Bull. Aust. Math. Soc. **107** (2023), 112–124
doi:10.1017/S0004972722000429

OSCILLATION OF IMPULSIVE LINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS SOLUTIONS

SIBEL DOĞRU AKGÖL

(Received 18 March 2022; accepted 24 March 2022; first published online 5 May 2022)

Abstract

Sufficient conditions are obtained for the oscillation of a general form of a linear second-order differential equation with discontinuous solutions. The innovations are that the impulse effects are in mixed form and the results obtained are applicable even if the impulses are small. The novelty of the results is demonstrated by presenting an example of an oscillating equation to which previous oscillation theorems fail to apply.

2020 *Mathematics subject classification*: primary 34C10; secondary 34A36, 34A37.

Keywords and phrases: second-order linear differential equation, impulsive differential equation, oscillation, discontinuous solution.

1. Introduction

We obtain sufficient conditions for the existence of oscillatory solutions of impulsive linear differential equations of the form

$$\begin{cases} (a(t)y')' + b(t)y = 0, & t \neq \tau_k, \\ \Delta y + a_k y = 0, & t = \tau_k, \\ \Delta a(t)y' + b_k y + c_k y' = 0, & t = \tau_k. \end{cases} \tag{1.1}$$

The functions $a(t) > 0$ and $b(t)$ are assumed to be piecewise left continuous on $[t_0, \infty)$ for some $t_0 \geq 0$; $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ are real sequences; $\tau_{k+1} > \tau_k$ for all $k = 1, 2, \ldots$; $\lim_{k\to\infty} \tau_k = \infty$ and $\Delta\varphi(\tau_k) = \varphi(\tau_k^+) - \varphi(\tau_k^-)$ with $\varphi(\tau_k^\pm) = \lim_{t\to\tau_k^\pm} \varphi(t)$. As the impulse effects contain both the solution and its derivative, they are said to be of mixed type. By separated impulse effects, we mean that the impulse effects contain either only the solution or only the derivative of the solution, for example, $\Delta y + a_k y = 0$ and $\Delta a(t)y' + c_k y' = 0$, where $t = \tau_k$.

For the sake of brevity, the notation $\underline{n}(t) := \inf\{k : \tau_k \geq t\}$, $\overline{n}(t) := \sup\{k : \tau_k < t\}$ and $\omega_k := (1 - c_k/a(\tau_k))/(1 - a_k)$ is used. The following hypotheses are assumed throughout the paper:

(H1) $1 - c_k/a(\tau_k) > 0$, $1 - a_k > 0$ and $b_k \le 0$, $k = 1, 2, \ldots$;

(H2) there exists a function $f(t) : [t_0, \infty) \to (0, \infty)$ such that $f'(t)$ exists on $[t_0, \infty)$ and

$$g(t) := b(t) + \alpha(t) f'(t) + \frac{(\alpha(t) f(t))^2}{a(t)} \ge 0,$$

where

$$\alpha(t) := \begin{cases} -\displaystyle\sum_{k=\underline{n}(t_0)}^{\overline{n}(t)} \frac{b_k}{f(\tau_k)(1 - c_k/a(\tau_k))} \prod_{i=k}^{\overline{n}(t)} \omega_i & \text{if } b_k \neq 0, \\ \displaystyle\prod_{k=\underline{n}(t_0)}^{\overline{n}(t)} \omega_k & \text{if } b_k = 0. \end{cases}$$

Differential equations containing impulse effects are practical tools to represent many evolutionary processes such as biological models, physical phenomena and engineering problems, and the corresponding theory is quite rich. Their qualitative theory has been investigated deeply by many researchers (see the famous books [4, 6]). It is well known that impulse effects can cause radical changes in the structure of the solution of a differential equation. For example, a nonoscillatory unforced differential equation may turn out to be oscillatory under impulsive conditions [2, 7, 9–11]. Since it is not easy to make a prediction, it is crucial to study the long-time behaviour of impulsive differential equations, in particular, their oscillatory properties. We refer to [1] for an excellent survey on the oscillation of differential equations under impulse effects and the papers [3, 8, 9] regarding self-adjoint impulsive differential equations with continuous solutions, namely, equations derived by setting $a_k = 0 = c_k$ in (1.1). There are only a few studies of their counterparts having discontinuous solutions (see [5, 7], where differential equations with separated impulse effects were considered). To the best of our knowledge, the only paper dealing with oscillation of equations with mixed impulse effects of the form (1.1) is [2], in which a Leighton-type oscillation theorem is produced.

2. Main results

We start with some auxiliary lemmas.

