TWO TERM CONDITIONS IN π EXACT COUPLES

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1. Introduction. In achieving his celebrated results on the homology groups of fibre spaces, J. P. Serre used the exact couple of a fibring defined by J. Leray. One of his main tools was the so-called two-term condition on the E^2 term of this exact couple, which, if satisfied, yielded exact sequences, such as those of Gysin and Wang (see **(5)**, Chapter IX). H. Federer, in **(3)**, defined an exact couple $\mathfrak{C}(X, Y, v)$ on the mapping space $M(X, Y) = \{f\colon X \to Y \mid X, Y$ are spaces and f is continuous$\}$ with the compact-open topology, where X is a locally finite CW complex and Y is arc-connected and n-simple for all n. The purpose of this paper is to find a two-term condition for the exact couple of H. Federer and to see what results can be derived from this condition.

In Chapter I we formulate a two-term condition for π exact couples, of which $\mathfrak{C}(X, Y, v)$ is a special case. We also give a necessary condition that the differential operator

$$d^i\colon E^i_{n,0} \to E^i_{n-1,i}$$

in $\mathfrak{C}^i(X, Y, v)$ be zero for $i \geqslant 2$.

In Chapter II we give necessary conditions on Y (Gap Theorem I) and on X (Gap Theorem II) that a two-term condition hold on $\mathfrak{C}(X, Y, v)$. These theorems yield exact sequences involving $\pi_i(M(X, Y), v)$ and $H^i(X, \pi_j(Y))$. Using these theorems, we then compute some of the homotopy groups of $M(X, Y)$ where $Y = \mathbf{U}$, the infinite unitary group, or $\mathbf{O}$, the infinite orthogonal group, and $\dim X \leqslant 4$ or 5, respectively.

CHAPTER I. PRELIMINARIES

2. π Exact couples. Let $\mathfrak{C} = \{D, E, i, j, k\}$ be an exact couple in the sense of Federer; i.e., D is a (not necessarily abelian) group, E is an abelian group, and i, j, and k are homomorphisms such that the following triangle is exact (see **(3)**):

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

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Suppose that D and E are bigraded, i.e.,

$$D = \sum \oplus D_{p,q}, \qquad E = \sum \oplus E_{p,q} \qquad (p, q \in Z, \text{ the integers}),$$

and i, j, k are homomorphisms such that $\deg i = (0, -1)$, $\deg j = (-1, 1)$, and $\deg k = (0, 0)$. It can easily be shown that $\deg i^n = (0, -1)$, $\deg j^n = (-1, n)$, $\deg k^n = (0, 0)$, and $\deg d^n = (-1, n)$, where $d^n = j^n \circ k^n$, in the derived couple $\mathfrak{C}^n$ (see **(5)**, Chapter VIII).

Definition 2.1. We call an exact couple $\mathfrak{C}$ satisfying the above a π *exact couple* if and only if $\mathfrak{C}$ satisfies:

(1) $D_{p,q} = 0$ if $p \leqslant 0, q < 0$,

(2) $E_{p,q} = 0$ if $q < 0$,

(3) there exists a positive integer k such that $E_{p,q} = 0$ for $q > k$.

Condition (3) ensures the finite convergence of the spectral sequence.

PROPOSITION 2.2. *Let $\mathfrak{C}$ be a π exact couple. Then*

(4) $E_{p,q} = 0$ *for* $p < 0$,

(5) *for any p, and if $q > k$, then $D_{p,q} \approx D_{p,k}$ via $i^{(q-k)}$ ($(q-k)$th iterate of i),*

(6) *for $n >$ $\max(q, k - q)$ we have*

$$E^n_{p,q} = E^{n+1}_{p,q} = \ldots = E^\infty_{p,q}.$$

Definition 2.3. Let $D_{p,k}$ be denoted by $\pi_p(\mathfrak{C})$. We filter $\pi_p(\mathfrak{C})$ as follows:

$$\pi_p(\mathfrak{C}) = \pi_{p,-1} \supset \pi_{p,0} \supset \pi_{p,1} \supset \ldots \supset \pi_{p,k-1} \supset \pi_{p,k} = 0,$$

where $\pi_{p,q} = \ker \{i^{(k-q)}: D_{p,k} \to D_{p,q}\}$ for $q < k$ and $\pi_{p,k} = 0$. This is called the filtration of $\pi_p(\mathfrak{C})$. See **(6)**.

The following proposition shows that $\pi_{p,q-1}$ is an extension of $\pi_{p,q}$ by $E^\infty_{p,q}$. The proof is given in **(3**, p. 351**)**.

PROPOSITION 2.4. $\pi_{p,q-1}/\pi_{p,q} \approx E^\infty_{p,q}$ *for* $q \leqslant k$.

3. The two-term condition. Let $\mathfrak{C}$ be a π exact couple.

Definition 3.1. We say that $\mathfrak{C}$ satisfies the two-term condition $\{\lambda, \mu; \nu\}$, where λ, μ, and ν are integers such that $\lambda \leqslant \mu$, $\nu \geqslant 1$, if and only if E^ν satisfies (a), (b), and (c) below.

(a) For each integer m such that $\lambda \leqslant m \leqslant \mu$, $E^\nu_{m,q} = 0$ unless $q = a_m$ or b_m, $a_m < b_m$.

(b) $E^\nu_{m-1,q} = 0$ if $q \geqslant b_m + \nu (\lambda \leqslant m \leqslant \mu)$.

(c) $E^\nu_{m+1,q} = 0$ if $q \leqslant a_m - \nu (\lambda \leqslant m \leqslant \mu)$.

THEOREM 3.2. *If $\mathfrak{C}$ satisfies the two-term condition $\{\lambda, \mu; \nu\}$, then the following sequence is exact:*

$$\begin{aligned} E^\nu_{\mu,b_\mu} &\xrightarrow{\phi_\mu} \pi_\mu(\mathfrak{C}) \xrightarrow{\psi_\mu} E^\nu_{\mu,a_\mu} \xrightarrow{\theta_\mu} \ldots \xrightarrow{\theta_{m+1}} E^\nu_{m,b_m} \\ &\xrightarrow{\phi_m} \pi_m(\mathfrak{C}) \xrightarrow{\psi_m} E^\nu_{m,a_m} \xrightarrow{\theta_m} E^\nu_{m-1,b_{m-1}} \xrightarrow{\phi_{m-1}} \ldots \\ &\xrightarrow{\psi_\lambda} E^\nu_{\lambda,a_\lambda}. \end{aligned}$$

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The proof of Theorem 3.2 is similar to the proof in (**5**, p. 240).

Note: It is of interest for §§7 and 10 to see how the homomorphism

$$\theta_m: E^{\nu}_{m,a_m} \longrightarrow E^{\nu}_{m-1,b_{m-1}}$$

is defined. Let $r_{m-1} = b_{m-1} - a_m$ for $m > \lambda$. Then

$$\theta_m = \begin{cases} d^{r_{m-1}} & \text{if } r_{m-1} \geqslant \nu, \\ 0 & \text{if } r_{m-1} < \nu. \end{cases}$$

COROLLARY 3.3. *Let* $0 \leqslant a < b \leqslant k$, *where* k *is that number such that* $E_{p,q} = 0$ *for* $q > k$. *If* $E^{\nu}_{p,q} = 0$ *unless* $q = a, b$ *for some* $\nu \geqslant 1$ *and all* $p \geqslant 0$, *then the following beginningless sequence is exact:*

$$\ldots \longrightarrow E^{\nu}_{p,b} \xrightarrow{\phi_p} \pi_p(\mathfrak{C}) \xrightarrow{\psi_p} E^{\nu}_{p,a} \xrightarrow{\theta_p} E^{\nu}_{p-1,b} \xrightarrow{\phi_{p-1}} \ldots$$
$$\xrightarrow{\phi_1} \pi_1(\mathfrak{C}) \xrightarrow{\psi_1} E^{\nu}_{1,a} \xrightarrow{\theta_1} E^{\nu}_{0,b} \longrightarrow 0.$$

COROLLARY 3.4. *Let* $\{b_i\}$ $(\lambda \leqslant i \leqslant \mu)$ *be a sequence of integers such that, for some* $\nu \geqslant 1$, $b_i - \nu < b_{i+1}$ *for* $\lambda \leqslant i < \mu$ *and* $b_i \leqslant k$ *for* $\lambda \leqslant i \leqslant \mu$. *If* $E^{\nu}_{p,q} = 0$ *unless* $q = b_p$ *for* $\lambda \leqslant p \leqslant \mu$ *and if* $E_{\lambda-1,q} = 0$ *if* $q \geqslant b_\lambda + \nu$, *then*

$$\pi_p(\mathfrak{C}) \approx E^{\nu}_{p,b_p} \qquad (\lambda \leqslant p \leqslant \mu).$$

Proof. Letting

$$a_p = \begin{cases} 0 & \text{if } b_p \neq 0 \\ -1 & \text{if } b_p = 0 \end{cases} \qquad (\lambda \leqslant p \leqslant \mu)$$

we obtain a two-term condition $\{\lambda, \mu; \nu\}$ and $E^{\nu}_{p,a_p} = 0$ for $\lambda \leqslant p \leqslant \mu$. Theorem 3.2 does the rest.

COROLLARY 3.5. *If* $E^{\nu}_{p,q} = 0$ *for* $1 \leqslant \lambda \leqslant p \leqslant \mu$, *then* $\pi_p(\mathfrak{C}) = 0$ *for* $\lambda \leqslant p \leqslant \mu$.

4. Extended two-term conditions. Let $\mathfrak{C}$ be a π exact couple. The next two theorems give conditions under which the exact sequence of Theorem 3.2 may be extended one extra term.

THEOREM 4.1 (Left Extended Two-Term Condition). *Suppose* $\mathfrak{C}$ *satisfies the two-term condition* $\{\lambda, \mu; \nu\}$. *In addition suppose that for* $p = \lambda - 1$ *and* $q \geqslant a_\lambda + \nu$, $E^{\nu}_{p,q} = 0$ *unless* $q = q_0$ $(a_\lambda + \nu \leqslant q_0 < b_\lambda + \nu)$ *and for* $p = \lambda - 2$, $E^{\nu}_{p,q} = 0$, $q_0 + \nu \leqslant q < q_0 + r$ *where* $r = q_0 - a_\lambda$ (see Figure 4.1). *Then the sequence*

$$E^{\nu}_{\mu,b_\mu} \xrightarrow{\phi_\mu} \ldots \xrightarrow{\phi_\lambda} \pi_\lambda(\mathfrak{C}) \xrightarrow{\psi_\lambda} E^{\nu}_{\lambda,a_\lambda} \xrightarrow{d^r} E^{\nu}_{\lambda-1,q_0}$$

is exact.

