\mathrm{VN}(\Gamma)$ into $\mathrm{VN}(S, \Gamma)$ and P is the conditional expectation from $\mathrm{VN}(S, \Gamma)$ to $I(\mathrm{VN}(\Gamma))$, brought back to $\mathrm{VN}(\Gamma)$. The net PT_iI of operators on $\mathrm{VN}(\Gamma)$ satisfies $\|PT_iI\|_{CB} \le C$ and $PT_iIx \to x$ in weak-* for all $x \in \mathrm{VN}(\Gamma)$. Furthermore, for each i, PT_iI is supported on $F^i \subset \Gamma$. It follows that $\Lambda(\mathrm{VN}(\Gamma)) \le C$, and hence $\Lambda(\mathrm{VN}(\Gamma)) \le \mathrm{C}(S, \Gamma)$ as asserted.

4. *Proof of Theorem* 1.2

By Theorem 2.3, it suffices to prove:

THEOREM 4.1. *Let $\Gamma \to K$ be an embedding of a (countable) discrete group in a (separable) pro-finite group K. Let Γ act on K via this embedding. Then* $\Lambda(\mathrm{VN}(K, \Gamma)) = \Lambda(\mathrm{VN}(\Gamma))$.

Proof. We have a natural embedding $\mathrm{VN}(\Gamma) \to \mathrm{VN}(K, \Gamma)$ and hence [**4**, Proposition 6.3] $\Lambda(\mathrm{VN}(\Gamma)) \le \Lambda(\mathrm{VN}(K, \Gamma))$. We now show the reverse inequality. By Theorem 3.1, and Lemma 2.7, for any $\varepsilon > 0$ we can find a net T_i of operators on $\mathrm{VN}(K, \Gamma)$ such that $T_i m \to m$ in weak-* for all m, $\|T_i\|_{CB} \le \Lambda(\mathrm{VN}(\Gamma)) + \varepsilon$, and each T_i has support on a constant field of finite subsets of Γ. We can choose a sequence of compact open subgroups $K_n \subset K$ with $K_{n+1} \subset K_n$ and $\bigcap K_m = \{e\}$. Let $E_n: L^\infty(K) \to L^\infty(K)^{K_n} \cong L^\infty(K/K_n)$ be the canonical conditional expectation given by averaging over K_n cosets. We extend E_n to a map defined on operators on $\mathrm{VN}(K, \Gamma)$ which are supported on a constant field of finite subsets of Γ by setting

$$E_n\left(\sum_{\gamma \in \Gamma} \pi(\gamma) a_\gamma\right) = \sum_{\gamma \in \Gamma} \pi(\gamma)(E_n a_\gamma).$$

Then E_n extends to a conditional expectation $\mathrm{VN}(K, \Gamma) \to \mathrm{VN}(K/K_n, \Gamma) \hookrightarrow \mathrm{VN}(K, \Gamma)$ [**10**]. Thus, E_n is completely positive and $\|E_n\|_{CB} = 1$. Since K/K_n is finite, it is clear that $E_n T_i$ has finite dimensional range for each n, i. Furthermore

$E_nT_im \to T_im$ for each i and each $m \in \mathrm{VN}(K, \Gamma)$ since $\bigcup L^\infty(K)^{K_n}$ is dense in $L^\infty(K)$. Finally, $\|E_nT_i\|_{CB} \leq \|T_i\|_{CB}$. Thus, $\Lambda(\mathrm{VN}(K, \Gamma)) \leq \Lambda(\mathrm{VN}(\Gamma)) + \varepsilon$, and this proves the theorem. □