LEMMA 2.1. *Let*

$$x(t) := y(t) \exp\left\{-\int_{t_0}^{t} \frac{\alpha(s) f(s)}{a(s)}\, ds\right\}, \tag{2.1}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

where $y(t)$ is a solution of (1.1). Then, $x(t)$ is a solution of the differential equation with separated impulse effects:

$$\begin{cases} (a(t)x')' + 2\alpha(t)f(t)x' + g(t)x = 0, & t \neq \tau_k, \\ \Delta x + a_k x = 0, & t = \tau_k, \\ \Delta(a(t)x') + c_k x' = 0, & t = \tau_k. \end{cases} \tag{2.2}$$

PROOF. Let $y(t)$ be a solution of (1.1). For $t \neq \tau_k$,

$$a(t)y'(t) = [a(t)x'(t) + \alpha(t)f(t)x(t)]\exp\left\{\int_{t_0}^{t} \frac{\alpha(s)f(s)}{a(s)}\,ds\right\}$$

and

$$(a(t)y'(t))' + b(t)y(t) = [(a(t)x'(t))' + 2\alpha(t)f(t)x'(t) + g(t)x(t)]\exp\left\{\int_{t_0}^{t} \frac{\alpha(s)f(s)}{a(s)}\,ds\right\},$$

which clearly implies that

$$(a(t)x'(t))' + 2\alpha(t)f(t)x'(t) + g(t)x(t) = 0, \quad t \neq \tau_k. \tag{2.3}$$

For $t = \tau_k$, $k = 1, 2, \ldots$, it is easy to see that

$$\Delta x|_{t=\tau_k} + a_k x(\tau_k) = [\Delta y|_{t=\tau_k} + a_k y(\tau_k)]\exp\left\{-\int_{t_0}^{\tau_k} \frac{\alpha(s)f(s)}{a(s)}\,ds\right\} = 0. \tag{2.4}$$

Noting that

$$\alpha(\tau_k^+) = \omega_k\alpha(\tau_k) - \frac{b_k}{f(\tau_k)(1 - a_k)},$$

we see that

$$\begin{aligned} &\Delta(a(t)x')|_{t=\tau_k} + c_k x'(\tau_k) \\ &= \left(\Delta(a(t)y' - \alpha(t)f(t)y)|_{t=\tau_k} + c_k\left[y'(\tau_k) - \frac{\alpha(\tau_k)f(\tau_k)y(\tau_k)}{a(\tau_k)}\right]\right)\exp\left\{-\int_{t_0}^{\tau_k} \frac{\alpha(s)f(s)}{a(s)}\,ds\right\} \\ &= y(\tau_k)\left(-b_k - f(\tau_k)\left[(1 - a_k)\alpha(\tau_k+) - \alpha(\tau_k) + \frac{c_k\alpha(\tau_k)}{a(\tau_k)}\right]\right)\exp\left\{-\int_{t_0}^{\tau_k} \frac{\alpha(s)f(s)}{a(s)}\,ds\right\} \\ &= 0. \end{aligned} \tag{2.5}$$

Thus, from (2.3)–(2.5), we conclude that $x(t)$ is a solution of (2.2). □

LEMMA 2.2. *Let $x(t)$ be a nonoscillatory solution of (2.2). If*

$$\lim_{n\to\infty} \frac{1}{a(\tau_n)} \sum_{i=0}^{n-1} \prod_{j=0}^{i} \frac{1}{\omega_j} \int_{\tau_i}^{\tau_{i+1}} g(t)\,dt = \infty, \tag{2.6}$$

then $x(t)x'(t)$ is ultimately negative.

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

PROOF. Suppose that $x(t)$ is ultimately positive. First, we will show that $x'(t)$ is nonoscillatory. We assume on the contrary that $x'(t)$ is oscillatory. Then, there is some $k \in \mathbb{N}$ and $t_a \in (\tau_k, \tau_{k+1}]$ such that $x'(t_a) = 0$. Thus, in view of (2.2),

$$a(t_a)x''(t_a) = -g(t_a)x(t_a) < 0, \tag{2.7}$$

which implies that there is some interval $(t_a, t_a + \delta)$, $\delta > 0$, in which $x'(t)$ is decreasing. Hence,

$$x'(t) < 0, \quad t \in (t_a, t_a + \delta). \tag{2.8}$$

Now, assume that t_a is the first root and x' has another root in the same interval, that is, there is some $t_b \in (t_a, \tau_{k+1})$ such that $x'(t_b) = 0$. From (2.8), this implies that $x''(t_b) \geq 0$. However, from (2.7), we see that $a(t_b)x''(t_b) < 0$ which leads to a contradiction. Hence, $x'(t)$ cannot have a root in (t_a, τ_{k+1}), that is, $x'(t) < 0$ there. This implies that

$$x'(\tau_{k+1}^+) = (1 - c_{k+1}/a(\tau_{k+1}))x'(\tau_{k+1}) < 0. \tag{2.9}$$

If we again suppose that there is some t_c such that $x'(t_c) = 0$, from (2.7), we obtain $x''(t_c) < 0$ which implies $x'(t) < 0$, $t \in (t_c - \delta, t_c + \delta)$, contradicting $x'(t_c) = 0$. Hence, $x'(t) < 0$ on $(\tau_{k+1}, \tau_{k+2}]$. By similar arguments, it can be seen that

$$x'(t) < 0, \quad t \in (\tau_{k+i}, \tau_{k+i+1}], \ i \in \mathbb{N}.$$

Fix some $T \geq t_0$ and let $\tau_k \geq T$. If $x'(t) < 0$ on $(\tau_k, \tau_{k+1}]$, from (2.9), $x'(\tau_{k+1}^+) < 0$ and, by the above discussion, $x'(t) < 0$ on $(\tau_{k+i}, \tau_{k+i+1}]$, for all $i \in \mathbb{N}$. Thus, $x'(t) < 0$ on $[T, \infty)$.