THEOREM 4.2 (Right Extended Two-Term Condition). *Let* $\mathfrak{C}$ *satisfy the two-term condition* $\{\lambda, \mu; \nu\}$. *In addition, let* $E^{\nu}_{p,q} = 0$ *for* $p = \mu + 1$ *and*

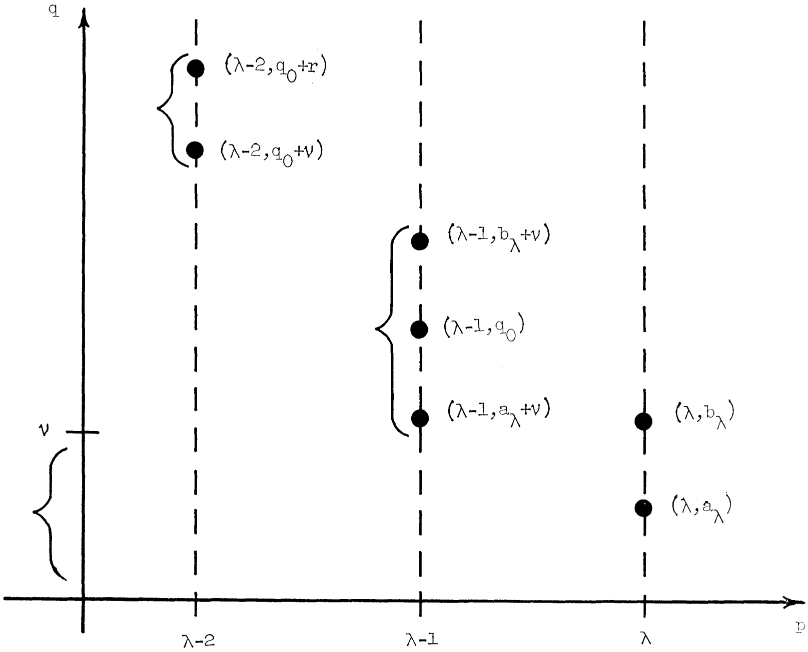

FIGURE 4.1. E^ν

$q \leqslant b_\mu - \nu$ unless $q = q_0$ ($q_0 > a_\mu - \nu$) and $E^\nu_{p,q} = 0$ for $p = \mu + 2$, $q_0 - \nu \geqslant q > q_0 - r$ where $r = b_\mu - q_0$. Then

$$E^\nu_{\mu+1,q_0} \xrightarrow{d^r} E^\nu_{\mu,b_\mu} \xrightarrow{\phi_\mu} \pi_\mu(\mathfrak{C}) \xrightarrow{\psi_\mu} \ldots \to E^\nu_{\lambda,a_\lambda}$$

is exact.

5. Description of $\mathfrak{C}(X, Y, v)$. Let X be a *locally finite CW complex*, Y be an *arc-connected space*, and v be a map from X to Y. In this section we give a description of the exact couple $\mathfrak{C}(X, Y, v)$ (see **(3)**).

Let X^n be the n-dimensional skeleton of X. Consider $M(X^n, Y) = \{f\colon X^n \to Y \mid f \text{ is a map}\}$. Let U_j be the arc-component of $M(X^j, Y)$ containing $v_j = v|X^j$. Define the map

$$r\colon U_j \to U_{j-1}$$

by $r(f) = f|X^{j-1}$ ($f \in M(X^j, Y)$). Since X is locally finite, r is a fibring in the sense of Serre (see **(5)**); i.e., r satisfies the covering homotopy theorem for polyhedra. Let $F_j = r^{-1}(v_{j-1}) = \{f \in U_j \mid f|X^{j-1} = v_{j-1}\}$. F_j is a fibre of r.

The usual sequence

$$\text{(5.1)} \qquad \ldots \to \pi_i(F_j, v_j) \xrightarrow{k_*} \pi_i(U_j, v_j) \xrightarrow{r_*} \pi_i(U_{j-1}, v_{j-1}) \xrightarrow{\partial} \pi_{i-1}(F_j, v_j) \to \ldots$$

for the fibring r is exact.

Define

$$D = \sum_{p,q} \oplus D_{p,q}$$

where $D_{p,q} = \pi_p(U_q, v_q)$ if $p, q \geqslant 0$ and $D_{p,q} = 0$ otherwise, and

$$E = \sum_{p,q} \oplus E_{p,q}$$

where $E_{p,q} = \pi_p(F_q, v_q)$ if $p, q \geqslant 0$ and $E_{p,q} = 0$ otherwise. Then (5.1) becomes

$$(5.2) \qquad \ldots \to E_{i,j} \xrightarrow{k} D_{i,j} \xrightarrow{i} D_{i,j-1} \xrightarrow{j} E_{i-1,j} \to \ldots,$$

where $k = k_*$, $i = r_*$, and $j = \partial$. This makes $\{D, E, i, j, k\}$ an exact couple. We denote this exact couple by $\mathfrak{C}(X, Y, v)$.

We state the following theorem for future reference. The proof is given in **(3)**.

THEOREM 5.3. *If X is a locally finite CW complex and if Y is arc-connected and simple ($=$ n-simple for all $n > 0$), then*

(a) γ: $E_{p,q} \approx C^q(X, \pi_{p+q}(Y))$, *the group of q-dimensional cellular cochains on X with coefficients in $\pi_{p+q}(Y)$, for $p \geqslant 1$. If $p = 0$, then $E_{0,q} \approx$ subgroup of* $C^q(X, \pi_q(Y))$; see **(3**, p 345**)**.

(b) *The following diagram is commutative for $p \geqslant 1$ and $q \geqslant 0$.*

$$\begin{array}{ccc}
E_{p,q} & \xrightarrow{\quad d = j \circ k \quad} & E_{p-1,q+1} \\
\gamma \big\downarrow \approx & & \gamma \big\downarrow \approx \\
C^q(X, \pi_{p+q}(Y)) & \xrightarrow{(-1)^q \delta} & C^{q+1}(X, \pi_{p+q}(Y))
\end{array}$$

Thus if $p \geqslant 1$, $q \geqslant 0$, then

$$E^2_{p,q} \approx H^q(X, \pi_{p+q}(Y)).$$

Also, $E^2_{0,q} \approx$ subgroup of $H_q(X, \pi_q(\mathrm{Y}))$; see **(3**, p. 351**)**.

PROPOSITION 5.4. *If X is a locally finite CW complex of dimension k, Y arc-connected and simple, and $v \in M(X, Y)$, then $\mathfrak{C}(X, Y, v)$ is a π exact couple.*

Proof. Theorem 5.3 implies that $\pi_1(F_j, v_j) = E_{1,j}$ and $\pi_0(F_j, v_j) = E_{0,j}$ are abelian, as required by §2. We must show that (1), (2), and (3) of Definition 2.1 are true. $D_{0,q} = \pi_0(U_q, v_q) = 0$ since U_q is arc-connected and $D_{p,q} = 0$ by definition if $p, q < 0$. This proves (1). (2) is true because $E_{p,q} = 0$ for $q < 0$ by definition. The dimension $X = k$ implies that $E_{p,q} = 0$ for $q > k$ since

$$E_{p,q} = \pi_p(\{f \in M(X, Y) | f = v\}, v) = 0.$$

This proves (3) and the proposition.

We note that if dimension $X = k$, then $\pi_m(\mathfrak{C}) = D_{m,k} = \pi_m(M(X, Y), v)$.

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6. The differential operator d^i. The main purpose of this section is to prove the following theorem.

THEOREM 6.1. *In $\mathfrak{C}(X, Y, v)$, with X an arc-connected, locally finite CW complex, Y arc-connected and simple, and $v \in M(X, Y)$ the constant map, the differential operator*

$$d^i: E^i_{n,0} \to E^i_{n-1,i}$$

is zero for any n and $i \geqslant 2$.

In order to simplify the rather long proof of this theorem, we first give several lemmas. These lemmas are easily proved using Theorem 5.3.

LEMMA 6.2. *For all $i \geqslant 1$, $E^i_{n,0} \equiv D^i_{n,0}$ and the homomorphism*

$$k^i: E^i_{n,0} \to D^i_{n,0}$$

is the identity.

LEMMA 6.3. *$E^2_{n,0} \approx$ the diagonal of $\prod_{x \in X^0} (\pi_n(Y))_x$ under γ for $n \geqslant 1$.*

We embed Y into U_i via the map ξ_i: $Y \to U_i$ defined by $\xi_i(y) = c_y{}^i$: $X^i \to Y$ such that $c_y{}^i(x) = y$ for all $x \in X^i$. Let v: $X \to Y$ be the constant map such that $v(X) = y_0$. Then since Y is arc-connected, $c_y{}^i$ is homotopic to v_i for all $y \in Y$. Thus $c_y{}^i \in U_i$.

Proof of 6.1. The differential operator $d^i = j^i \circ k^i$, by definition. By Lemma 6.2,

$$E^i_{n,0} = D^i_{n,0} \quad \text{and} \quad k^i: E^i_{n,0} \to D^i_{n,0} \text{ the identity.}$$

Thus d^i is essentially j^i. The homomorphism j^i is defined as follows. If

$$y \in \ker d^{l-1} \subset E^{l-1}_{n-1,i} \qquad (2 \leqslant l < i),$$

then let $\eta(y)$ denote the homology class of y in $E^l_{n-1,i}$. Let $x \in D^i_{n,0}$. Then, by **(5**, p. 232**)**,

$$\begin{aligned} j^i(x) &= \eta\{j^{i-1}(i^{-1}(x))\} = \eta^{(2)}\{j^{i-2}(i^{-2}(x))\} = \dots \\ &= \eta^{(i-1)}\{j(i^{-i+1}(x))\}, \end{aligned}$$

where $\eta^{(l)}$ is the lth iterate of the process of taking homology classes and

$$i^{-l}(x) = \{y \in D_{n,l} | \, i^{(l)}(y) = x, \text{ where } i^{(l)} = i \circ i \circ \dots \circ i \ (l \text{ times})\}.$$

First we give the proof for $i = 2$. At the end we indicate the easy extension to the cases $i > 2$. Since d^2 is essentially j^2, we must show that

$$j^2: D^2_{n,0} \to E^2_{n-1,2}$$

is zero. This homomorphism is defined by the following diagram:

$$
\begin{tikzcd}
\pi_n(Y) \approx H^0(X, \pi_n(Y)) \arrow[r, "\gamma", "\cong"'] & E^2_{n,0} = \ker\{j\colon \pi_n(U_0, v_0) \to \pi_{n-1}(F_1, v_1)\} = D^2_{n,0} & \\
& \pi_n(U_1, v_1) \arrow[u, "i", "\text{(onto)}"'] \arrow[r, "j"] & \ker\{d\colon \pi_{n-1}(F_2, v_2) \to \pi_{n-2}(F_3, v_3)\} \arrow[d, "\eta"] \\
& & E^2_{n-1,2} \\
\pi_n(U_2, v_2) \arrow[r, "t_*"] & \pi_n(U_2, F_2, v_2) \arrow[uu, "r_{2*}", "(\approx)"'] \arrow[uur, "\partial"] &
\end{tikzcd}
$$

In the diagram r_{2*} is induced by the restriction r_2: $(U_2, F_2, v_2) \to (U_1, v_1)$, t_* is induced by the inclusion t: $(U_i, v_i) \subset (U_i, F_i, v_i)$, ∂ is the boundary operator in the homotopy sequence of the pair (U_2, F_2), $i = r'_{1*}$ is induced by the restriction r'_1: $(U_1, v_1) \to (U_0, v_0)$, and η is as above. By definition $j = \partial \circ r_{2*}{}^{-1}$, where r_{2*} is an isomorphism by (**5**, p. 118). The homomorphism i is onto by exactness. Beginning with $E^2_{n,0}$ and following the diagram to $E^2_{n-1,2}$ defines $d^2 = j^2 = \eta \circ j \circ i^{-1}$.