5. *Proof of Theorems* 1.3 *and* 1.4

Let $P \to M$ be the frame bundle of M. Let $A(M) \subset \mathrm{Diff}(M)$ be the subgroup leaving the Γ-invariant connection invariant. There is a $A(M)$-invariant smooth Riemannian metric on the manifold P [**9**], and we let $\mathrm{Iso}(P)$ be the isometry group of P with respect to this metric. We thus have inclusions $\Gamma \to A(M) \to \mathrm{Iso}(P)$. $\mathrm{Iso}(P)$ and $A(M)$ are (not necessarily connected) Lie groups and $A(M) \subset \mathrm{Iso}(P)$ is closed. The group $\mathrm{Iso}(P)$ acts properly on P, and Γ will act properly on P if and only if Γ is closed in $\mathrm{Iso}(P)$. For the Levi-Civita connection on a compact Riemannian M, this is the case if and only if $\Gamma \cap \mathrm{Iso}(M)$ is finite, as follows easily from a classical theorem of Yano [**9**, p. 125]. To prove Theorems 1.3, 1.4, it suffices to show:

Lemma 5.1. *Assume the hypotheses of Theorem* 1.3.
(a) *If* Γ *acts properly on* P, *then* $n \leq m$.
(b) *If* Γ *preserves the Levi-Civita connection, does not act properly, and* $m = 2$, *then* Γ *preserves a smooth Riemannian metric on* M.

We first turn to the proof of assertion (b) of the lemma. We shall need the notion of algebraic hull of a measurable cocycle. We refer the reader to [**15**] for a detailed discussion of this notion in a geometric framework. Here, we recall some salient features. If we choose a measurable trivialization of P (or equivalently, a measurable section of P), then the Γ action on $P \cong M \times GL(n)$ can be expressed as $\gamma(m, g) = (\gamma m, \alpha(\gamma, m)g)$ where $\alpha: \Gamma \times M \to GL(n)$ satisfies the cocycle equation. Choosing a different measurable section is equivalent to choosing a cocycle equivalent to α. If there is a measure μ on M, quasi-invariant and ergodic for the Γ-action, then up to conjugacy there is a unique algebraic subgroup $H \subset GL(n)$ such that α is equivalent to a cocycle taking all values in H, but not equivalent to a cocycle taking all values in a proper algebraic subgroup of H. Then H is called the algebraic hull of α(or of the action on the principal bundle P). (It depends on the measure μ.) Our approach to proving assertion (b) is based on the following lemma.

Lemma 5.2. *Assume the hypotheses of* 5.1. *Suppose there is a finite* Γ*-invariant ergodic measure on* M *such that every algebraic Kazhdan subgroup of the algebraic hull of the action on* P *is compact. Then there is a smooth* Γ*-invariant Riemannian metric on* M.

Proof. Since Γ is a Kazhdan group and the algebraic hull is Kazhdan by [**18**], the algebraic hull is compact. As in [**16**, Lemma 4.11], this implies that there is a finite Γ-invariant measure on P. The argument of [**16**, Theorem 5.4] then implies the lemma.

We also need the following two facts.

Lemma 5.3. *Any proper algebraic Kazhdan subgroup of* $\mathrm{Sp}(1, 2)$ *is compact.*

Proof. This holds for simple subgroups by examination, from which the result for general algebraic subgroups follows easily.

LEMMA 5.4. *Let* $\Gamma_0 \subset \Gamma$ *be a non-trivial torsion-free normal subgroup. Then* Γ_0 *is not abelian.*

Proof. Let $\bar{\Gamma}_0$ be the Zariski closure of Γ_0 in Sp $(1, n)$. Then Γ normalizes $(\bar{\Gamma}_0)^0$, and hence Sp $(1, n)$ does as well by the Borel density theorem [**14**]. Since $\bar{\Gamma}_0$ is infinite, we have $\bar{\Gamma}_0 = Sp\,(1, n)$, and hence Γ_0 is not abelian.

