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INVARIANCE OF TORSION AND THE BORSUK CONJECTURE

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1. Introduction. The following results of Whitehead and Wall are well-known applications of the algebraic K-theoretic functors K_0 and K_1 to basic homotopy questions in topology.

THEOREM 1 [**20**]. *If $f: X \to Y$ is a homotopy equivalence between compact* CW *complexes, then there is a torsion $\tau(f)$ in the algebraically-defined Whitehead group* Wh $\pi_1(Y)$ *which vanishes if and only if f is a simple homotopy equivalence.*

THEOREM 2 [**18**]. *If X is an arbitrary space which is finitely dominated (i.e., homotopically dominated by a compact polyhedron), then there is an obstruction $\sigma(X)$ in the algebraically-defined reduced projective class group $\tilde{K}_0\pi_1(X)$ which vanishes if and only if X is homotopy equivalent to some compact polyhedron.*

If we direct sum over components, then the above statements make good sense even if the spaces involved are not connected. Also the theory in [**18**] was explicitly worked out for spaces having the homotopy type of a CW complex. However it is well-known that finitely dominated spaces always have the homotopy type of a CW complex (for example, see [**14**]).

Applications of these theorems to the study of combinatorial and topological manifolds (also infinite-dimensional manifolds) is well-documented in the literature of the past 15 years. However these theorems are not sharp enough to settle the following two basic conjectures.

Invariance of Torsion [**15**]. Any topological homeomorphism of compact CW complexes is a simple homotopy equivalence.

Borsuk Conjecture [**3**]. Any compact metric ANR is homotopy equivalent to some compact polyhedron.

The first conjecture was settled affirmatively in [**4**] via a proof which ran through Q-manifold theory. Subsequently there have been other proofs in [**5**], [**9**] and [**13**]. The second conjecture was settled affirmatively in [**19**] via a proof which again ran through Q-manifold theory. Subsequently there have been other proofs in [**10**] and [**13**] for the case of

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finite-dimensional ANRs, but the only proof of the general case still runs through Q-manifold theory. In any case, the proof of either conjecture still remains a difficult program.

In the wake of such elaborate proofs of these conjectures it seems like a good idea to seek sharper versions of Theorems 1 and 2 so that the conjectures become straightforward corollaries. This is precisely what we have done in Theorems 1′ and 2′ below. These improvements are certainly not the best possible, but they are equal to the task without carrying along too much excess baggage.

For our improvement of Theorem 1 we will need the following definition which generalizes the notion of a homotopy equivalence. For convenience all spaces here and in the sequel will be metric. A diagram $X \xrightarrow{f} Y \xrightarrow{p} B$ is said to be a $p^{-1}(\epsilon)$-*equivalence* provided that there is a map $g: Y \to X$ and homotopies $\varphi_t: gf \simeq \mathrm{id}_X$, $\theta_t: fg \simeq \mathrm{id}_Y$ so that $pf\varphi_t: X \to B$ and $p\theta_t: Y \to B$ are ϵ-homotopies (i.e., the track of each point has diameter $< \epsilon$). For simplicity we will call f a $p^{-1}(\epsilon)$-equivalence (with inverse g), and refer to $f\varphi_t: X \to Y$ and $\theta_t: Y \to Y$ as $p^{-1}(\epsilon)$-homotopies. If we have $B = Y$ and $p = \mathrm{id}_Y$, then "$p^{-1}(\epsilon)$" is replaced by "ϵ."

THEOREM 1′. *Let B be a compact space which is locally simply connected. There exists an $\epsilon > 0$ so that if $X \xrightarrow{f} Y \xrightarrow{p} B$ is a $p^{-1}(\epsilon)$-equivalence, where X and Y are compact* CW *complexes, then the torsion $\tau(f)$ of Theorem* 1 *lies in the kernel of the induced homomorphism*

$$p_*: \mathrm{Wh}\pi_1(Y) \to \mathrm{Wh}\pi_1(B).$$

If $B = \{\text{point}\}$, then Theorem 1′ says nothing new. To see how Theorem 1′ implies the Invariance of Torsion note that if $f: X \to Y$ is a topological homeomorphism and $B = Y$, then f is clearly an ϵ-equivalence, for every $\epsilon > 0$. Taking $p = \mathrm{id}_Y$ in Theorem 1′ we conclude that $\tau(f) \in \mathrm{Ker}(p_*) = 0$, and so f must be a simple homotopy equivalence.

For our improvement of Theorem 2 we will need the following definition which generalizes the notion of a finite domination. A diagram $X \xrightarrow{p} B$ is said to be $p^{-1}(\epsilon)$-*finitely dominated* provided that there is a compact polyhedron K and maps $K \underset{u}{\overset{d}{\rightleftarrows}} X$ so that there is a $p^{-1}(\epsilon)$-homotopy $\theta_t: du \simeq \mathrm{id}_X$. For simplicity we say that X is $p^{-1}(\epsilon)$-finitely dominated, and if this is so for some ϵ, we say that X is finitely dominated.