Now to show that $j^2 = d^2 = 0$. Since X is arc-connected, $E^2_{n,0} \approx \pi_n(Y)$. Consider $\alpha \in \pi_n(Y)$ such that $\alpha \neq 0$. Let v: $X \to Y$ be the constant map into $\{y_0\}$. Choose $f \in \alpha$ such that f: $(S^n, 1) \to (Y, y_0)$ is a map. Since

$$E^2_{n,0} = \ker\{j\colon D_{n,0} \to E_{n-1,1}\} \approx \text{diagonal of } \prod_{x \in X^0} (\pi_n(Y))_x$$

by Lemma 6.3, then α corresponds isomorphically to g_α: $X^0 \to \pi_n(Y)$ such that $g_\alpha(x) = \alpha$ for each $x \in X^0$. $g_\alpha \in \pi_n(U_0, v_0)$ is represented by the map f': $(S^n, 1) \to (U_0, v_0)$ where $[f'(s)](x) = f(s)$ $(s \in S^n, x \in X^0)$. Note that $f' = \xi_0 \circ f$, where ξ_0: $Y \subset U_0$ is defined above.

Define the map f'': $(S^n, 1) \to (U_1, v_1)$ by $f'' = \xi_1 \circ f$, ξ_1: $Y \subset U_1$. $f''(1) = \xi_1(y_0) = c_{y_0}{}^1 = v_1$. We shall show that $r'_1 \circ f'' = f'$. We claim that for any i (if r'_i: $(U_i, v_i) \to (U_{i-1}, v_{i-1})$ is the restriction)

$$(*) \qquad r'_i \circ \xi_i = \xi_{i-1}.$$

Thus $r'_1 \circ f'' = r'_1 \circ \xi_1 \circ f = \xi_0 \circ f = f'$. Therefore the homotopy class of f'', denoted $[f'']$, is such that $r'_{1*}([f'']) = [f'] = \alpha$. Thus $i^{-1}(\alpha)$ contains $[f'']$.

Since $j(i^{-1}(\alpha)) = j([f'']) = \partial \circ r_{2*}{}^{-1}([f''])$, we must find a map representing $r_{2*}{}^{-1}([f''])$ in $\pi_n(U_2, F_2, v_2)$. Consider $f''' = \xi_2 \circ f$: $(S^n, 1) \to (U_2, r_2)$. This map is such that if r_2: $(U_2, F_2, v_2) \to (U_1, v_1)$ and if f''' is considered as a map from $(I^n, I^{n-1}, J^{n-1}) \to (U_2, F_2, v_2)$ as follows:

$$(I^n, I^{n-1}, J^{n-1}) \xrightarrow{\rho} (S^n, 1) \xrightarrow{f'''} (U_2, v_2) \xrightarrow{t} (U_2, F_2, v_2),$$

where ρ pinches the boundary S^{n-1} of I^n to a point, then

$$
\begin{aligned}
r_2 \circ (t \circ f''' \circ \rho) &= r_2 \circ (t \circ \xi_2 \circ f \circ \rho) \\
&= r'_2 \circ \xi_2 \circ f \circ \rho \qquad (r'_2 = r_2 \circ t) \\
&= \xi_1 \circ f \circ \rho = f'' \circ \rho \qquad (\text{by } (*)).
\end{aligned}
$$

Thus $r_{2*}([t \circ f''' \circ \rho]) = [f'' \circ \rho] = [f'']$. Thus

$$j([f'']) = \partial[t \circ f''' \circ \rho] = \partial \circ t_*([f''' \circ \rho]) = 0$$

since $\partial \circ t_* = 0$ by the exactness of the homotopy sequence. Therefore $j^2(\alpha) = d^2(\alpha) = 0$.

The proof for $i > 2$ is now clear. Since

$$E^i_{n,0} \subset E^2_{n,0} \qquad \text{for } i \geqslant 2,$$

any $\alpha \in E^i_{n,0}$ can be represented by an element f' as above. Then $i^{-i+1}(\alpha)$ is represented by $f^{(i)} = \xi_{i-1} \circ f\colon (S^n, 1) \to (U_{i-1}, v_{i-1})$. Therefore

$$\begin{aligned} j(i^{-i+1}(\alpha)) &= \partial \circ r_{i*}{}^{-1}([f^{(i)}]) = \partial([t \circ f^{(i+1)} \circ \rho]) \\ &= \partial \circ t_*([f^{(i+1)} \circ \rho]) = 0. \end{aligned}$$

This proves Theorem 6.1.

We next give a theorem that will be useful in §10 for showing that certain exact sequences split.

Let us first define some maps. Let

$$\hat{j}\colon Y \subset M(X, Y), \qquad \hat{k}\colon Y \subset M(X^0, Y)$$

be the constant injections of Y into the respective mapping spaces, where X^0 is the set of vertices of X. Pick any x_0 in X^0. Let

$$i_{x_0}\colon M(X, Y) \to Y, \qquad p_{x_0}\colon M(X^0, Y) \to Y$$

be the projections; i.e., $i_{x_0}(f) = f(x_0)$ for any f in $M(X, Y)$.

Consider the following maps from $\mathfrak{C}(X, Y, v)$:

$$D_{m,k} \xrightarrow{i^{(k-1)}} D_{m,1} \xrightarrow{\bar{\imath}} D^2_{m,0} \xrightarrow{\bar{p}_{x_0\#}} \Pi_m(Y, y_0)$$

where $m > 0$, k is the dimension of X, $\bar{\imath} = i$, and

$$\bar{p}_{x_0\#} = p_{x_0\#}|D^2_{m,0}.$$

Throughout the rest of this section we assume that $v\colon X \to Y$ is such that $v(x) = y_0$ for all x in X. We note that, by Lemmas 6.3 and 6.2, $\bar{p}_{x_0\#}$ is an *isomorphism* provided X is *connected.*

THEOREM 6.4. *Let X be a k-dimensional, connected, locally finite CW complex, Y be simple and connected, and $v\colon X \to Y$ be the constant map to y_0. Then*

$$(\bar{p}_{x_0\#} \circ \bar{\imath} \circ i^{(k-1)}) \circ \hat{j}_\# = \textit{identity on } \Pi_m(Y, y_0).$$

Preliminary to the proof we give several lemmas.

LEMMA 6.5. *$\hat{k}_\#\colon \Pi_i(Y, y_0) \to \Pi_i(M(X^0, Y), \hat{k}(y_0)) \overset{\psi}{\approx} \prod_{x \in X^0} (\Pi_i(Y, y_0))_x$ is the diagonal injection.*

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This follows because if α is a member of $\Pi_i(Y, y_0)$, then

$$\psi(\hat{k}_{\#}(\alpha)) = \prod_{x\in X^0} (p_{x\#}(\hat{k}_{\#}(\alpha)))_x = \prod_{x\in X^0} [(p_x \circ \hat{k})_{\#}(\alpha)]_x = \prod_{x\in X^0} (\alpha)_x.$$

LEMMA 6.6. *The following triangle commutes:*

$$\begin{tikzcd} D_{m,k} \arrow[d, "i_{x_0\#}"'] \arrow[dr, "i^{(k)}"] & \\ & D_{m,0} \\ \Pi_m(Y, y_0) \arrow[ur, "\hat{k}_{\#}"'] & \end{tikzcd}$$

Thus $\hat{k} \circ i_{x_0}$ $(x_0 \in X^0)$ induces $i^{(k)}$.

Proof. It is clear that $i_{x_0\#} = p_{x_0\#} \circ i^{(k)}$. Thus the triangle looks like this:

$$\begin{tikzcd} D_{m,k} \arrow[d, "i^{(k)}"'] \arrow[dr, "i^{(k)}"] & \\ D_{m,0} \arrow[d, "p_{x_0\#}"'] & D_{m,0} \\ \Pi_m(Y, y_0) \arrow[ur, "\hat{k}_{\#}"'] & \end{tikzcd}$$

Consider $\alpha \in D_{m,k}$. Then

$$i^{(k)}(\alpha) \in D^2_{m,0} = \text{diagonal of} \prod_{x\in X^0} (\Pi_m(Y, y_0))_x$$

by Lemmas 6.2 and 6.3. Thus

$$\hat{k}_{\#} \circ i_{x_0\#}(\alpha) = \hat{k}_{\#} \circ p_{x_0\#} \circ i^{(k)}(\alpha) = i^{(k)}(\alpha)$$

because $\hat{k}_{\#} \circ p_{x_0\#}|\text{dia} \prod_{x\in X^0} (\Pi_m(Y, y_0))_x = \text{identity}$ by Lemma 6.5.

Proof of 6.4. Since $i^{(k)}$ is induced by $\hat{k} \circ i_{x_0}$, and $i_{x_0} \circ \hat{j} = \text{identity on } Y$,

$$\begin{aligned} (\bar{p}_{x_0\#} \circ \bar{\imath} \circ i^{(k-1)}) \circ \hat{j}_{\#} &= (\bar{p}_{x_0\#} \circ (\hat{k} \circ i_{x_0})_{\#}) \circ \hat{j}_{\#} \\ &= (p_{x_0} \circ \hat{k})_{\#} \circ (i_{x_0} \circ \hat{j})_{\#} \\ &= \text{identity on } \Pi_m(Y, y_0). \end{aligned}$$

This proves Theorem 6.4.

CHAPTER II. TWO-TERM CONDITIONS IN $\mathfrak{C}(X, Y, v)$

7. Two theorems. In this section we show that if Y is n-connected and X of dimension k, then $M(X, Y)$ is $n - k$ connected. Also, if $\pi_i(Y) = 0$ for $n > n_0$, then $\pi_i(M(X, Y), v) = 0$ for $n > n_0$.

THEOREM 7.1. *Let X be a locally finite CW complex such that the dimension of $X = k \geqslant 1$. Let Y be n-connected such that $n \geqslant k$. Then*

(i) *$M(X, Y)$ is $n - k$ connected,*
(ii) $\pi_{n-k+1}(M(X, Y), v) \approx H^k(X, \Pi_{n+1}(Y))$,
(iii) *the sequence*

$$H(k-2, n+1) \xrightarrow{d^2} H(k, n+2) \xrightarrow{\phi} \bar{\pi}_{n-k+2} \xrightarrow{\psi} H(k-1, n+1) \to 0$$

is exact, where $H(i, j) = H^i(X, \Pi_j(Y))$ and $\bar{\pi}_m = \Pi_m(M(X, Y), v)$.