Proof of 5.1(b) By passing to a group of finite index, we can assume Γ is torsion free. Let $\bar{\Gamma}$ be the closure (usual Lie group topology) of Γ in $A(M)$. The hypotheses of (b) imply that $\Gamma \neq \bar{\Gamma}$. Since the Γ-invariant connection on M is Riemannian, the identity component $A(M)^0$ is compact [**9**, p. 125]. Let $K = (\bar{\Gamma})^0 \subset A(M)^0$. Let $K_1 \subset K$ be a connected semisimple normal subgroup such that K/K_1 is abelian. Clearly $K_1 \subset K$ is a characteristic subgroup. Let $\Gamma_0 = \Gamma \cap K$ and $\Gamma_1 = \Gamma \cap K_1$. Since Γ is torsion free, either Γ_1 is trivial or infinite. If it is trivial, then Γ_0 projects injectively into K/K_1, and hence Γ_0 is abelian. However it is clearly normal in Γ which implies by Lemma 5.4 that Γ_0 is trivial. This in turn would imply that $\Gamma \subset A(M)$ is discrete, contradicting the assumption $\Gamma \neq \bar{\Gamma}$. Therefore, we may assume Γ_1 is infinite.

Since $\Gamma_0 \subset \Gamma$ is normal and $K_1 \subset K$ is characteristic, it follows that $\Gamma_1 \subset \Gamma$ is normal. We let $Q = \bar{\Gamma}_1 \subset K_1$. Then Q is normalized by Γ, and hence by K. In particular, $Q \subset K_1$ is normal, and hence Q is semisimple (and non-trivial). For $m \in M$, let $Q_m \subset Q$ be the stabilizer of m in Q, and let $q_m \subset q$ be the corresponding inclusion of Lie algebras. For each $m \in M$, we have a natural identification of q/q_m with the tangent space at m of the Q-orbit of m in M, i.e. with a subspace $V_m \subset TM_m$. Since Q is compact and semisimple, we can fix an Aut (q)-invariant positive definite inner product on q. This induces in a canonical way an inner product on q/q_m, and hence on V_m. Furthermore, for any $\gamma \in \Gamma$ (or more generally for any element of $N_{A(M)}(Q)$) and any $m \in M$, $d\gamma_m \colon V_m \to V_{\gamma_m}$ and the following diagram commutes:

$$\begin{array}{ccc} q/q_m & \xrightarrow{A(\gamma)} & q/q_{\gamma m} = q/A(\gamma)(q_m) \\ \downarrow & & \downarrow \\ V_m & \xrightarrow{d\gamma_m} & V_{\gamma m} \end{array}$$

where γ acts on q via the automorphism $A(\gamma)$. Since the inner product on q is Aut (q) invariant, it follows that we have a Γ-invariant assignment to each $m \in M$ of an inner product on V_m. (We remark that $m \to V_m$ may not be of constant dimension, so the assignment may not be globally smooth. However, since dim $Q > 0$ and Q acts faithfully by definition, we have dim $V_m > 0$ for m in a non-trivial open set. Moreover, the assignment of inner product is clearly measurable.) It follows that for each Γ-ergodic component μ of the Γ action on M (with respect to the smooth measure class), the algebraic hull H will act orthogonally on a subspace V with dim $V =$ dim V_m for μ a.e. m in this ergodic component. In particular, for a set of ergodic components of positive measure, dim $V > 0$. We also have $H \subset$ Sp (1, 2) since the Γ-action preserves a Sp (1, 2)-structure. Since the structural representation of Sp (1, 2) is irreducible, we deduce that $H \subset \mathrm{Sp}(1, 2)$ is proper.