THEOREM 2′. *Let B be a compact space which is locally simply connected. There exists an $\epsilon > 0$ so that if $X \xrightarrow{p} B$ is $p^{-1}(\epsilon)$-finitely dominated, then*

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the obstruction $\sigma(X)$ of Theorem 2 lies in the kernel of the induced homomorphism $p_*: \tilde{K}_0\pi_1(X) \to \tilde{K}_0\pi_1(B)$.

If $B = \{\text{point}\}$, then Theorem 2′ says nothing new. To see how Theorem 2′ implies the Borsuk Conjecture let X be a compact metric ANR. It is well-known (and easy to prove) that X is locally simply connected and ϵ-finitely dominated, for all $\epsilon > 0$. Thus in Theorem 2′ we may take $B = X$ and $p = \mathrm{id}_X$ to conclude that $\sigma(X) = 0$.

Finally we observe that the parameter space B which appears in Theorem 1′ above does not have to be a locally simply connected compactum. In fact it can be any compactum for which there exists a locally simply connected compactum B' and a map $q: B \to B'$ such that the induced homomorphism $q_*: \mathrm{Wh}\pi_1(B) \to \mathrm{Wh}\pi_1(B')$ is one-to-one. A similar statement is true for Theorem 2′, with the condition on $q: B \to B'$ being that $q_*: \tilde{K}_0\pi_1(B) \to \tilde{K}_0\pi_1(B')$ is one-to-one. For example, we only have to assume that B is homotopy equivalent to a compactum which is locally simply connected.

2. Proof of theorem 1′. Our proof of Theorem 1′ depends on the following result of Steve Ferry.

LEMMA 2.1 [**11**]. *Let Y be a compact* CW *complex. There exists an $\epsilon > 0$ so that if X is a compact* CW *complex and $f: X \to Y$ is an ϵ-equivalence, then f is a simple homotopy equivalence.*

Remarks on Proof. The proof of this result is given in [**11**] and it uses so-called Q-manifold theory, which is too difficult to describe in a short space.

A generalization of this result, which uses no Q-manifold theory, is given in [**6**]. A weaker version follows from the apparatus of [**13**], where ϵ depends on both X and Y. Lying at the heart of any of these proofs is the usual torus geometry and the difficult calculation $\mathrm{Wh}\pi_1(T^n) = 0$ of [**2**].

The following result enables us to replace the parameter space B in Theorem 1′ by a polyhedron.

LEMMA 2.2. *If Theorem 1′ is true for B any compact polyhedron, then it is also true for B any locally simply connected compactum.*

Proof. Consider a $p^{-1}(\epsilon)$-equivalence $X \xrightarrow{f} Y \xrightarrow{p} B$, where B is locally simply connected. Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be an open cover of B so that the U_i have small diameter and let $B' = N(U_i)$ be the nerve of $\mathcal{U}$ as described, for example, in [**8**, p. 172]. Then there is a map $q: B \to B'$ so that if v_i is the vertex of B' associated with U_i and $\mathrm{St}(v_i)$ is the open star of v_i, then

$$q^{-1}(\mathrm{St}(v_i)) \subset U_i.$$

If B_2' is the 2-skeleton of B', then it is easy to construct a map $r: B_2' \to B$ so that $r(v_i) \in U_i$ and $r(\Delta)$ has small diameter, for all simplices Δ in B_2'. Then there are induced homomorphisms $\pi_1(B) \underset{r_\#}{\overset{q_\#}{\rightleftarrows}} \pi_1(B')$.

ASSERTION. $r_\# q_\# = \mathrm{id}$.

Proof. If $\sigma: S^1 \to B$ is a loop, then $q\sigma: S^1 \to B'$ can be homotoped to $\tau: S^1 \to B_2'$ by homotoping it into the faces of the higher dimensional simplices. Then $r\tau: S^1 \to B$ is close to σ, and they are therefore homotopic.

By the functorality of Wh, q_*: Wh $\pi_1(B) \to$ Wh $\pi_1(B')$ is one to one. If ϵ is small enough, then we have $\tau(f) \in \mathrm{Ker}((qp)_*) = \mathrm{Ker}(p_*)$, and we are done.

Remark. The above result is also true for Theorem 2'.

For the proof of Theorem 1' let $X \xrightarrow{f} Y \xrightarrow{p} B$ be a $p^{-1}(\epsilon)$-equivalence. By Lemma 2.2 we may assume that B is a compact polyhedron. We will show that if ϵ is small enough (and dependent only on B), then $p_*(\tau(f)) = 0$. This will first be done for the somewhat easier case in which X is also a polyhedron, then we use this to obtain the general case.