Proof. Since Y is n-connected and $\dim X \leqslant n$, it is easy to see that $M(X, Y)$ is arc-connected. By Theorem 5.3, we have

$$E^2_{p,q}(X, Y, v) \approx H^q(X, \pi_{p+q}(Y)).$$

Thus $E_{p,q} = 0$ for $q \leqslant k$ and $p + q \leqslant n$. This implies that $E^2_{p,q} = 0$ for $0 \leqslant p \leqslant n - k$. Corollary 3.5 then gives (i). For (ii) and (iii), we let $a_{n-k+i} = k - 1$ and $b_{n-k+i} = k$ for $i = 1, 2$. See Figure 7.1. This gives a two-term

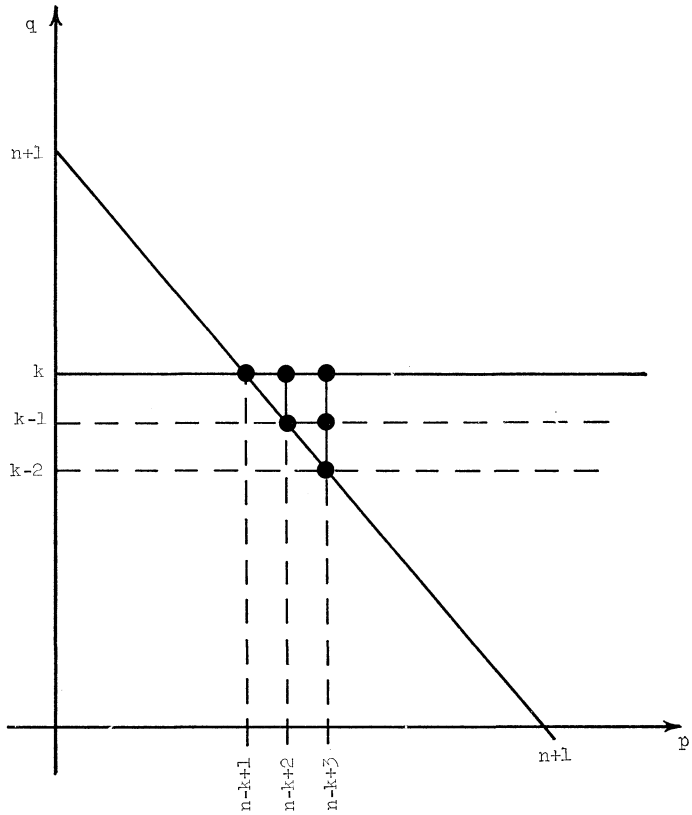

FIGURE 7.1. $E^2(X, Y, v)$

condition $\{n-k+1, n-k+2; 2\}$. Thus Theorem 3.2 implies that the sequence

$$H(k, n+2) \xrightarrow{\phi_{n-k+2}} \bar{\pi}_{n-k+2} \xrightarrow{\psi_{n-k+2}} H(k-1, n+1) \xrightarrow{\theta_{n-k+2}} H(k, n+1)$$
$$\xrightarrow{\phi_{n-k+1}} \bar{\pi}_{n-k+1} \xrightarrow{\psi_{n-k+1}} H(k-1, n)$$

is exact, where $H(i,j) = H^i(X, \pi_j(Y))$ and $\bar{\pi}_l = \pi_l(M(X, Y), v)$. Since $k - k + 1 = 1 < 2$, the note following Theorem 3.2 implies that $\theta_{n-k+2} = 0$. $H(k-1, n) = 0$ because $\pi_n(Y) = 0$. Thus $H(k, n+1) \approx \bar{\pi}_{n-k+1}$. Let $q_0 = k - 2$. Then

$$E^2_{n-k+3,q} = 0 \quad \text{for } q < k-2 \quad \text{and} \quad E^2_{n-k+4,q} = 0 \quad \text{for } q < k-4.$$

Thus the conditions of Theorem 4.2 hold and the following is exact.

$$H(k-2, n+1) \xrightarrow{d^2} H(k, n+2) \xrightarrow{\phi_{n-k+2}} \bar{\pi}_{n-k+2} \xrightarrow{\psi_{n-k+2}} H(k-1, n+1) \to 0.$$

This proves the theorem.

THEOREM 7.2. *Let Y be simple, arc-connected such that $\pi_i(Y) = 0$ for $i > m$ and X be a locally finite CW complex of dimension k. Then*

(i) $\pi_i(M(X, Y), v) = 0$ *for* $i > m$.

(ii) $\pi_m(M(X, Y), v) \approx H^0(X, \pi_m(Y))$.

(iii) *The following sequence is exact:*

$$0 \longrightarrow H(1, m) \xrightarrow{\phi_{m-1}} \bar{\pi}_{m-1} \xrightarrow{\psi_{m-1}} H(0, m-1) \xrightarrow{d^2} H(3, m).$$

The proof of Theorem 7.2 is similar to that of 7.1. Part (i) was known to Thom (see (**8**)). We note that if $v: X \to Y$ is constant, then $d^2 = 0$ in (iii) by Theorem 6.1, provided X is connected. Then, Theorem 6.4 implies the following corollary.

COROLLARY 7.3. *In Theorem* 7.2, *if X is connected and $v: X \to Y$ is a constant map, then* (iii) *reads as follows:*

$$\pi_{m-1}(M(X, Y), v) \approx H^1(X, \pi_m(Y)) \oplus \pi_{m-1}(Y).$$

8. Gap Theorem I. In this section we obtain two-term conditions on $\mathfrak{C}(X, Y, v)$ by placing restrictions on the homotopy groups of Y.

Let $\{a_i\}$ $(1 \leqslant i < \infty)$ be a strictly increasing sequence of positive integers. Let $\{\pi_{a_i}\}$ $(1 \leqslant i < \infty)$ be a sequence of groups such that π_{a_i} is *abelian* for $i > 1$ and for $i = 1$ if $a_1 > 1$. If $a_1 = 1$, then π_1 is not necessarily abelian.

Definition. An arc-connected space Y is said to be of *homotopy kind* $\{\pi_{a_i}, a_i\}$ $(1 \leqslant i < \infty)$ if and only if $\pi_{a_i}(Y) \approx \pi_{a_i}$ $(1 \leqslant i < \infty)$ and $\pi_j(Y) = 0$ if $j \notin \{a_i\}$ $(1 \leqslant i < \infty)$.

Thus $K(\pi, n)$ has homotopy kind (type) $\{\pi, n\}$, $K(\pi_m, m) \times K(\pi_n, n)$ $(m < n)$ has homotopy kind $\{\pi_m, m; \pi_n, n\}$, and any arc-connected space X has homotopy kind $\{\pi_i(X), i\}$ $(1 \leqslant i < \infty)$.

THEOREM 8.1. *Let X be a locally finite CW complex of dimension k. Let Y be of homotopy kind $\{\pi_m, m; \pi_n, n\}$ where $m < n$, π_m and π_n are abelian, and if $m = 1$, π_1 acts simply on π_n. Then*

(i) $\pi_i(M(X, Y), v) = 0$ *for* $i > n$,

(ii) $\pi_i(M(X, Y), v) \approx H^{n-i}(X, \pi_n)$ *for* $m < i \leqslant n$, *and*

(iii) *if* $\bar{\pi}_i = \pi_i(M(X, Y), v)$ *and* $H^i(X, \pi_j) = H(i, j)$, *the following sequence is exact*:

$$
\begin{aligned}
&0 \longrightarrow H(n-m, n) \xrightarrow{\phi_m} \bar{\pi}_m \xrightarrow{\psi_m} H(0, m) \xrightarrow{\theta_m} H(n-m+1, n) \xrightarrow{\phi_{m-1}} \ldots \\
(8.2)\quad &\xrightarrow{\theta_{i+1}} H(n-i, n) \xrightarrow{\phi_i} \bar{\pi}_i \xrightarrow{\psi_i} H(m-i, m) \xrightarrow{\theta_i} H(n-i+1, n) \xrightarrow{\phi_{i-1}} \ldots \\
&\xrightarrow{\theta_2} H(n-1, n) \xrightarrow{\phi_1} \bar{\pi}_1 \xrightarrow{\psi_1} H(m-1, m) \xrightarrow{\theta_1} E^2_{0,n} \longrightarrow 0,
\end{aligned}
$$

where $E^2_{0,n} \subset H(n, n)$ (see Figure 8.1).

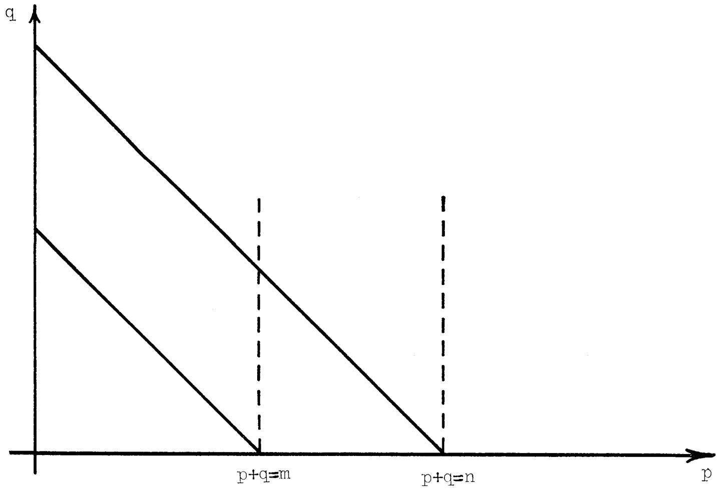

FIGURE 8.1. $E^2(X, Y, v)$

Proof. By Theorem 5.3, $E^2_{p,q} \approx H^q(X, \pi_{p+q}(Y))$ for $p \geqslant 1$ and

$$E^2_{0,q} \subset H^q(X, \pi_q(Y)).$$

Thus $E_{p,q} = 0$ for $p + q > n$. Proposition 5.4 and Corollary 3.5 imply that $\pi_i(M(X, Y), v) = 0$ for $i > n$. Let $b_i = n - i$ for $m < i \leqslant n$. So $b_i - 2 = n - i - 2 < n - i - 1 = b_{i+1}$. Thus the hypotheses of Corollary 3.4 are fulfilled for $m + 1 \leqslant i \leqslant n$ and hence

$$\pi_i(M(X, Y), v) \approx H^{n-i}(X, \pi_n) \qquad \text{for } m < i \leqslant n.$$

For $0 \leqslant i \leqslant m + 1$, let $a_i = m - i$, $b_i = n - i$. This gives a two-term condition $\{0, m + 1; 2\}$ on $\mathfrak{C}(X, Y, v)$. Theorem 3.2 thus implies that the above sequence (8.2) is exact. The zeros on each end result because $D_{0,k} = E^2_{m+1,-1} = 0$. Note that if $v\colon X \to Y$ is constant and X connected, then $\theta_m = 0$ by Theorem 6.1 and $\pi_m(M(X, Y), v) \approx H^{n-m}(X; \pi_n(Y)) \oplus \pi_m(Y)$ by Theorem 6.4.

The following corollary gives the effect of $k = \dim X$ on the exact sequence (8.2).

COROLLARY 8.3. *With the same hypothesis as Theorem* 8.1, *let us assume further that*

(i) $k \geqslant n$. *Then* (8.2) *stands as it is.*

(ii) $m \leqslant k < n$. *Then* (8.2) *ends as follows:*

$$0 \longrightarrow H(n-m, n) \longrightarrow \ldots \longrightarrow H(k, n) \xrightarrow{\phi_{n-k}} \bar{\pi}_{n-k} \xrightarrow{\psi_{n-k}} H\,(m-n+k, m) \longrightarrow 0$$

and $\pi_i(M(X, Y), v) \approx H^{m-i}(X, \pi_m)$ *for* $1 \leqslant i < n-k$ (see Figure 8.2).

(iii) $n - m \leqslant k < m$. *Then* (8.2) *ends as in* (ii) *above*, $\pi_i(M(X, Y), v) \approx H^{m-i}(X, \pi_m)$ *for* $m-k < i < n-k$ *and* $\pi_i(M(X, Y), v) = 0$ *for* $1 \leqslant i < m-k$ (see Figure 8.4).

(iv) $k < n - m$. *Then* $\pi_i(M(X, Y), v) = 0$ *for* $1 \leqslant i < m-k$ *and* $m < i < n-k$ *and* $\pi_i(M(X, Y), v) \approx H^{j-i}(X, \pi_j)$ *for* $j-k \leqslant i \leqslant j$ $(j = m, n)$ (see Figure 8.3).

Proof. We note that case (iii) is vacuous if $n - m \geqslant m$. The proof consists of taking the exact sequence (8.2) and using the relative position of $k = \dim X$ to determine zeros in it.

COROLLARY 8.4. *If X is a locally finite CW complex of dimension k and $Y = K(\pi, n)$, the Eilenberg–MacLane space, where π is abelian, then*

$$\pi_i(M(X, K(\pi, n)), v) \approx H^{n-i}(X, \pi)$$

for $n - k \leqslant i \leqslant n$ *and* $= 0$ *for* $i > n$ *or* $i < n-k$.