An application of Lemmas 5.2, 5.3 then completes the proof of Lemma 5.1(b). □

Proof of Lemma 5.1. (a) Let $P' \to M$ denote the Γ-invariant G-structure on M, where $G = \mathrm{Sp}(1, m)$. We first recall that since Γ acts properly on P', for any compact sets $K, K' \subset P'$, $\{(\gamma, p) \in \Gamma \times K' \mid \gamma p \in K\}$ is precompact in $\Gamma \times K'$ [**11**]. We can choose a bounded measurable section of the projection $P' \to M$, i.e. the image of the section is precompact. It follows that the corresponding trivialization $P' \cong M \times G$ preserves precompact sets. We let $\alpha: \Gamma \times M \to G$ be the cocycle defined by the action on P' via the above trivialization. We then have:

LEMMA 5.5. (a) *For any compact set $K \subset G$, there exists $N \in \mathbb{Z}^+$ such that for all $m \in M$,*

$$\mathrm{card}\,(\{\gamma \in \Gamma \mid \alpha(\gamma, m) \in K\}) \leq N.$$

(b) *For any $\gamma \in \Gamma$, $\{\alpha(\gamma, m) \mid m \in M\}$ is precompact.*

Another consequence of the fact that Γ acts properly is that P'/Γ is Hausdorff. In particular, the action is smooth in the sense of ergodic theory [**14**], or tame in the terminology of [**15**]. This latter condition is equivalent to the existence of a fundamental set $D \subset P'$ for the Γ-action, i.e. a measurable set D such that $\{\gamma D \mid \gamma \in \Gamma\}$ are distinct and P is the disjoint union of these sets [**14**, Appendix A]. Lemma 5.1(a) follows from Lemma 2.2, Theorem 2.3 and the following general result.

THEOREM 5.6. *Let Γ be a (countable) discrete group acting on a measure space (X, μ) where μ is finite and Γ-invariant. Suppose G is a locally compact group and $\alpha: X \times \Gamma \to G$ is a (measurable) cocycle such that*

(i) *the corresponding action of Γ on $X \times G$ (given by $(x, g)\gamma = (x\gamma, g\alpha(x, \gamma))$) is tame;*

(ii) *For each compact $K \subset G$, there is an integer N such that for all $x \in X$,*

$$\mathrm{card}\,\{\gamma \in \Gamma \mid \alpha(x, \gamma) \in K\} \leq N; \text{ and}$$

(iii) *for each $\gamma \in \Gamma$, $\{\alpha(x, \gamma) \mid x \in X\}$ is precompact. Then $\Lambda(\Gamma) \leq \Lambda(G)$ (where these are as in Lemma 2.2).*

For the proof, we shall need the following lemma which ensures that we can choose a fundamental set with a special property.

LEMMA 5.7. *Assume the hypotheses of 5.6. Then there is a fundamental set $D \subset X \times G$ for the Γ-action such that if ν is a finite measure on D in the class of the restriction of the product measure on $X \times G$ to D, then the projection of ν to a measure on X is in the same measure class as μ. (We only require here that D be a fundamental set modulo null sets,* i.e. *fundamental in the measure algebra.)*

Proof. This is basically an exercise in measure theory. We sketch the arguments when G is a continuous group, leaving details of verification and case of G discrete to the reader. For a set $A \subset X \times G$, and $x \in X$, we set $A^x = \{g \in G \mid (x, g) \in A\}$. Let $X_A = \{x \in X \mid \mathrm{Haar}\,(A^x) > 0\}$. It suffices to show that we can find a fundamental set D with X_D conull in X. Fix any fundamental set A. Since $\bigcup_{\gamma \in \Gamma} A\gamma = X \times G$, we must have $\bigcup_{\gamma\in\Gamma}(X_A)\cdot\gamma = X$. We can thus choose a sequence of measurable sets $X_i \subset X_A$ (not necessarily disjoint) and a sequence γ_i in Γ such that X is the disjoint union $X = \bigcup_i X_i\gamma_i$ where $X_1 = X_A$, $\gamma_1 = e$. We fix these sequences together with an ordering on them. Let $n: X_A \to \{\infty\}\cup\mathbb{Z}^+$ be $n(x) = \text{card}\{i \mid x\in X_i\}$. For each j, we can choose subsets (measurable) $B_{k,j}\subset A\cap(n^{-1}(j)\times G)$, $k\le j$ (or $k<j$ for $j=\infty$), such that for each $x\in n^{-1}(j)$, $\{B^x_{k,j}\}(1\le k\le j)$ is a partition of A^x into disjoint sets of positive Haar measure. For each i and $x\in X_i\gamma_i$, let $k(i,x) = \text{card}\{r\le i \mid x\gamma_i^{-1}\in X_r\}$. Thus, $k(i,x)\le n(x\gamma_i^{-1})$. Define $D^x = B^{x\gamma_i^{-1}}_{k(i,x),n(x\gamma_i^{-1})}\cdot\gamma_i$. Then $D = \bigcup_{x\in X} D^x$ is the required fundamental set. □