Conversely, if $x'(t) > 0$ on $(\tau_k, \tau_{k+1}]$, then $x'(\tau_{k+1}^+)(1 - c_{k+1}/a(\tau_{k+1}))x'(\tau_{k+1}) > 0$. However, we know that $x'(t)$ has no root in $(\tau_{k+i}, \tau_{k+i+1}]$, for all $n \in \mathbb{N}$. Thus, $x'(t) > 0$ on $[T, \infty)$. Hence, $x'(t)$ is nonoscillatory.

Our next aim is to show that $x'(t)$ is ultimately negative, that is, there is some $T_* \geq t_0$ such that $x'(t) < 0$ for $t \geq T_*$. Suppose on the contrary, there exists $k \in \mathbb{N}$ such that $x'(\tau_k) > 0$ for $\tau_k \geq T_*$. Then,

$$x'(\tau_k^+) = (1 - c_k/a(\tau_k))x'(\tau_k) > 0,$$

and so $x'(t) > 0$ for $t \geq \tau_k$. From (H1), $\alpha(t) > 0$ on $[t_0, \infty)$. Thus, we can write

$$(a(t)x'(t))' = -2\alpha(t)f(t)x'(t) - g(t)x(t) < -g(t)x(t) \leq 0, \tag{2.10}$$

which shows that $a(t)x'(t)$ is decreasing on each interval $[\tau_{k+i-1}, \tau_{k+i})$, $i \in \mathbb{N}$. Now, we need to prove that, for $n \geq 1$,

$$x'(\tau_{k+n}) \leq \frac{1}{a(\tau_{k+n})} \prod_{j=0}^{n-1} \left(1 - \frac{c_{k+j}}{a(\tau_{k+j})}\right) \left\{ a(\tau_k)x'(\tau_k) - x(\tau_k) \sum_{i=0}^{n-1} \prod_{j=0}^{i} \omega_{k+j} \int_{\tau_{k+i}}^{\tau_{k+i+1}} g(t)\, dt \right\}. \tag{2.11}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

Integrating (2.10) on $(\tau_i, \tau_{i+1}]$,

$$a(\tau_{k+1})x'(\tau_{k+1}) \le a(\tau_k^+)x'(\tau_k^+) - \int_{\tau_k}^{\tau_{k+1}} g(t)x(t)\,dt.$$

Since $x'(t) > 0$ for $t \ge \tau_k$, it follows that x is increasing on $(\tau_k, \tau_{k+1}]$. Hence,

$$\begin{aligned} a(\tau_{k+1})x'(\tau_{k+1}) &\le (a(\tau_k) - c_k)x'(\tau_k) - x(\tau_k^+)\int_{\tau_k}^{\tau_{k+1}} g(t)\,dt \\ &= \Big(1 - \frac{c_k}{a(\tau_k)}\Big)\Big\{a(\tau_k)x'(\tau_k) - \frac{1}{\omega_k}x(\tau_k)\int_{\tau_k}^{\tau_{k+1}} g(t)\,dt\Big\}. \end{aligned} \tag{2.12}$$

Integrating (2.10) on $(\tau_{k+1}, \tau_{k+2}]$ and using (2.12),

$$\begin{aligned} &a(\tau_{k+2})x'(\tau_{k+2}) \\ &\quad \le (a(\tau_{k+1}) - c_{k+1})x'(\tau_{k+1}) - x(\tau_{k+1}^+)\int_{\tau_{k+1}}^{\tau_{k+2}} g(t)\,dt \\ &\quad \le \Big(1 - \frac{c_{k+1}}{a(\tau_{k+1})}\Big)\Big\{\Big(1 - \frac{c_k}{a(\tau_k)}\Big)\Big[a(\tau_k)x'(\tau_k) - \frac{1}{\omega_k}x(\tau_k)\int_{\tau_k}^{\tau_{k+1}} g(t)\,dt\Big] \\ &\qquad - \frac{1}{\omega_{k+1}}x(\tau_k^+)\int_{\tau_{k+1}}^{\tau_{k+2}} g(t)\,dt\Big\} \\ &\quad = \Big(1 - \frac{c_{k+1}}{a(\tau_{k+1})}\Big)\Big(1 - \frac{c_k}{a(\tau_k)}\Big)\Big\{a(\tau_k)x'(\tau_k) - x(\tau_k)\Big[\frac{1}{\omega_k}\int_{\tau_k}^{\tau_{k+1}} g(t)\,dt \\ &\qquad + \frac{1}{\omega_k\omega_{k+1}}\int_{\tau_{k+1}}^{\tau_{k+2}} g(t)\,dt\Big]\Big\}. \end{aligned}$$