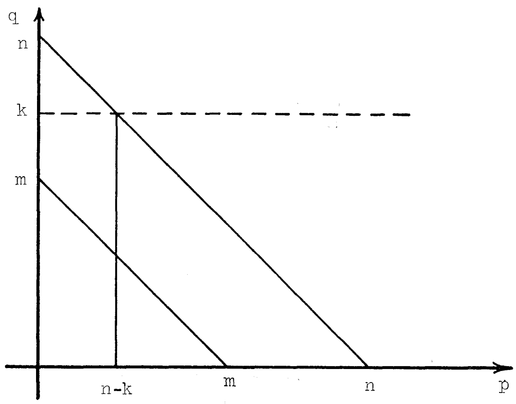

FIGURE 8.2. $E^2(X, Y, v)$

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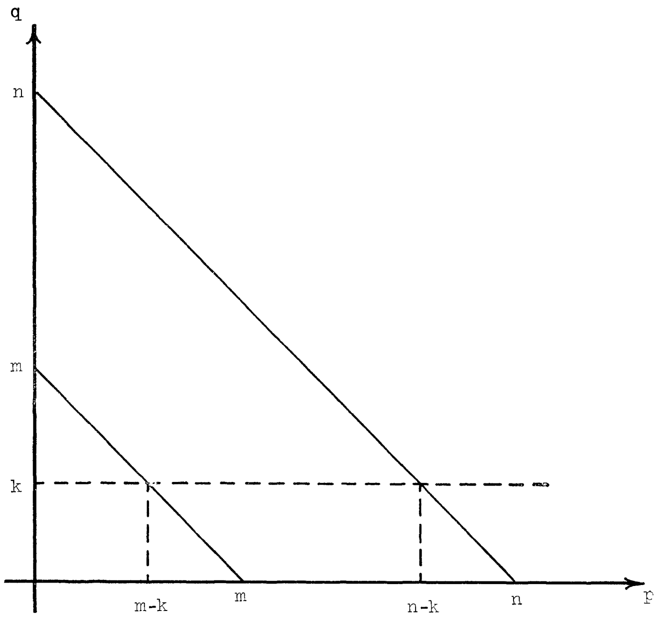

FIGURE 8.3. $E^2(X, Y, v)$

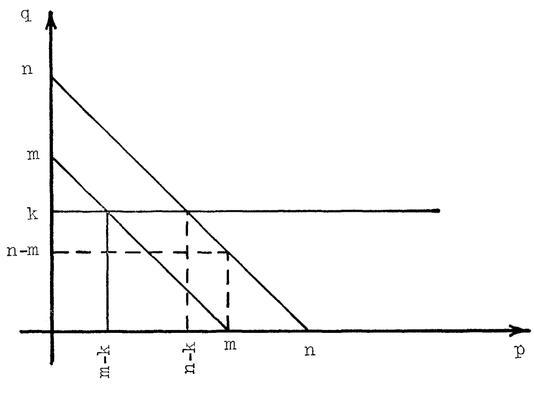

FIGURE 8.4. $E^2(X, Y, v)$

This result was known to Federer (3) and Thom (8).

Theorem 8.1 generalizes to the following theorem, utilizing regular gaps in the homotopy groups of Y.

THEOREM 8.5 (Gap Theorem I). *Let m, n, k be positive integers such that $m < n$. Define the sequence $\{a_i\}$ $(1 \leqslant i < \infty)$ by $a_i = m + j + jk$ if $i = 2j + 1$ and $a_i = n + j + jk$ if $i = 2j + 2$. Let $\{\pi_{a_i}\}$ $(1 \leqslant i < \infty)$ be a sequence of*

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abelian groups (if $m = 1$, we let π_1 act simply on π_{a_i} for $i \geqslant 1$). Let Y be an arc-connected space of homotopy kind

$$\{\pi_{a_1}, a_1; \pi_{a_2}, a_2; \pi_{a_3}, a_3; \ldots\} = \{\pi_m, m; \pi_n, n; \pi_{m+1+k}, m+1+k; \ldots\}$$

and let X be a locally finite CW complex of dimension k. Then the following sequences Σ_i are exact for $1 < i < \infty$, where $\pi_l(M(X, Y), v) = \bar{\pi}_l$, $H^j(X, \pi_p(Y)) = H(j, p)$, and α is any integer such that $a_{i-1} + 1 < \alpha < a_i$ (see Figure 8.5).

$$0 \longrightarrow H(a_{i+1} - a_i, a_{i+1}) \xrightarrow{\phi_{a_i}} \bar{\pi}_{a_i} \xrightarrow{\psi_{a_i}} H(0, a_i)$$

$$\xrightarrow{\theta_{a_i}} H(a_{i+1} - (a_i - 1), a_{i+1}) \xrightarrow{\phi_{a_i - 1}} \ldots$$

$$(\Sigma_i) \quad \ldots \xrightarrow{\theta_{\alpha+1}} H(a_{i+1} - \alpha, a_{i+1}) \xrightarrow{\phi_\alpha} \bar{\pi}_\alpha \xrightarrow{\psi_\alpha} H(a_i - \alpha, a_i)$$

$$\xrightarrow{\theta_\alpha} H(a_{i+1} - (\alpha - 1), a_{i+1}) \xrightarrow{\phi_{\alpha-1}} \ldots$$

$$\ldots \xrightarrow{\theta_{a_{i-1}+2}} H(k, a_{i+1}) \xrightarrow{\phi_{a_{i-1}+1}} \bar{\pi}_{a_{i-1}+1} \xrightarrow{\psi_{a_{i-1}+1}} H(a_i - (a_{i-1} + 1), a_i) \rightarrow 0.$$

If $i = 1$, then the sequence Σ_1 is given by (8.2) *and depends on k as in Theorem* 8.3.

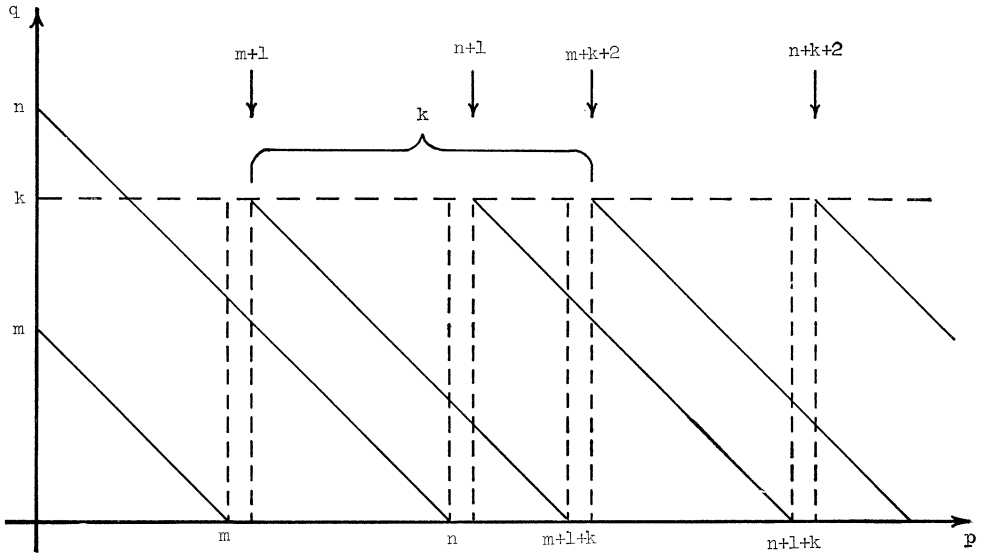

FIGURE 8.5. $E^2(X, Y, v)$

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Proof. Σ_1 is the exact sequence (8.2). Let $i > 1$. We then have the situation pictured in Figure 8.6, where

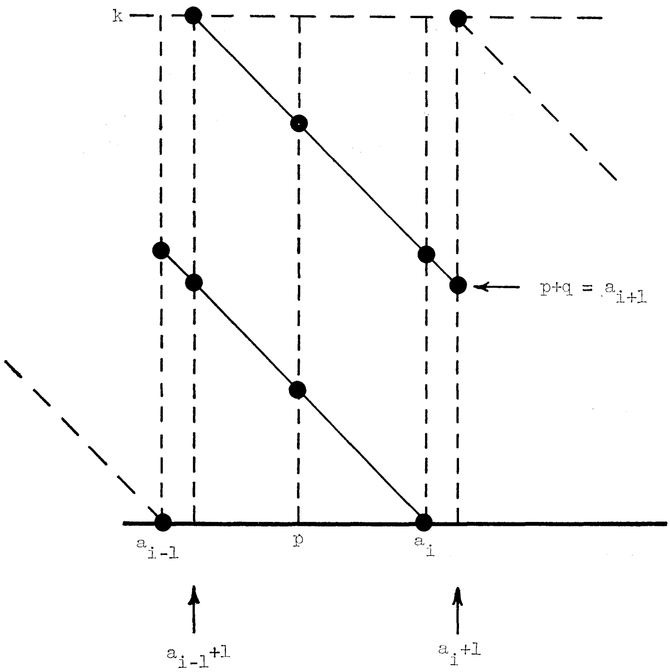

FIGURE 8.6. $E^2(X, Y, v)$

(i) $E^2_{a_{i-1},q} = 0$ unless $q = 0$ or $a_i - a_{i-1}$;
(ii) $E^2_{p,q} = 0$, if $a_{i-1} + 1 \leqslant p \leqslant a_i$, unless $q = a_i - p$ or $q = a_{i+1} - p$;
(iii) $E^2_{a_i+1,q}$ unless $q = k$ or $q = a_{i+1} - (a_i + 1)$.

Let

$$c_p{}^i = \begin{cases} 0 & \text{if } p = a_{i-1}, \\ a_i - p & \text{if } a_{i-1} + 1 \leqslant p \leqslant a_i, \\ a_{i+1} - (a_i + 1) & \text{if } p = a_i + 1 \end{cases}$$

and

$$d_p{}^i = \begin{cases} a_i - a_{i-1} & \text{if } p = a_{i-1}, \\ a_{i+1} - p & \text{if } a_{i-1} + 1 \leqslant p \leqslant a_i, \\ k & \text{if } p = a_i + 1. \end{cases}$$

Thus for $a_{i-1} \leqslant p \leqslant a_i + 1$, $c_p{}^i < d_p{}^i$. This gives a two-term condition $\{a_{i-1}, a_i + 1; 2\}$ on $\mathfrak{C}(X, Y, v)$. Then Theorems 3.2 and 5.3 imply the existence of Σ_i and the note following 3.2 gives the zeros on each end, because

$$d^i_{a_{i-1}} - c^i_{a_{i-1}+1} = (a_i - a_{i-1}) - (a_i - (a_{i-1} + 1)) = 1$$

and

$$d^i_{a_i} - c^i_{a_i+1} = (a_{i+1} - a_i) - (a_{i+1} - (a_i + 1)) = 1.$$

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This proves Theorem 8.5. Again, if X is connected and v is constant, then Theorem 6.1 $\Rightarrow \Theta_{a_i} = 0$ and Theorem 6.4 $\Rightarrow \bar{\pi}_{ai} \approx H(a_{i+1} - a_i, a_{i+1}) \oplus \pi_{a_i}(Y)$.

It is clear that, if $\dim X < k$, then Theorem 8.5 still holds. However, there will be many zeros in each Σ_i. An analysis of this situation would be similar to Theorem 8.3. It is also clear that the gaps in the homotopy groups of Y are the minimum ones for the given triplet (m, n, k) where $m < n$, $\dim X = k$. These gaps can be widened and 8.5 still holds. A statement of such a theorem would go as follows.