We now rephrase Lemma 5.7 in a convenient form. Since D is a fundamental set, the map $D\times\Gamma\to X\times G$, $(\xi,\gamma)\to\xi\cdot\gamma$ is a measure space bijection. Letting $x(\xi,\gamma)$, $g(\xi,\gamma)$ denote the corresponding components, we have

$$(*)\qquad \begin{cases} x(\xi,\gamma\bar\gamma) = x(\xi,\gamma)\bar\gamma \text{ and} \\ g(\xi,\gamma\bar\gamma) = g(\xi,\gamma)\alpha(x(\xi,\gamma),\bar\gamma). \end{cases}$$

Let D have the measure given by restriction of the product measure. Choose any $m\in L^1(D)$ with $m>0$ on D, such that for (almost) any $x\in X$, $\int_G m^x\,dg = 1$ where $m^x(g) = m(x,g)$ if $(x,g)\in D$, and $m^x(g) = 0$ if $(x,g)\notin D$. Such an m exists by virtue of Lemma 5.7. We then have the relation:

$$(**)\qquad \int_X f(x)\,dx = \int_D f(x(\xi,e))m(\xi)\,d\xi$$

for any $f\in L^\infty(X)$.

We also need another well-known characterization of elements of $M_0(G)$.

LEMMA 5.8. [4]. *If G is a locally compact group, and $u\in C(G)$, define $Mu\in C(G\times G)$ by $Mu(\bar g, g) = u(g^{-1}\bar g)$. Then $u\in M_0(G)$ if and only if pointwise multiplication by Mu defines a bounded operator on the projective tensor product $L^2(G)\hat\otimes L^2(G)$, and in that case $\|u\|_{M_0(G)}$ is just the operator norm of multiplication by Mu on this tensor product.*

Proof of Theorem 5.6. Choose a net $u_i\in A_c(G)$ such that $u_i\to 1$ uniformly on compact sets and $\|u_i\|_{M_0(G)}\le C$ where $C>\Lambda(G)$. We shall produce a family $\nu_i\in A(\Gamma)$ such that $\nu_i\to 1$ pointwise and $\|\nu_i\|_{M_0(\Gamma)}\le C$. By Lemma 2.2, this suffices to prove the theorem.

Given $u\in C(G)$, let $u^*: X\times G\times X\times G\to C$ be given by $u^*(\bar x,\bar g,x,g) = u(g^{-1}\bar g)$ and $\tilde u: D\times\Gamma\times D\times\Gamma\to C$ be given by

$$\begin{aligned}\tilde u(\bar\xi,\bar\gamma,\xi,\gamma) &= u^*(x(\bar\xi,\bar\gamma), g(\bar\xi,\bar\gamma), x(\xi,\gamma), g(\xi,\gamma)) \\ &= u(g(\xi,\gamma)^{-1}g(\bar\xi,\bar\gamma)).\end{aligned}$$

If $u\in M_0(G)$, then one sees easily that the pointwise multiplier on $L^2(X\times G)\hat\otimes L^2(X\times G)$ given by

$$\sum_n \hat f_n\otimes f_n \mapsto u^*\sum_n \hat f_n\otimes f_n$$

is bounded with norm equal to $\|u\|_{M_0(G)}$. (This is essentially the argument of the proof of Lemma 3.2.) Via the measure space isomorphism $D\times\Gamma\to X\times G$, we deduce that the pointwise multiplier on $L^2(D\times\Gamma)\hat{\otimes}L^2(D\times\Gamma)$ given by

$$\sum_n \hat{\phi}_n\otimes\phi_n\mapsto\tilde{u}\sum_n\hat{\phi}_n\otimes\phi_n$$

is also bounded with norm equal to $\|u\|_{M_0(G)}$.