Now, suppose that (2.11) holds for $n = N$. Then, for $t \in (\tau_{k+N}, \tau_{k+N+1}]$,

$$\begin{aligned} &a(\tau_{k+N+1})x'(\tau_{k+N+1}) \\ &\quad \le (a(\tau_{k+N}) - c_{k+N})x'(\tau_{k+N}) - x(\tau_{k+N}^+)\int_{\tau_{k+N}}^{\tau_{k+N+1}} g(t)\,dt \\ &\quad \le \Big(1 - \frac{c_{k+N}}{a(\tau_{k+N})}\Big)\Big\{\prod_{j=0}^{N-1}\Big(1 - \frac{c_{k+j}}{a(\tau_{k+j})}\Big)\Big[a(\tau_k)x'(\tau_k) \\ &\qquad - x(\tau_k)\sum_{i=0}^{N-1}\prod_{j=0}^{i}\frac{1}{\omega_{k+j}}\int_{\tau_{k+i}}^{\tau_{k+i+1}} g(t)\,dt\Big] - \frac{1}{\omega_{k+N}}x(\tau_{k+N})\int_{\tau_{k+N}}^{\tau_{k+N+1}} g(t)\,dt\Big\} \\ &\quad \le \prod_{j=0}^{N}\Big(1 - \frac{c_{k+j}}{a(\tau_{k+j})}\Big)\Big\{a(\tau_k)x'(\tau_k) - x(\tau_k)\sum_{i=0}^{N}\prod_{j=0}^{i}\frac{1}{\omega_{k+j}}\int_{\tau_{k+i}}^{\tau_{k+i+1}} g(t)\,dt\Big\}, \end{aligned}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

where, in the last line, the estimate $x(\tau_{k+N}) \geq x(\tau_{k+N-1}^+) = (1 - a_{k+N-1})x(\tau_{k+N-1})$ is used N times. Thus, by induction on n, we see that (2.11) holds for any $n \geq 1$.

If we take the limit of both sides of (2.11) as $n \to \infty$, from (2.6), we see that $x'(\tau_{k+n}) < 0$ for sufficiently large values of n. However, this contradicts the assumption that $x'(t) > 0$ for $t \geq \tau_k$. Hence, $x'(\tau_k) < 0$ for $\tau_k \geq T_*$. Since $x'(t)$ has a constant sign for $t \geq \tau_k$, it follows that $x'(t) < 0$ for all $t \neq \tau_{k+n}$, $t \geq T_*$.

If $x(t)$ is ultimately negative, by repeating all the steps of the proof, it can be shown that there is some $T^* \geq t_0$ such that $x'(t) > 0$ for $t \geq T^*$. Thus, the proof is complete. □

THEOREM 2.3. *Suppose that (2.6) holds, and*

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} \prod_{j=0}^{i} \omega_j \int_{\tau_i}^{\tau_{i+1}} \mu(s, t_0)\, ds = \infty, \tag{2.13}$$

where

$$\mu(t, s) := \exp\Big\{-2 \int_s^t \frac{f(r)\alpha(r)}{a(r)}\, dr\Big\}.$$

Then, (2.2) is oscillatory.

PROOF. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (2.2). If we assume $x(t)$ is ultimately positive, namely, there exists some $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$, in view of Lemma 2.2, $x'(t) < 0$ for $t \geq T$ and $t \neq \tau_k$. Since $g(t) > 0$, from (2.2),

$$(a(t)x'(t))' + 2\alpha(t)f(t)x'(t) < 0, \quad t \geq T,\ t \neq \tau_k,$$

that is,

$$\frac{(a(t)x'(t))'}{a(t)x'(t)} + \frac{2\alpha(t)f(t)}{a(t)} > 0. \tag{2.14}$$

Define $\tau_k := \min\{\tau_j : \tau_j \geq T\}$. For $t \in (\tau_k, \tau_{k+1}]$, integration of (2.14) yields

$$\ln\Big(\frac{a(t)x'(t)}{a(\tau_k^+)x'(\tau_k^+)}\Big) + 2 \int_{\tau_k}^t \frac{\alpha(r)f(r)}{a(r)}\, dr > 0.$$

Since $x'(t) < 0$ for $t \geq T$, this implies that

$$x'(t) < x'(\tau_k^+)\mu(\tau_k, t) = \Big(1 - \frac{c_k}{a(\tau_k)}\Big)x'(\tau_k)\mu(\tau_k, t), \quad t \in (\tau_k, \tau_{k+1}]. \tag{2.15}$$

Setting $t = \tau_{k+1}$,

$$x'(\tau_{k+1}) < \Big(1 - \frac{c_k}{a(\tau_k)}\Big)x'(\tau_k)\mu(\tau_k, \tau_{k+1}). \tag{2.16}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

Integrating (2.15) on $(\tau_k, \tau_{k+1}]$,

$$x(\tau_{k+1}) < (1 - a_k)x(\tau_k) + \Big(1 - \frac{c_k}{a(\tau_k)}\Big)x'(\tau_k) \int_{\tau_k}^{\tau_{k+1}} \mu(\tau_k, s)\, ds. \tag{2.17}$$

Now, we apply the same procedure on $(\tau_{k+1}, \tau_{k+2}]$ and similarly obtain

$$x(\tau_{k+2}) < (1 - a_{k+1})x(\tau_{k+1}) + \Big(1 - \frac{c_{k+1}}{a(\tau_{k+1})}\Big)x'(\tau_{k+1}) \int_{\tau_{k+1}}^{\tau_{k+2}} \mu(\tau_{k+1}, s)\, ds. \tag{2.18}$$

Observe that $\mu(\tau_k, \tau_{k+1})\mu(\tau_{k+1}, s) = \mu(\tau_k, s)$. Thus, using (2.16) and (2.17) in (2.18),

$$\begin{aligned} &x(\tau_{k+2}) \\ &\quad < (1 - a_k)(1 - a_{k+1})\Big\{x(\tau_k) + x'(\tau_k)\Big[\omega_k \int_{\tau_k}^{\tau_{k+1}} \mu(\tau_k, s)\, ds + \omega_k\omega_{k+1} \int_{\tau_{k+1}}^{\tau_{k+2}} \mu(\tau_k, s)\, ds\Big]\Big\}. \end{aligned}$$