THEOREM 8.6. *Let X be a locally finite CW complex of* $\dim k$ *and* $\{(m_i, n_i)\}$ $(1 \leqslant i < \infty)$ *be a sequence of pairs of integers such that*

(i) $m_i < n_i < m_{i+1}$ *for all i,*

(ii) $m_{i+1} - m_i \geqslant k + 1$ *for all i,*

(iii) $n_{i+1} - n_i \geqslant k + 1$ *for all i.*

Let Y be an arc-connected simple space of homotopy kind

$$\{\pi_{m_1}, m_1; \pi_{n_1}, n_1; \pi_{m_2}, m_2; \pi_{n_2}, n_2; \pi_{m_3}, m_3; \ldots\}.$$

Then sequences Σ'_i, Σ''_i similar to Σ_i of Theorem 8.5 *are exact in each interval* $[m_i + 1, n_i]$ *and* $[n_i + 1, m_{i+1}]$, *respectively.*

9. Gap Theorem II. Let $\{a_i\}$ $(0 \leqslant i < N)$ (N an integer or $N = \aleph_0$) be a strictly ascending sequence of non-negative integers such that $a_0 = 0$; i.e., $a_0 = 0 < a_1 < a_2 < \ldots < a_n < \ldots$.

Definition. We call a CW complex X "a CW complex of cell type $\{a_i\}$ $(0 \leqslant i < N)$" if and only if

(i) X has cells of dimension a_i for each $i < N$,

(ii) X has no cells of dimension k for any $k \notin \{a_i\}$.

Examples. (a) S^n is a CW complex of cell type $\{0, n\}$.

(b) Δ^n, the unit n-simplex, is a CW complex of cell type $\{0, 1, 2, \ldots, n\}$.

(c) CP^n, the n-dimensional complex projective space, is a CW complex of cell type $\{0, 2, 4, \ldots, 2n\}$.

(d) If $Y_p^{n+1} = S^n \cup e^{n+1}$ where e^{n+1} is joined to S^n by the map $h: S^n \to S^n$ of degree $p \neq 0$, then Y_p^{n+1} is a CW complex of cell type $\{0, n, n + 1\}$.

THEOREM 9.1. (Gap Theorem II). *Let X be a CW complex of cell type* $\{0, a_1, \ldots, a_{n-1}, a_n\}$ *and Y be m-connected* $(m \geqslant 1)$. *Let σ_i denote the sequence*

$$H(a_n, i + a_n) \xrightarrow{\phi_i} \bar{\pi}_i \xrightarrow{\psi_i} H(a_{n-1}, i + a_{n-1})$$

and $\Sigma(i, j)$ $(i \leqslant j)$ denote the sequence

$$H(a_n, j + a_n) \xrightarrow{\phi_j} \bar{\pi}_j \xrightarrow{\psi_j} H(a_{n-1}, j + a_{n-1}) \xrightarrow{\theta_j} H(a_n, j - 1 + a_n)$$
$$\xrightarrow{\phi_{j-1}} \ldots \xrightarrow{\theta_{i+1}} H(a_n, i + a_n) \xrightarrow{\phi_i} \bar{\pi}_i \xrightarrow{\psi_i} H(a_{n-1}, i + a_{n-1}).$$

If $i = j$, *let* $\Sigma(i, i) = \sigma_i$. *Then*:

(1) *if* $a_{n-2} < m \leqslant a_{n-1}$, *the following sequences are exact*:

(a) *If* $a_{n-1} - a_{n-2} = 1$ (*thus* $m = a_{n-1}$), *then*

$$\sigma_1 \xrightarrow{\theta_1} E^2_{0,a_n} \to 0, \qquad \text{where } E^2_{0,a_n} \subset H(a_n, a_n).$$

If, in addition, $a_n - a_{n-1} = 1$, *then*

$$H(a_{n-2}, a_{n-2} + 2) \xrightarrow{d^2} \sigma_1 \to 0.$$

(b) *If* $a_{n-1} - a_{n-2} > 1$, $2a_{n-1} \leqslant a_n + a_{n-2}$, *and* $m - a_{n-1} > 1$, *then*

$$\Sigma(1, m - a_{n-2} - 1) \xrightarrow{\theta_1} E^2_{0,a_n} \to 0.$$

(c) *If* $a_{n-1} - a_{n-2} > 1$, $2a_{n-1} > a_n + a_{n-2}$, *and* $m - a_{n-2} > 1$, *then*

(i) *if* $a_n - a_{n-1} > 1$, *then*

$$H(a_{n-1}, m - a_{n-2} + a_{n-1}) \xrightarrow{d^{a_n - a_{n-1}}} \Sigma(1, m - a_{n-2} - 1) \xrightarrow{\theta_1} E^2_{0,a_n} \longrightarrow 0;$$

(ii) *if* $a_n - a_{n-1} = 1$, *then, for* $1 \leqslant i \leqslant m - a_{n-2} - 1$, $0 \to \sigma_i \to 0$.

(2) *If* $m > a_{n-1}$, *we have*

$$(*) \quad \pi_i(M(X, Y), v) \approx H^{a_n}(X, \pi_{i+a_n}(Y)) \qquad \text{for } 1 \leqslant i \leqslant m - a_{n-1} - 1 \quad (m - a_{n-1} > 1)$$

(*if* $m > a_n$ *also, then* $\bar{\pi}_i = 0$ *for* $1 \leqslant i \leqslant m - a_n$) *and the following sequences are exact*:

(a) *If* $a_{n-1} - a_{n-2} = 1$, *then*

$$\sigma_{m-a_{n-2}} \xrightarrow{\theta_{m-a_{n-2}}} H(a_n, m - a_{n-1} + a_n) \xrightarrow{\phi_{m-a_{n-1}}} \bar{\pi}_{m-a_{n-1}} \longrightarrow 0.$$

In addition, if $a_n - a_{n-1} = 1$, *we have*

$$H(a_{n-2}, m + 1) \xrightarrow{d^2} \sigma_{m-a_{n-2}} \to 0 \quad \text{and} \quad H(a_n, m + 1) \approx \bar{\pi}_{m-a_{n-2}-1}.$$

(b) *If* $a_{n-1} - a_{n-2} > 1$ *and* $2a_{n-1} \leqslant a_n + a_{n-2}$, *then*

$$\Sigma(m - a_{n-1} + 1, m - a_{n-2} - 1) \xrightarrow{\theta_{m-a_{n-1}+1}} H(a_n, m - a_{n-1} + a_n) \xrightarrow{\phi_{m-a_{n-1}}} \bar{\pi}_{m-a_{n-1}} \longrightarrow 0.$$

(c) *If* $a_{n-1} - a_{n-2} > 1$ *and* $2a_{n-1} > a_n + a_{n-2}$, *then*

(i) *if* $a_n - a_{n-1} > 1$, 2(b) *holds and extends to*

$$H(a_{n-1}, m - a_{n-2} + a_{n-1}) \xrightarrow{d^{a_n - a_{n-1}}} \Sigma(m - a_{n-1} + 1, m - a_{n-2} - 1) \longrightarrow \ldots;$$

(ii) *if* $a_n - a_{n-1} = 1$, *we have* $0 \to \sigma_i \to 0$ *for*

$$m - a_{n-2} - 1 \geqslant i \geqslant m - a_{n-1} + 1 \quad \text{and} \quad \phi_{m-a_{n-1}}: H(a_n, m + 1) \approx \bar{\pi}_{m-a_{n-1}}.$$

Proof. Under the conditions of 1(a), we have, by Theorem 5.3, $E^2_{0,q} = 0$ unless $q = a_n$, $E^2_{1,q} = 0$ unless $q = a_{n-1}, a_n$, and $E^2_{2,q} = 0$ unless $q = a_{n-2}, a_{n-1}, a_n$ (see Figure 9.1). Thus, if $c_i = a_{n-1}$ and $d_i = a_n$ for $i = 0, 1$, a two-term condition $\{0, 1; 2\}$ is satisfied and Theorem 3.2 implies 1(a). The result for $a_n - a_{n-1} = 1$ is given by Theorem 7.1(iii).

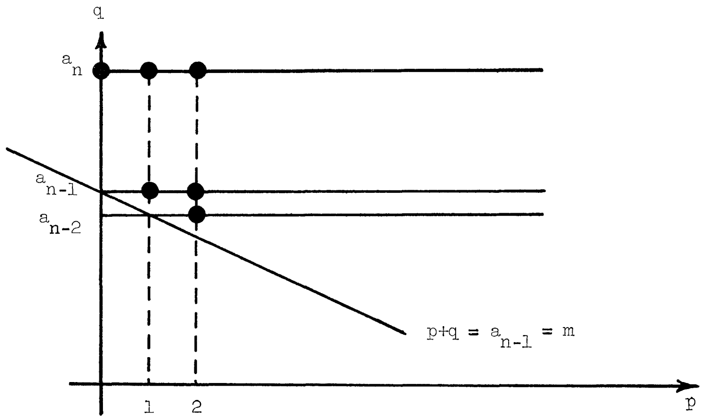

FIGURE 9.1. $E^2(X, Y, v)$

Suppose $a_{n-1} - a_{n-2} > 1$. Since $E^2_{p,q} = 0$ unless $q = a_n, a_{n-1}$ $(0 \leqslant p \leqslant m - a_{n-2})$ we clearly have the two-term condition $\{0, m - a_{n-2} - 1; 2\}$ satisfied (see Figure 9.2). Then Theorem 3.2 and $\pi_0(\mathfrak{C}) = 0$ give 1(b). We note that the two-term condition $\{0, m - a_{n-2}; 2\}$ does not hold because

$$a_{n-1} - a_{n-2} \geqslant 2 \Rightarrow E^2_{m-a_{n-2}+1,\, a_{n-2}} \text{ is not necessarily zero,}$$

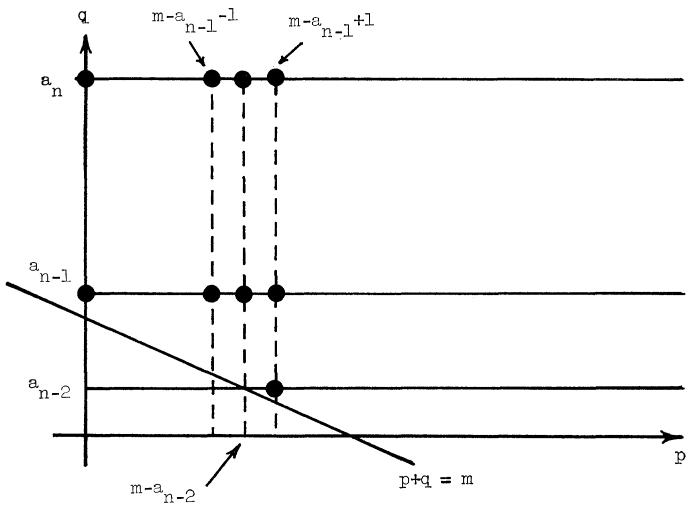

FIGURE 9.2. $E^2(X, Y, v)$

which violates Definition 3.1(c). Since $2a_{n-1} - a_n \leqslant a_{n-2}$, then the hypotheses of Theorem 4.2 are not necessarily fulfilled and thus 1(b) can be extended no further.

1(c) follows because if $2a_{n-1} - a_n > a_{n-2}$, then the conditions on Theorem 4.2 (with $q_0 = a_{n-1}$) are satisfied and if $a_n - a_{r-1} = 1$, then $\theta_i = 0$ in 1(b); by the note following Theorem 3.2.