Let $g: Y \to X$ be a $p^{-1}(\epsilon)$-inverse of f as described in the definition and assume that g is cellular (with respect to some subdivision of X). Now form the *mapping cylinder* $M(g)$, which is the space obtained from the disjoint union $(Y \times [0, 1]) \amalg X$ by identifying $(y, 1)$ with $g(y)$, for all $y \in Y$. We may write $M(g) = (Y \times [0, 1)) \cup X$ and identify Y with $Y \times \{0\}$, called the *top* of $M(g)$. The *base* of $M(g)$ is X, and there is a collapse to the base, $c: M(g) \to X$, defined by retracting each ray $(\{y\} \times [0, 1)) \cup \{g(y)\}$ to $\{g(y)\}$. It is well-known that $M(g)$ can be given a CW structure so that X and Y are subcomplexes [**7**, p. 6]. Note that the composition $M(g) \xrightarrow{c} X \xrightarrow{f} Y$ is still a $p^{-1}(\epsilon)$-equivalence and, of course, $\tau(fc) = \tau(f)$. By this construction we are therefore led to the following simpler case:

Y is a compact subcomplex of X and $f: X \to Y$ is a $p^{-1}(\epsilon)$-equivalence so that the inclusion $Y \hookrightarrow X$ is a $p^{-1}(\epsilon)$-inverse.

There is a standard recipe for homotoping a weak deformation retraction to a strong deformation retraction as given, for example, on page 31 of [**17**]. Applying this recipe to f we get a homotopy of f to a strong deformation retraction $f': X \to Y$. Moreover there is a $(pf')^{-1}(\epsilon')$-homotopy $f' \simeq \mathrm{id}_X$ rel Y, where ϵ' depends only on ϵ and it is small if ϵ is small. So all we have to do is prove that $p_*(\tau(f')) = 0$.

Now assume that p is cellular (with respect to some subdivision of B) and form the CW complex $M(p)$. Define $X_1 = X \cup_Y M(p)$, obtained by sewing X and $M(p)$ together along Y, and define $f_1: X_1 \to B$ to be the following composition:

$$f_1: X \cup M(p) \xrightarrow{f' \cup \mathrm{id}} M(p) \xrightarrow{c} B.$$

It is clear that $f' \cup \mathrm{id} : X \cup M(p) \to M(p)$ is a $c^{-1}(\epsilon')$-equivalence and $c: M(p) \to B$ is a δ-equivalence, for every $\delta > 0$. Then f_1 is an ϵ'-equivalence, and if ϵ' is small enough we have $\tau(f_1) = 0$ by Lemma 2.1. It follows from § 6 of [**7**] that $\tau(f_1) = p_*(\tau(f'))$, and we are done for the case in which X is a polyhedron. To deduce the general case from the polyhedral case it clearly suffices to establish the following

ASSERTION. *If X is a compact* CW *complex, then for every $\epsilon_1 > 0$ there is a compact polyhedron X_1 and an ϵ_1-equivalence $f_1: X_1 \to X$.*

Proof. Assuming the result to be true for X, all we have to do is prove that it is also true for $X \cup_\varphi D^n$, the space obtained by attaching an n-cell to X via an attaching map $\varphi: S^{n-1} \to X$. So let $f_1: X_1 \to X$ be an ϵ_1-equivalence of a compact polyhedron to X with inverse $g_1: X \to X_1$. We can write

$$X \cup_\varphi D^n = M(\varphi) \cup D^n,$$

which is the space obtained by sewing the mapping cylinder $M(\varphi)$ to D^n along S^{n-1}. Let $c: M(\varphi) \to X$ be the collapse to the base and assume that $g_1 c: M(\varphi) \to X_1$ can be approximated by embeddings. This assumption loses no generality because X_1 can be replaced by $X_1 \times D^k$, for any k-cell.

Let $h: M(\varphi) \to X_1$ be such an embedding for which $h|S^{n-1}: S^{n-1} \to X_1$ is PL (i.e., piecewise linear). Then the composition

$$X_1 \xrightarrow{f_1} X \hookrightarrow M(\varphi) \xrightarrow{h} h(M(\varphi))$$

is a weak deformation retraction to $h(M(\varphi))$, and as in [**17**, p. 31] we can homotop it to a retraction $r: X_1 \to h(M(\varphi))$ which is homotopic to id_{X_1} rel $h(M(\varphi))$ via a $f_1^{-1}(\epsilon_2)$-homotopy, where ϵ_2 is small if ϵ_1 is small. Let $X_2 = X_1 \cup_h D^n$ and note that the following composition is an ϵ_2-equivalence:

$$X_2 = X_1 \bigcup_h D^n \xrightarrow{r \cup \mathrm{id}} h(M(\varphi)) \bigcup_h D^n \xrightarrow{h^{-1} \cup \mathrm{id}} M(\varphi) \cup D^n$$
$$= X \bigcup_\varphi D^n.$$

3. Proof of theorem 2′. Our strategy is to reduce the proof of Theorem 2′ to an application of Theorem 1′. We are given a diagram $X \xrightarrow{p} B$ which is $p^{-1}(\epsilon)$-finitely dominated. We will show that if $\epsilon > 0$ is small enough (and dependent only on B), then $p_*(\sigma(X)) = 0$.