To obtain a pointwise multiplier operator on $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$, we first inject $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$ into $L^2(D\times\Gamma)\hat{\otimes}L^2(D\times\Gamma)$, then multiply by $\tilde{u}$, then project back into $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$ using a trace map. More precisely, define I: $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)\to L^2(D\times\Gamma)\hat{\otimes}L^2(D\times\Gamma)$ by the formula

$$I\left(\sum_n\hat{\eta}_n\otimes\eta_n\right)=\sum_n(m^{1/2}\otimes\hat{\eta}_n)\otimes(m^{1/2}\otimes\eta_n).$$

Since

$$\left(\int_{D\times\Gamma}|m^{1/2}(\xi)\eta(\gamma)|^2\,d\xi\,d\gamma\right)^{1/2}=\left(\int_D m(\xi)\,d\xi\right)^{1/2}\|\eta\|_2=\|\eta\|_2,$$

we have $\|I\|\le 1$. Similarly, define P: $L^2(D\times\Gamma)\hat{\otimes}L^2(D\times\Gamma)\to L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$ by

$$P\left(\sum_n\hat{\phi}_n\otimes\phi_n\right)=\sum_n\int_D\hat{\phi}_n(\xi,\cdot)\otimes\phi_n(\xi,\cdot)\,d\xi.$$

To see that this defines an operator with $\|P\|\le 1$, it suffices to show that for any $\hat{\phi},\phi\in L^2(D\times\Gamma)$, $P(\hat{\phi}\otimes\phi)\in L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$, and that $\|P(\hat{\phi}\otimes\phi)\|\le\|\hat{\phi}\|_2\|\phi\|_2$. Fix an orthogonal basis $\{e_j\}$ for $L^2(D)$. Then we can write

$$\hat{\phi}(\xi,\gamma)=\sum_j\langle\hat{\phi},e_j\rangle(\gamma)e_j(\xi)$$

where

$$\langle\hat{\phi}_j,e_j\rangle(\gamma)=\int_D\hat{\phi}(\xi,\gamma)\bar{e}_j(\xi)\,d\xi$$

and

$$\sum_j\int_\Gamma|\langle\hat{\phi}_j,e_j\rangle(\gamma)|^2\,d\gamma=\|\hat{\phi}\|_2^2.$$

We have a similar expression

$$\phi(\xi,\gamma)=\sum_j\langle\phi,e_j\rangle(\gamma)e_j(\xi).$$

Then

$$\langle\hat{\phi},e_j\rangle,\langle\phi,e_j\rangle\in L^2(\Gamma),$$
$$P(\hat{\phi}\otimes\phi)=\sum_j\langle\hat{\phi},e_j\rangle\otimes\langle\phi,e_j\rangle,$$

and

$$\sum_j\|\langle\hat{\phi},e_j\rangle\|_2\|\langle\phi,e_j\rangle\|_2\le\left(\sum_j\|\langle\hat{\phi},e_j\rangle\|_2^2\right)^{1/2}\left(\sum_j\|\langle\phi,e_j\rangle\|_2^2\right)^{1/2}=\|\hat{\phi}\|_2\|\phi\|_2.$$

Thus, $\|P\|\le 1$ as asserted. □

Consider now the map $P\tilde{u}I$, where we identify $\tilde{u}$ with the pointwise multiplication operator it defines. This is a bounded map on $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$ of norm at most