If $m > a_{n-1}$, then $E^2_{p,q} = 0$ unless $q = a_n$ for $1 \leqslant p \leqslant m - a_{n-1} - 1$ $(m - a_{n-1} > 1)$ and Corollary 3.4 implies (*). 2(a) follows because the two-term condition $\{m - a_{n-1}, m - a_{n-2}; 2\}$ is satisfied with $c_i = a_{n-1}, d_i = a_n$ for $m - a_{n-1} \leqslant i \leqslant m - a_{n-2}$ and because

$$E^2_{m-a_{n-1},a_{n-1}} \approx H^{a_{n-1}}(X, \pi_m(Y)) = 0.$$

2(b) follows because the two-term condition $\{m - a_{n-1} + 1, m - a_{n-2} - 1; 2\}$ is satisfied with $c_i = a_{n-1}, d_i = a_n$ in this range (see Figure 9.3). As above,

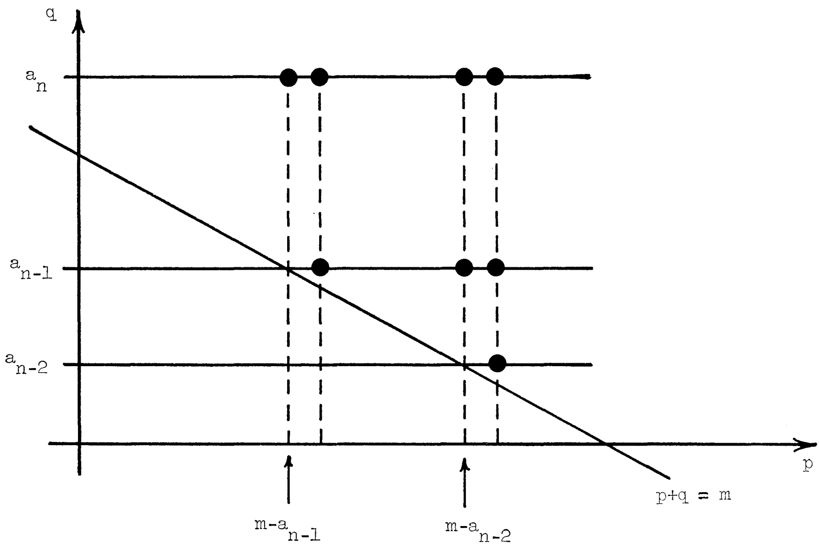

FIGURE 9.3. $E^2(X, Y, v)$

Theorem 4.2 does not necessarily hold because $2a_{n-1} - a_n \leqslant a_{n-2}$. 2(c) follows for the same reason that 1(c) does. This proves the theorem.

Theorem 9.1 would perhaps be more applicable if, instead of having to know the cell structure of X, one only needed to know the homology groups of X. The next two corollaries give such extensions. Let the statement "X is of homology kind $\{0, a_1, a_2, \ldots, a_{n-1}, a_n\}$" mean $H_i(X) = 0$ if and only if $i \notin \{a_j\}$ $(1 \leqslant j \leqslant n)$.

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COROLLARY 9.2. *Let X be a locally finite CW complex of dimension k and suppose X is of homology kind $\{0, a_1, \ldots, a_{n-1}, a_n\}$ where $H_i(X)$ is a free abelian group for $i = a_n, a_{n-1}, a_{n-2}$. Then the conclusions of Theorem 9.1 hold without change for any m-connected Y.*

Proof. $E^2_{p,q} \approx H^q(X, \pi_{p+q}(Y)) \approx \mathrm{Hom}(H_q(X), \pi_{p+q}(Y)) \oplus \mathrm{Ext}(H_{q-1}(X), \pi_{p+q}(Y))$. (See **2.**) Thus, since $\mathrm{Ext}(A, B) = 0$ if A is free (see **(4)**), $E^2_{q,q} \approx \mathrm{Hom}(H_q(X), \pi_{p+q}(Y))$ for $q \geqslant a_{n-2}$. Therefore $E^2_{pq,} = 0$ for $q \geqslant a_{n-2}$ unless $q = a_{n-2}, a_{n-1}, a_n$. Thus Theorem 9.1 holds without change. This proves the corollary.

COROLLARY 9.3. *Suppose X is a locally finite, k-dimensional CW complex such that X is of homology kind $\{0, a_1, \ldots, a_{n-1}, a_n\}$. Then, if Y is m-connected,*

(i) *if $a_{n-1} + 1 = a_n$, the conclusions of Theorem 9.1 hold for the triplet $\{a_n + 1, a_n, a_n - 1\}$ replacing the triplet $\{a_n; a_{n-1}, a_{n-2}\}$ in the statement of Theorem 9.1;*

(ii) *if $a_{n-1} + 1 < a_n$, the conclusions of Theorem 9.1 hold for the triplet $\{a_n + 1, a_n, a_{n-1} + 1\}$ replacing the triplet $\{a_n, a_{n-1}, a_{n-2}\}$ in the statement of Theorem 9.1.*

Proof. $E^2_{p,q} \approx \mathrm{Hom}(H_q(X), \pi_{p+q}(Y)) \oplus \mathrm{Ext}(H_{q-1}(X), \pi_{p+q}(Y))$. Thus if $a_{n-1} + 1 = a_n$, then $E^2_{p,q}$ is possibly non-zero for $q = a_n - 1, a_n, a_n + 1$. Therefore Theorem 7.1 implies the result. If $a_n - a_{n-1} > 1$, then the three largest values of q for which $E^2_{p,q} \neq 0$ are $a_n + 1$, a_n, and $a_{n-1} + 1$ since

$$\begin{aligned} E^2_{p,a_n+1} &\approx \mathrm{Ext}(H_{a_n}(X), \pi_{p+a_n+1}(Y)) \\ \text{and}\quad E^2_{p,a_{n-1}+1} &\approx \mathrm{Ext}(H_{a_{n-1}}(X), \pi_{p+a_{n-1}+1}(Y)). \end{aligned}$$

Thus Theorem 9.1 holds for these values.

10. Examples. In this section we apply the gap theorems to some special cases. As an example of Gap Theorem I, let us consider $Y = \mathbf{U}$, the unitary group. Let $Z =$ the integers. We recall that $\pi_i(\mathbf{U}) = Z$ if i is odd and is zero otherwise.

PROPOSITION 10.1. *Let $Y = \mathbf{U}$ and $v: X \to Y$ be constant. Then, if X is a connected locally finite CW complex of*

(i) *dimension* 1: $\pi_i(M(X, \mathbf{U}), v) \approx \begin{cases} H^0(X) & (i \text{ odd}), \\ H^1(X) & (i \text{ even}); \end{cases}$

(ii) *dimension* 2: $\pi_{2i}(M(X, \mathbf{U}), v) \approx H^1(X)$ $(i = 1, 2, \ldots)$ *and* $\bar{\pi}_{2i-1} \approx H^2(X) \oplus Z$ $(i = 2, 3, \ldots)$ *and* $0 \to H^2(X) \to \bar{\pi}_1 \to Z \to 0$ *is exact;*

(iii) *dimension* 3: $\bar{\pi}_{2i-1} \approx H^2(X) \oplus Z$ $(i = 2, 3, \ldots)$ *and the sequences*

$$0 \longrightarrow H^3(X) \xrightarrow{\phi_{2i}} \bar{\pi}_{2i} \xrightarrow{\psi_{2i}} H^1(X) \longrightarrow 0, 0 \longrightarrow H^2(X) \longrightarrow \bar{\pi}_1 \longrightarrow z \longrightarrow 0$$

are exact $(i = 1, 2, \ldots)$;

(iv) *dimension* 4: *the sequences*

$$0 \longrightarrow H^3(X) \xrightarrow{\phi_{2i}} \bar{\pi}_{2i} \xrightarrow{\psi_{2i}} H^1(X) \xrightarrow{d^3} H^4(X) \quad (i = 1, 2, \ldots)$$

are exact (see Figure 10.1).

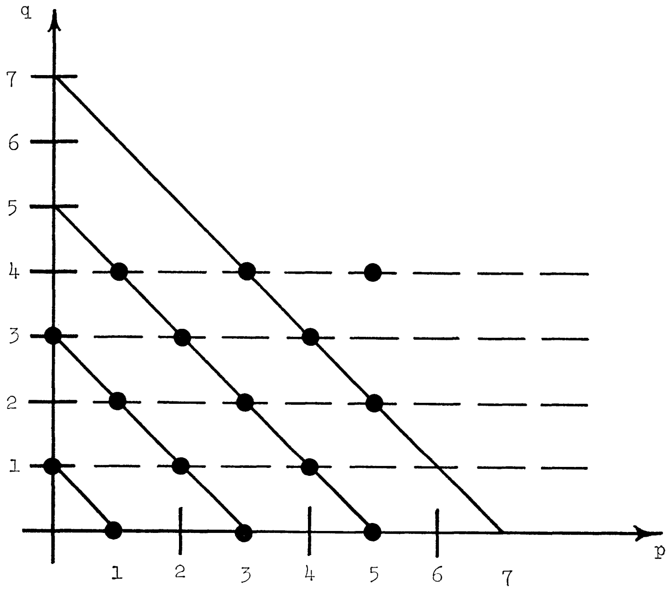

FIGURE 10.1. $E^2(X, U, v)$

Proof. (i) follows from Corollary 8.3(iv). (iii) follows from Theorem 8.5 (Gap Theorem I) with $m = 1$ and $n = k = 3$. The sequences Σ_i for $i \geqslant 2$ are as follows:

$$0 \longrightarrow H^2(X) \xrightarrow{\phi_{2i-1}} \bar{\Pi}_{2i-1} \xrightarrow{\psi_{2i-1}} H^0(X) \xrightarrow{\theta_{2i-1}} H^3(X) \xrightarrow{\phi_{2i-2}} \bar{\Pi}_{2i-2} \xrightarrow{\psi_{2i-2}} H^1(X) \longrightarrow 0.$$

Since Θ_{2i-1} is induced by d^3, Theorem 6.1 implies that $\Theta_{2i-1} = 0$ for $i = 1, 2, 3, \ldots H^0(X) = Z$ gives the isomorphism. (ii) follows from (iii) because the dimension of $X = 2$ implies that $H^3(X) = 0$.

(iv) follows because, for each $j = 2i$, we have the two-term condition $\{2i, 2i; 2\}$ and extended two-term conditions on the right and left. This proves the proposition.

Next let $Y = \mathbf{O}$, the infinite orthogonal group. It is well known that $\Pi_i(\mathbf{O}) \approx \Pi_{i+8}(\mathbf{O})$ and $\Pi_0(\mathbf{O}) = Z_2$, $\Pi_1(\mathbf{O}) \approx Z_2$, $\Pi_2(\mathbf{O}) = \Pi_4(\mathbf{O}) = \Pi_5(\mathbf{O}) = \Pi_6(\mathbf{O}) = 0$, $\Pi_3(\mathbf{O}) \approx Z \approx \Pi_7(0)$. If $\mathbf{O}_+$ is a component of $\mathbf{O}$, then we have the following proposition.