Suppose that

$$x(\tau_{k+n}) < \prod_{j=0}^{n-1}(1 - a_{k+j})\Big\{x(\tau_k) + x'(\tau_k) \sum_{i=0}^{n-1} \prod_{j=0}^{i} \omega_{k+j} \int_{\tau_{k+i}}^{\tau_{k+i+1}} \mu(\tau_k, s)\, ds\Big\} \tag{2.19}$$

for $n = N$. Then, in a similar way to the proof of (2.18), it can be shown that (2.19) holds for $n = N + 1$. Thus, by induction, the inequality (2.19) is true for any $n \geq 1$.

Applying (2.13) in (2.19) leads to the contradiction that $x(\tau_n) < 0$ for sufficiently large values of n. Hence, $x(t)$ is oscillatory. □

THEOREM 2.4. *Suppose that (2.6) holds and that there exists a continuous function $h(t) : [t_0, \infty) \to (0, \infty)$ such that $h'(t)$ exists on $[t_0, \infty)$ and $a(t)h'(t) \geq 2\alpha(t)f(t)h(t)$. If*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \prod_{j=1}^{i-1} \omega_j \int_{\tau_{i-1}}^{\tau_i} h(s)g(s)\, ds = \infty \tag{2.20}$$

and

$$\limsup_{n\to\infty} \int_{\tau_i}^{\tau_{i+1}} \frac{dt}{a(t)h(t)} \geq 1, \tag{2.21}$$

then (2.2) is oscillatory.

PROOF. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (2.2). We may assume $x(t) > 0$ for $t \geq T$ for some $T \geq t_0$. Then, by Lemma 2.2, $x'(t) < 0$ for $t \geq T$. Define $\tau_k := \min\{\tau_j : \tau_j \geq T\}$. Multiplying the first line of (2.2) by $h(t)/x(t)$, and then integrating it on $(\tau_k, t]$, for $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{aligned} &\frac{h(t)a(t)x'(t)}{x(t)} - \frac{h(\tau_k)a(\tau_k)x'(\tau_k^+)}{x(\tau_k^+)} + \int_{\tau_k}^{t} a(s)h(s)\Big(\frac{x'(s)}{x(s)}\Big)^2 ds \\ &\qquad - \int_{\tau_k}^{t} [a(s)h'(s) - 2\alpha(s)f(s)h(s)]\frac{x'(s)}{x(s)}\, ds + \int_{\tau_k}^{t} h(s)g(s)\, ds = 0, \end{aligned} \tag{2.22}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

which implies that

$$\frac{h(t)a(t)x'(t)}{x(t)} < \frac{h(\tau_k)a(\tau_k)x'(\tau_k^+)}{x(\tau_k^+)} - \int_{\tau_k}^{t} h(s)g(s)\,ds,$$

and so

$$\frac{h(\tau_{k+1})a(\tau_{k+1})x'(\tau_{k+1})}{x(\tau_{k+1})} < \omega_k \frac{h(\tau_k)a(\tau_k)x'(\tau_k)}{x(\tau_k)} - \int_{\tau_k}^{\tau_{k+1}} h(s)g(s)\,ds.$$

For $t \in (\tau_{k+1}, \tau_{k+2}]$, similarly, we can show

$$\begin{aligned}\frac{h(t)a(t)x'(t)}{x(t)} &< \frac{h(\tau_{k+1})a(\tau_{k+1})x'(\tau_{k+1}^+)}{x(\tau_{k+1}^+)} - \int_{\tau_{k+1}}^{t} h(s)g(s)\,ds\\ &< \omega_{k+1}\Big\{\omega_k \frac{h(\tau_k)a(\tau_k)x'(\tau_k)}{x(\tau_k)} - \int_{\tau_k}^{\tau_{k+1}} h(s)g(s)\,ds\Big\} - \int_{\tau_{k+1}}^{t} h(s)g(s)\,ds.\end{aligned}$$

The last inequality holds for $t = \tau_{k+2}$. Using similar arguments and applying induction on n, it is not hard to prove that for $n \geq 1$,

$$\frac{h(\tau_{k+n})a(\tau_{k+n})x'(\tau_{k+n})}{x(\tau_{k+n})} < \frac{h(\tau_k)a(\tau_k)x'(\tau_k)}{x(\tau_k)} \prod_{j=0}^{n-1} \omega_{k+j} - \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \omega_{k+j} \int_{\tau_{k+i-1}}^{\tau_{k+i}} h(s)g(s)\,ds.$$

However, from Lemma 2.2, $x(t)x'(t) < 0$. So, using (2.20),

$$\frac{h(\tau_n)a(\tau_n)x'(\tau_n)}{x(\tau_n)} < -\sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \omega_j \int_{\tau_{i-1}}^{\tau_i} h(s)g(s)\,ds \to -\infty, \quad \text{as } n \to \infty. \tag{2.23}$$