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$\|u\|_{M_0(G)}$. We now explicitly compute $P\tilde{u}I$. We have

$$P\tilde{u}I(\hat{\eta}\otimes\eta)(\bar{\gamma},\gamma)=\int_D (\tilde{u}I(\hat{\eta}\otimes\eta))(\xi,\bar{\gamma},\xi,\gamma)\,d\xi$$

$$=\int_D u(g^{-1}(\xi,\gamma)g(\xi,\bar{\gamma}))I(\hat{\eta}\otimes\eta)(\xi,\bar{\gamma},\xi,\gamma)\,d\xi$$

$$=\int_D u(\alpha(x(\xi,\gamma),\gamma^{-1}\bar{\gamma}))m(\xi)\hat{\eta}(\bar{\gamma})\eta(\gamma)\,d\xi$$

(by equation (*))

$$=\left(\int_D u(\alpha(x(\xi,e)\gamma,\gamma^{-1}\bar{g}))m(\xi)\,d\gamma\right)\hat{\eta}(\bar{g})\eta(\gamma)$$

$$=\left(\int_X u(\alpha(x\gamma,\gamma^{-1}\bar{\gamma}))\,dx\right)\hat{\eta}(\bar{\gamma})\eta(\gamma)$$

(by equation (**))

$$=\left(\int_X u(\alpha(x,\gamma^{-1}\bar{\gamma}))\,dx\right)\hat{\eta}(\bar{\gamma})\eta(\gamma)$$

(by Γ-invariance of dx)

$$=\nu(\gamma^{-1}\bar{\gamma})\hat{\eta}(\bar{\gamma})\eta(\gamma)$$

where

$$\nu(\gamma)=\int_X u(\alpha(x,\gamma))\,dx.$$

I.e. $P\tilde{u}I$ is pointwise multiplication by M_ν on $L^2(\Gamma)\hat{\otimes}L^2(\Gamma)$. It follows from Lemma 5.8 that $\nu\in M_0(\Gamma)$ and $\|\nu\|_{M_0(\Gamma)}\leq\|u\|_{M_0(G)}$. Furthermore, if u is supported on a compact set $K\subset G$,

$$\int_\Gamma|\nu(\gamma)|\,d\gamma\leq\int_{X\times\Gamma}|u(\alpha(x,\gamma))|\,dx\,d\gamma\leq N\cdot\|u\|_\infty$$

where N is as in (ii) of the hypotheses. Hence $\nu\in l^1(\Gamma)\subset A(\Gamma)$.

Now let u_i be as in the first paragraph of the proof. Let $\nu_i(\gamma)=\int_X u_i(\alpha(x,\gamma))\,dx$. Then we have seen $\nu_i\in A(\Gamma)$ and $\|\nu_i\|_{M_0(\Gamma)}\leq\|u_i\|_{M_0(G)}\leq C$. Finally, since $u_i\to 1$ uniformly on compact sets, condition (iii) of the hypotheses implies $\nu_i\to 1$ pointwise on Γ. This completes the proof of Theorem 5.6.

6. *Proof of Theorem* 1.5 *and Corollary* 1.6

Suppose that Γ acts properly on the universal covering space $\tilde{M}$ of a compact manifold M, and commutes with the fundamental group $\pi_1(M)$, and further that there is a finite Γ-invariant measure for the quotient action on M. We choose a Borel section $\sigma: M\to\tilde{M}$ such that $\sigma(M)$ is precompact, and then, measure-theoretically, we identify $\tilde{M}$ with $M\times\pi_1(M)$. There is a measurable cocycle $\alpha: M\times\Gamma\to\pi_1(M)$ such that the action of Γ on $\tilde{M}$ can be identified with the action of Γ on $M\times\pi_1(M)$ by the formula

$$(m,g)\gamma=(m\gamma,g\alpha(m,\gamma))\quad\forall m\in M,\quad\forall g\in\pi_1(M),\quad\forall\gamma\in\Gamma.$$