PROPOSITION 10.2. *Let $Y = \mathbf{O}_+$, $v: X \to Y$ be constant, and $M = M(X, \mathbf{O}_+)$. In addition, let X be a connected, locally finite CW complex of*

(i) *dimension* 1: *then, if $\Pi_j(M, v)$ is denoted by $\bar{\Pi}_j$,*

$$\bar{\Pi}_{1+8i} \approx Z_2,$$
$$\bar{\Pi}_{2+8i} \approx H^1(X) \approx \bar{\Pi}_{6+8i},$$
$$\bar{\Pi}_{3+8i} \approx Z,$$
$$\bar{\Pi}_{4+8i} = \bar{\Pi}_{5+8i} = 0,$$
$$\bar{\Pi}_{7+8i} \approx H^1(X; Z_2) \oplus Z,$$
$$\bar{\Pi}_{8+8i} \approx H^1(X; Z_2) \oplus Z_2 \qquad (i = 0, 1, 2, \ldots).$$

(ii) *dimension* 2: *then,*

$$\bar{\Pi}_{1+8i} \approx H^2(X) \oplus Z_2 \quad (i = 1, 2, \ldots), \quad 0 \to H^2(X) \to \bar{\Pi}_1 \to Z_2 \to 0$$
is exact,
$$\bar{\Pi}_{2+8i} \approx H^1(X),$$
$$\bar{\Pi}_{3+8i} \approx Z,$$
$$\bar{\Pi}_{4+8i} = 0,$$
$$\bar{\Pi}_{5+8i} \approx H^2(X),$$
$$0 \to H^2(X; Z_2) \xrightarrow{\phi} \bar{\Pi}_{6+8i} \xrightarrow{\psi} H^1(X) \to 0 \text{ is exact},$$
$$\bar{\Pi}_{8+8i} \approx H^1(X; Z_2) \oplus Z_2 \qquad (i = 0, 1, 2, \ldots);$$

(iii) *dimension* 3: *then*

$$\bar{\Pi}_{1+8i} \approx H^2(X) \oplus Z_2 \quad (i = 1, 2, \ldots), \qquad 0 \to H^2(X) \to \bar{\Pi}_1 \to Z_2 \to 0$$
is exact,
$$\bar{\Pi}_{2+8i} \approx H^1(X),$$
$$\bar{\Pi}_{3+8i} \approx Z,$$
$$\bar{\Pi}_{4+8i} \approx H^3(X),$$
$$H^1(X) \xrightarrow{d^2} H^3(X; Z_2) \xrightarrow{\phi} \bar{\Pi}_{5+8i} \xrightarrow{\psi} H^2(X) \to 0 \qquad (i = 0, 1, \ldots);$$

(iv) *dimension* 4: *then*

$$\bar{\Pi}_{1+8i} \approx H^2(X) \oplus Z_2 \quad (\mathrm{i} = 1, 2, \ldots), \qquad 0 \to H^2(X) \to \bar{\Pi}_1 \to z_2 \to 0$$
is exact,
$$\bar{\Pi}_{2+8i} \approx H^1(X),$$
$$\bar{\Pi}_{3+8i} \approx H^4(X) \oplus Z,$$
$$H^2(X) \xrightarrow{d^2} H^4(X; Z_2) \xrightarrow{\phi} \bar{\Pi}_{4+8i} \xrightarrow{\psi} H^3(X) \to 0 \qquad (i = 0, 1, \ldots);$$

(v) *dimension* 5: *then*

$$\bar{\Pi}_{1+8i} \approx H^2(X) \oplus Z_2 \quad (i = 1, 2, \ldots), \qquad 0 \to H^2(X) \to \bar{\Pi}_1 \to Z_2 \to 0$$
is exact,
$$0 \to H^5(X) \xrightarrow{\phi} \bar{\Pi}_{2+8i} \xrightarrow{\psi} H^1(X) \to 0 \qquad (i = 0, 1, \ldots).$$

The proof is similar to that of Proposition 10.1 except that any sequence involving $H^0(X; \Pi_i(\mathbf{O}_+))$ splits by Theorem 6.4. See Figure 10.2.

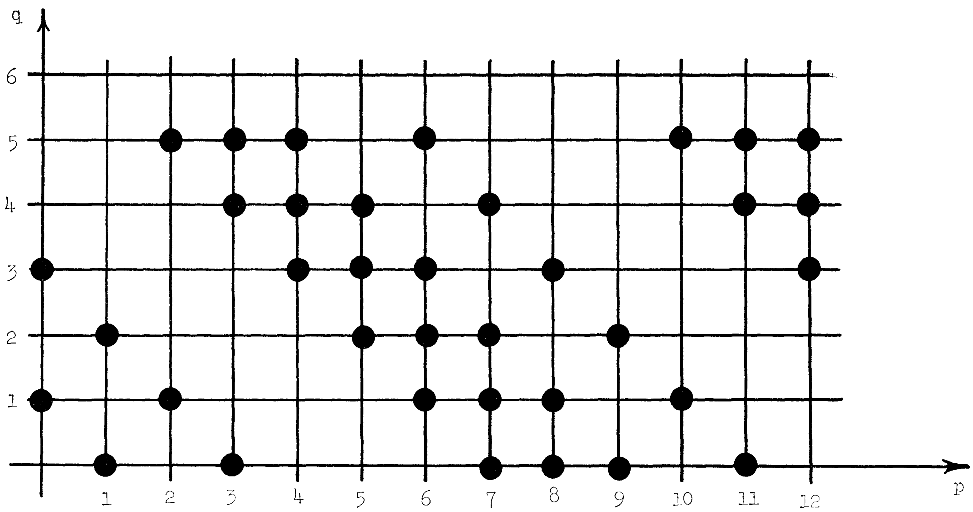

FIGURE 10.2. $E^2(X, \mathbf{O}_+, v)$

As an example of Gap Theorem II, we prove

PROPOSITION 10.3. *Let* $X = \mathrm{CP}^n$, *the n-dimensional complex projective space* (real dim $= 2n$) *and* $Y = S^m$ *such that* $m > 2n$, *then*

(i) $\pi_i(M(\mathrm{CP}^n, S^m), v) = 0$ *for* $1 \leqslant i \leqslant m - 1 - 2n$

(*vacuous if* $m - 1 - 2n = 0$);

(ii) $\pi_{m-2n}(M(\mathrm{CP}^n, S^m), v) \approx H^{2n}(\mathrm{CP}^n)$;

(iii) $H^{2n}(\mathrm{CP}^n, \pi_{m+2}(S^m)) \xrightarrow{\phi} \pi_{m-2n+2}(M(\mathrm{CP}^n, S^m), v)$

$\xrightarrow{\psi} Z \xrightarrow{\theta} H^{2n}(\mathrm{CP}^n, \pi_{m+1}(S^m)) \xrightarrow{\phi} \pi_{m-2n+1}(M(\mathrm{CP}^n, S^m), v) \to 0$

is exact (see Figure 10.3).

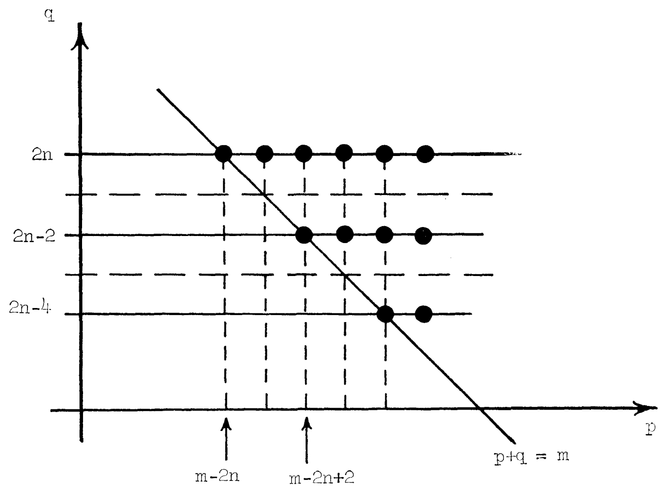

FIGURE 10.3. $E^2(\mathrm{CP}^n, S^m, v)$

Proof. The proposition follows directly from Corollary 9.2, since $H^i(\mathrm{CP}^n) \approx Z$ for $i = 0, 2, 4, \ldots, 2n$ and is zero otherwise.

As another example, we consider $M(C, Y)$, where C is a closed surface. If C is orientable of genus p ($p = 0, 1, 2, \ldots$), written O_p, then

$$H^i(O_p) \approx \begin{cases} Z & (i = 0, 2), \\ \bigoplus_{j=1}^{2p} (Z)_j & (i = 1; p = 0 \Rightarrow H_1(O_0) = 0), \\ 0 & (i > 2). \end{cases}$$

If C is non-orientable, then C is either P^2, the two-dimensional projective space, with p handles, written P_p^2, or K, the Klein bottle, with p handles, written K_p (see (**7**)). In this case

$$H^i(P_p^2) \approx \begin{cases} Z & (i = 0), \\ \bigoplus_{1}^{2p} (Z) & (i = 1), \\ Z_2 & (i = 2), \\ 0 & (i > 2), \end{cases}$$

and

$$H^i(K_p) \approx \begin{cases} Z & (i = 0), \\ \bigoplus_{1}^{2p+1} (Z) & (i = 1), \\ Z_2 & (i = 2), \\ 0 & (i > 2). \end{cases}$$

PROPOSITION 10.4. *Let C be a closed surface. Then C is a complex of cell type* $\{0, 1, 2\}$.

(i) *If Y is simply connected, then*

$$H^0(C, \pi_2(Y)) \xrightarrow{d^2} H^2(C, \pi_3(Y)) \xrightarrow{\phi_1} \pi_1(M(C, Y), v) \xrightarrow{\psi_1} H^1(C, \pi_2(Y)) \to 0.$$

(ii) *If Y is n-connected for $n \geqslant 2$, then*

$$H^0(C, \pi_{n+1}(Y)) \xrightarrow{d^2} H^2(C, \pi_{n+2}(Y)) \xrightarrow{\phi_n} \bar{\pi}_n \xrightarrow{\psi_n} H^1(C, \pi_{n+1}(Y)) \to 0,$$

$\pi_i(M(C, Y), v) = 0$ *for* $1 \leqslant i \leqslant n - 2 (n > 2)$ *and*

$$H^2(C, \pi_{n+1}(Y)) \approx \pi_{n-1}(M(C, Y), v).$$

The proof of this is immediate from Gap Theorem II. As an example, let

(i) $Y = S^2$. Then $\pi_2(S^2) \approx Z \approx \pi_3(S^2)$. Let $v\colon C \to S^2$ be constant. Then

$$0 \to Z \to \pi_1(M(O_p, S^2), v) \to \bigoplus_1^{2p} (Z) \to 0,$$

$$0 \to Z_2 \to \pi_1(M(P_p^2, S^2), v) \to \bigoplus_1^{2p} (Z) \to 0,$$

$$0 \to Z_2 \to \pi_1(M(K_p, S^2), v) \to \bigoplus_1^{2p+1} (Z) \to 0.$$

These groups are usually non-abelian. However, if $p = 0$, we see that

$$\pi_1(M(S^2, S^2), v) \approx Z, \qquad \pi_1(M(P^2, S^2), v) \approx Z_2.$$

(ii) $Y = S^n$ for $n > 2$. Then $\pi_n(S^n) \approx Z$, $\pi_{n+1}(S^n) \approx Z_2$, and v the constant map implies that

$$\pi_{n-1}(M(O_p, S^n), v) \approx \bigoplus_1^{2p} (Z) \oplus Z_2,$$

$$\pi_{n-2}(M(O_p, S^n), v) \approx Z,$$

$$\pi_{n-1}(M(P_p^2, S^n), v) \approx \bigoplus_1^{2p} (Z) \oplus Z_2,$$

$$\pi_{n-2}(M(P_p^2, S^n), v) \approx Z_2,$$

$$\pi_{n-1}(M(K_p, S^n), v) \approx \bigoplus_1^{2p+1} (Z) \oplus Z_2,$$

$$\pi_{n-2}(M(K_p, S^n), v) \approx Z_2,$$

and $M(C, S^n)$ is $n - 3$ connected. If $p = 0$, then

$$\pi_{n-1}(M(S^2, S^n), v) \approx Z_2 \approx \pi_{n-1}(M(P^2, S^n), v).$$

For similar computations, see **(1)**.

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