Now, from (2.22),

$$\frac{h(t)a(t)x'(t)}{x(t)} + \int_{\tau_k}^{t} a(s)h(s)\Big(\frac{x'(s)}{x(s)}\Big)^2 ds < \frac{h(\tau_k)a(\tau_k)x'(\tau_k^+)}{x(\tau_k^+)} - \int_{\tau_k}^{t} h(s)g(s)\,ds$$

and from (2.23), this implies that there is a sufficiently large τ_ℓ so that

$$\frac{h(t)a(t)x'(t)}{x(t)} + \int_{\tau_\ell}^{t} a(s)h(s)\Big(\frac{x'(s)}{x(s)}\Big)^2 ds \leq -1,$$

for $t \geq \tau_\ell$. Hence,

$$\frac{h(t)a(t)x'(t)}{x(t)} \leq -1 - \int_{\tau_\ell}^{t} a(s)h(s)\Big(\frac{x'(s)}{x(s)}\Big)^2 ds. \tag{2.24}$$

Since $x'(t) < 0$,

$$\frac{x'(t)}{x(t)} \leq a(t)h(t)\Big(\frac{x'(t)}{x(t)}\Big)^2\Big(1 + \int_{\tau_\ell}^{t} \Big(\frac{x'(s)}{x(s)}\Big)^2 ds\Big)^{-1}.$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

Integrating the last inequality on $(\tau_\ell, t]$ yields

$$\ln\left(\frac{x'(\tau_\ell^+)}{x(t)}\right) \le \ln\left(1 + \int_{\tau_\ell}^{t} \left(\frac{x'(s)}{x(s)}\right)^2 ds\right).$$

From (2.24), it follows that

$$\frac{x'(\tau_\ell^+)}{x(t)} \le -a(t)h(t)\frac{x'(t)}{x(t)}, \quad \text{that is,} \quad x'(t) \le -\frac{x'(\tau_\ell^+)}{a(t)h(t)}.$$

Now, we integrate the last expression on $(\tau_\ell, \tau_{\ell+1}]$ to obtain

$$x(\tau_{\ell+1}) \le x(\tau_\ell^+)\left(1 - \int_{\tau_\ell}^{\tau_{\ell+1}} \frac{dt}{a(t)h(t)}\right).$$

Thus, using the hypothesis (2.21), it is easy to see

$$\limsup_{\ell\to\infty} x(\tau_{\ell+1}) \le 0,$$

which leads to a contradiction because of the assumption that $x(t)$ is ultimately positive. Hence, $x(t)$ is oscillatory. □

Now, we can easily establish the following oscillation criteria for (1.1).

THEOREM 2.5. *If all hypotheses of Theorem 2.3 hold, (1.1) is oscillatory.*

THEOREM 2.6. *If all hypotheses of Theorem 2.4 hold, (1.1) is oscillatory.*

In view of (2.1) and Lemma 2.1, it can be seen that Theorems 2.5 and 2.6 follow directly from Theorems 2.3 and 2.4, respectively.

3. Examples

In this section, we describe some examples to illustrate our results. For each example, the graphs show the discontinuities in a small interval and the oscillating behaviour on a larger interval.

EXAMPLE 3.1. Consider the impulsive differential equation

$$\begin{cases} \left(\dfrac{1}{t^2}y'\right)' + \dfrac{t+2}{(t+1)^2}y = 0, & t \ne k, t \ge 2, \\[2ex] \Delta y + \dfrac{k+3}{2(k+2)}y = 0, \ \ \Delta\left(\dfrac{1}{t^2}y'\right) - \dfrac{3}{4k^3(k+1)(k+2)}y + \dfrac{1}{2k^2}y' = 0, & t = k, \ k > 2. \end{cases} \tag{3.1}$$

Let $t_0 = 2$. Clearly,

$$\tau_k = k, \ a(t) = \frac{1}{t^2}, \ b(t) = \frac{t+2}{(t+1)^2}, \ a_k = \frac{k+3}{2(k+2)}, \ b_k = -\frac{3}{4k^3(k+1)(k+2)}, \ c_k = \frac{1}{2k^2},$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

and so, $1 - a_k = (k+1)/(2(k+2)) > 0$ and $1 - c_k/a(\tau_k) = 1/2$. Thus, (H1) holds. If we choose $f(t) = 3/(2t^3(t+1))$, from $\omega_k = (k+2)/(k+1)$,

$$\alpha(t) = \sum_{k=2}^{\overline{n}(t)} \frac{1}{k+2} \prod_{i=k}^{\overline{n}(t)} \frac{i+2}{i+1} = (\overline{n}(t)+2) \sum_{k=2}^{\overline{n}(t)} \frac{1}{(k+2)(k+1)} = \frac{\overline{n}(t)-1}{3}.$$

For $t \in (i, i+1]$, $t \geq 2$, we have $\overline{n}(t) = i+1$ and $\alpha(t) = i/3$, which implies that

$$\begin{aligned} g(t) &= b(t) + \alpha(t) f'(t) + \frac{(\alpha(t) f(t))^2}{a(t)} \\ &> \frac{t+2}{(t+1)^2} - \frac{4t+3}{2t^3(t+1)^2} + \frac{(t-1)^2}{4t^4(t+1)^2} > \frac{1}{t+1} > 0. \end{aligned}$$