Since Γ acts properly on $\tilde{M}$, the Γ-action on $M \times \pi_1(M)$ is 'tame' in the terminology of [**15**] ('smooth' in [**14**]). Further, if K is a finite subset of $\pi_1(M)$, there is an integer N such that

$$\operatorname{card}\{\gamma \in \Gamma \colon \alpha(m, \gamma) \in K\} \leq N \quad \forall m \in M.$$

Finally, if $\gamma \in \Gamma$, then $\{\alpha(m, \gamma) \colon m \in M\}$ is precompact. All these claims follow from the reasoning used to prove Lemma 5.5 and in the discussion immediately following that result. Theorem 5.6, with $\pi_1(M)$ in place of G, now applies to prove Theorem 1.5.

Gromov [7] has shown if a semisimple group H with finite fundamental group and no compact factors acts real analytically on a compact manifold M and preserves a connection, then the natural action of $\tilde{H}$ on $\tilde{M}$ (which commutes with $\pi_1(M)$) is proper. We suppose $H = \mathrm{Sp}(1, n)$, and then $H = \tilde{H}$, i.e., H itself acts properly on $\tilde{M}$. The restriction of this action to a lattice Γ in H is a fortiori proper, and Theorem 1.5 may be applied to show that $\Lambda(\pi_1(M)) \geq \Lambda(\Gamma)$. Since $\Lambda(\Gamma) = \Lambda(H) = 2n-1$ (see § 2), Corollary 1.6 is proved.

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REFERENCES

[1] A. Connes, J. Feldman & B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergod. Th. & Dynam. Sys.* **1** (1981), 431-450.
[2] A. Connes & V. Jones. A II_1-factor with two non-conjugate Cartan subalgebras. *Bull. A.M.S.* **6** (1982) 211-212.
[3] J. de Canniere & U. Haagerup. Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups. *Amer. J. Math.* **107** (1984), 455-500.
[4] M. Cowling & U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Preprint.
[5] H. Dye. On groups of measure preserving transformations. *Amer. J. Math.* **81** (1959), 119-159.
[6] J. Feldman & C. C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras, II. *Trans. Amer. Math. Soc.* **234** (1977), 325-359.
[7] M. Gromov. Rigid transformation groups. To appear.
[8] U. Haagerup. Group C^*-algebras without the completely bounded approximation property. Preprint.
[9] S. Kobayashi. *Transformation Groups in Differential Geometry* (Springer: New York, 1972).
[10] J. Packer. On the embedding of subalgebras corresponding to quotient actions in group measure space factors. Preprint.
[11] R. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. Math.* **73** (1961), 295-323.
[12] R. J. Zimmer. Strong rigidity for ergodic actions of semisimple Lie groups. *Ann. Math.* **112** (1980), 511-529.
[13] R. J. Zimmer. On the cohomology of ergodic actions of semisimple Lie groups and discrete subgroups. *Amer. J. Math.* **103** (1981), 937-950.
[14] R. J. Zimmer. *Ergodic Theory and Semisimple Groups* (Birkhauser: Boston, 1984).

[15] R. J. Zimmer. Ergodic theory and the automorphism group of a G-structure, in C. C. Moore, ed., *Group Representations, Ergodic Theory, Operator Algebras, and Mathematical Physics* (Springer: New York, 1987) 247–278.

[16] R. J. Zimmer. Lattices in semisimple groups and invariant geometric structures on compact manifolds, R. Howe, ed., *Discrete Groups in Geometry and Analysis* (Birkhauser: Boston, 1987) 152–210.

[17] R. J. Zimmer. Actions of lattices in semisimple groups preserving a geometric structure of finite type. *Ergod. Th. & Dynam. Sys.* **5** (1985), 301–306.

[18] R. J. Zimmer. Kazhdan groups acting on compact manifolds. *Invent. Math.* **75** (1984), 425–436.

[19] R. J. Zimmer. Representations of fundamental groups of manifolds with a semisimple transformation group. *J. Amer. Math. Soc.*, to appear.