Thus, (H2) also holds and

$$\frac{1}{a(\tau_n)} \sum_{i=0}^{n-1} \prod_{j=0}^{i} \frac{1}{\omega_j} \int_{\tau_i}^{\tau_{i+1}} g(t)\, dt > n^2 \sum_{i=0}^{n-1} \frac{1}{i+2} \ln\Big(\frac{i+2}{i+1}\Big).$$

If we take the limit of both sides, we see that (2.6) is satisfied. However, since $\alpha(t) < t/3$ for $t \geq 2$, we have the estimate

$$\mu(s, t_0) = \exp\Big\{-2 \int_2^s \frac{f(r)\alpha(r)}{a(r)}\, dr\Big\} > \exp\Big\{-\int_2^s \frac{1}{r+1}\, dr\Big\} = \frac{3}{s+1}.$$

Hence, from

$$\sum_{i=0}^{n-1} \prod_{j=0}^{i} \omega_j \int_{\tau_i}^{\tau_{i+1}} \mu(s, t_0)\, ds > 3 \sum_{i=0}^{n-1} (i+2) \int_i^{i+1} \frac{1}{s+1}\, ds = 3 \sum_{i=0}^{n-1} (i+2) \ln\Big(\frac{i+2}{i+1}\Big),$$

we see that (2.13) also holds. Thus, by Theorem 2.5, (3.1) is oscillatory. This conclusion is illustrated in Figure 1.

EXAMPLE 3.2. Consider the impulsive differential equation

$$\begin{cases} (e^{-t} y')' + (1 + e^{-t}) y = 0, & t \neq k, t \geq 2, \\ \Delta y - e^k y = 0, \quad \Delta(e^{-t} y') - \dfrac{1}{4(k^2 - k)} y - y' = 0, & t = k, k > 2. \end{cases} \tag{3.2}$$

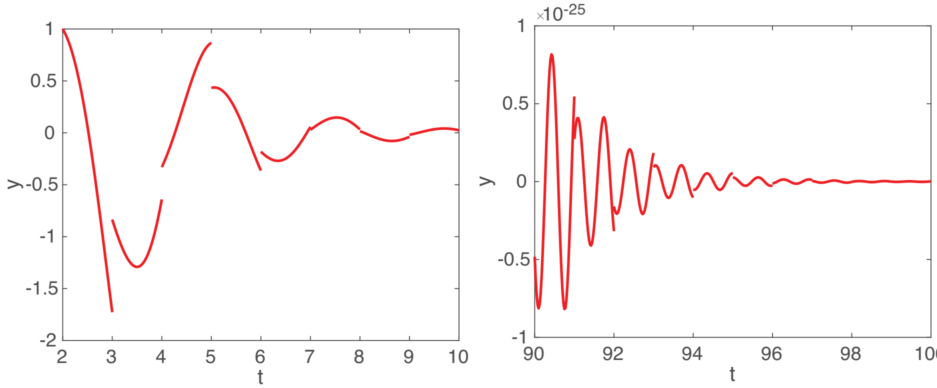
Let $t_0 = 2$. Clearly,

$$\tau_k = k, \; a(t) = e^{-t}, \; b(t) = 1 + e^{-t}, \; a_k = -e^k, \; b_k = -\frac{1}{4(k^2 - k)}, \; c_k = -1.$$

Thus, (H1) holds. Since $\omega_k = 1$, by choosing $f(t) = e^{-t}$, we get

$$\alpha(t) = \frac{1}{4} \sum_{k=2}^{\overline{n}(t)} \frac{e^k}{(1 + e^k)(k^2 - k)}. \tag{3.3}$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press



(a) Graph of $y(t)$, $t \in [2, 10]$.

(b) Graph of $y(t)$, $t \in [90, 100]$.

FIGURE 1. Illustrations for Example 3.1.

From $2/5 < e^k/(1+e^k) < 1$, we may write $1/10t < \alpha(t) < 1/2$. Then

$$g(t) > 1 + e^{-t} - \frac{1}{2e^t} + \frac{1}{100t^2 e^t} > 1,$$

so that

$$\frac{1}{a(\tau_n)} \sum_{i=0}^{n-1} \prod_{j=0}^{i} \frac{1}{\omega_j} \int_{\tau_i}^{\tau_{i+1}} g(t)\,dt > e^n \frac{(n-1)n}{2}.$$

Taking the limit of both sides, it is easily seen that (2.6) holds.

Now, if we take $h(t) = e^t$, we can write $2\alpha(t)f(t)h(t) < 1 = a(t)h'(t)$, and

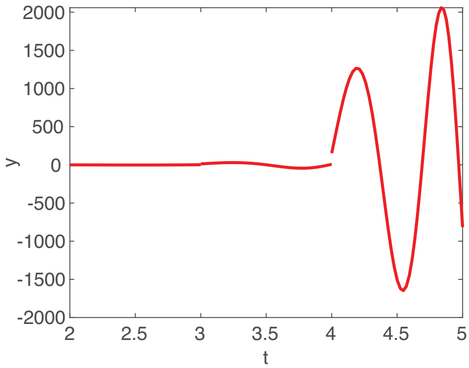
$$\int_i^{i+1} \frac{dt}{a(t)h(t)} = 1,$$

which shows that (2.21) holds. Finally, to check the hypothesis (2.20), we write

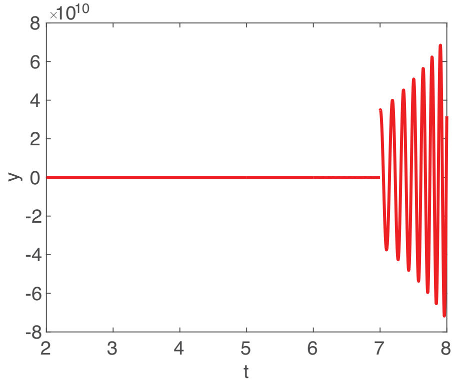
$$\sum_{i=1}^{n} \prod_{j=1}^{i-1} \omega_j \int_{\tau_{i-1}}^{\tau_i} h(s)g(s)\,ds > \sum_{i=1}^{n} \int_{i-1}^{i} e^s\,ds = e^n - 1.$$

Clearly, the last expression tends to infinity as $n \to \infty$. Hence, by means of Theorem 2.6, (3.2) is oscillatory. The oscillation behaviour can also be seen in Figure 2.

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press



(a) Graph of $y(t)$, $t \in [2, 5]$.



(b) Graph of $y(t)$, $t \in [2, 8]$.

FIGURE 2. Illustrations for Example 3.2.

4. Concluding remarks

The following remark demonstrates the novelty of Theorem 2.5 and hence also Theorem 2.3.

REMARK 4.1. As far as we know, the only paper that deals with the oscillation of (1.1) is [2]. If we attempt to apply the Leighton-type theorem [2, Theorem 2.1] to (3.1), we compute

$$\int_2^t b(s) \prod_{k=2}^{\overline{n}(s)} \frac{1}{\omega_k}\, ds + \sum_{k=2}^{\overline{n}(t)} \frac{b_k}{1 - c_k/a(\tau_k)} \prod_{j=2}^{k} \frac{1}{\omega_j}$$
$$= 3 \int_2^t \frac{(s+2)}{(s+1)^2(\overline{n}(s)+2)} - \frac{9}{2} \sum_{k=2}^{\overline{n}(t)} \frac{1}{k^3(k+1)(k+2)^2},$$

which is finite. Hence, the hypothesis of Theorem 2.1 in [2] is not satisfied. As we have shown in Example 3.1, our result shows that this system oscillates.

Finally, the next remark shows the usefulness of Theorem 2.6 as an alternative to Theorem 2.5.

REMARK 4.2. In Example 3.2, from (3.3), we have $\alpha(t) = c$, where $1/5 < c < 1/2$. Thus,

$$\mu(s, t_0) = \exp\left\{-c \int_2^s \left(1 - \frac{1}{s}\right) ds\right\} = s^c e^{-cs},$$

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press

where c is a suitable positive constant. This implies that

$$\sum_{i=0}^{n-1} \prod_{j=0}^{i} \omega_j \int_{\tau_i}^{\tau_{i+1}} \mu(s, t_0)\, ds = \sum_{i=0}^{n-1} s^c e^{-cs} < \infty.$$

Thus, the hypothesis (2.13) does not hold, and so Theorem 2.5 cannot be applied to the system (3.2).

References

[1] R. P. Agarwal, F. Karakoc and A. Zafer, 'A survey on oscillation of impulsive ordinary differential equations', *Adv. Difference Equ.* **2010** (2010), Article no. 354841.

[2] S. D. Akgöl and A. Zafer, 'Leighton and Wong type oscillation theorems for impulsive differential equations', *Appl. Math. Lett.* **121** (2021), Article no. 107513.

[3] D. D. Bainov, Y. I. Domshlak and P. S. Simeonov, 'Sturmian comparison theory for impulsive differential inequalities and equations', *Arch. Math.* **67** (1996), 35–49.

[4] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, Advances in Mathematics for Applied Sciences, 28 (World Scientific, Singapore, 1995).

[5] C. Guo and Z. Xu, 'On the oscillation of second order linear impulsive differential equations', *Differ. Equ. Appl.* **2**(3) (2010), 319–330.

[6] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Oscillation Theory of Impulsive Differential Equations* (World Scientific, Singapore, 1989).

[7] Z. Luo and J. Shen, 'Oscillation of second order linear differential equations with impulses', *Appl. Math. Lett.* **20**(1) (2007), 75–81.

[8] A. Özbekler and A. Zafer, 'Principal and nonprincipal solutions of impulsive differential equations with applications', *Appl. Math. Comput.* **216** (2010), 1158–1168.

[9] J. Sugie, 'Interval oscillation criteria for second-order linear differential equations with impulsive effects', *J. Math. Anal. Appl.* **479**(1) (2019), 621–642.

[10] J. Sugie and K. Ishihara, 'Philos-type oscillation criteria for linear differential equations with impulsive effects', *J. Math. Anal. Appl.* **470**(2) (2019), 911–930.

[11] K. Wen, Y. Zeng, H. Peng and L. Huang, 'Philos-type oscillation criteria for second-order linear impulsive differential equation with damping', *Bound. Value Probl.* **2019** (2019), Article no. 111.

SIBEL DOĞRU AKGÖL, Department of Mathematics,
Atılım University, 06830 İncek, Ankara, Turkey
e-mail: sibel.dogruakgol@atilim.edu.tr

https://doi.org/10.1017/S0004972722000429 Published online by Cambridge University Press