We begin the proof of Theorem 2′ by introducing some notation. Let K be a compact polyhedron which $p^{-1}(\epsilon)$-dominates X and choose maps $K \underset{u}{\overset{d}{\rightleftarrows}} X$ as in the definition. Let $e: K \to K$ be a PL map which is close to $ud: K \to K$. For n large let A_n be the space formed from the disjoint union

$$(*) \quad (K \times [0, 1]) \amalg (K \times [1, 2]) \amalg \ldots \amalg (K \times [n-1, n]),$$

where we identify (x, n) in $K \times [n-1, n]$ with $(e(x), 0)$ in $K \times [0, 1]$, and identify (x, i) in $K \times [i-1, i]$ with $(e(x), i)$ in $K \times [i, i+1]$, for $1 \leqq i \leqq n-1$. Then A is a compact polyhedron which is a circular chain of copies of the mapping cylinder $M(e)$.

Let S^1 be represented by the quotient space $[0, n]/\{0, n\}$, where notation is chosen so that the positive direction on $[0, n]$ corresponds to the counterclockwise direction on S^1. We will identify $[i-1, i]$ with its image in S^1. There is a natural projection of the disjoint union (*) onto $[0, n]$ which sends $K \times [i-1, i]$ to $[i-1, i]$. This factors through the appropriate identifications to give us a map $\varphi: A_n \to S^1$. Note that each $\varphi^{-1}([i-1, i])$ is a copy of $M(e)$ with $\varphi^{-1}(i-1)$ being the top of $\varphi^{-1}([i-1, i])$.

Let $p_1 = p \times \mathrm{id}: X \times S^1 \to B \times S^1$. It follows from the proof of Proposition 3.1 of [**12**] that there is a homotopy equivalence $r: A_n \to X \times S^1$ for which the composition

$$A_n \xrightarrow{r} X \times S^1 \xrightarrow{\text{proj}} S^1$$

is close to φ. This closeness depends only on the size of n, and the existence of r depends only on the fact that $du \simeq \mathrm{id}$. Since we are given a $p^{-1}(\epsilon)$-homotopy $du \simeq \mathrm{id}$, the proof of Proposition 3.1 of [**12**] also yields that r is a $p_1^{-1}(\epsilon_1)$-equivalence, where ϵ_1 depends only on ϵ and it is small if ϵ is small. (The construction given in [**12**] specifically applies to a circular chain of the mapping cylinders $M(ud)$. But such a chain is homotopy equivalent to A_n because ud is close to e.) Let $r_1: X \times S^1 \to A_n$ be a $p_1^{-1}(\epsilon_1)$-inverse of r.

In a similar manner let B_n be the space formed from the disjoint union (*) by identifying $(x, 0)$ in $K \times [0, 1]$ with $(e(x), n)$ in $K \times [n-1, n]$, and identifying (x, i) in $K \times [i, i+1]$ with $(e(x), i)$ in $K \times [i-1, i]$, for $1 \leqq i \leqq n-1$. Then B_n is a compact polyhedron which is also a circular chain of copies of $M(e)$. There is also a natural projection-induced map $\theta: B_n \to S^1$ so that each $\theta^{-1}([i-1, i])$ is a copy of $M(e)$

with $\theta^{-1}(i-1)$ being the base of $\theta^{-1}([i-1, i])$. As with the case of A_n there is a $p_1^{-1}(\epsilon_1)$-equivalence $s: B_n \to X \times S^1$, with inverse $s_1: X \times S^1 \to B_n$, such that the composition

$$B_n \xrightarrow{s} X \times S^1 \xrightarrow{\text{proj}} S^1$$

is close to θ.

Let $f = s_1 r: A_n \to B_n$ and let $p' = p_1 s: B_n \to B \times S^1$. It is easy to see that f is a $(p')^{-1}(4\epsilon_1)$-equivalence. For ϵ_1 small we conclude by Theorem 1′ that $(p')_*(\tau(f)) = 0$, thus

$$(p_1)_*(s_*(\tau(f))) = 0.$$

By [**1**, Chapter XII] we have functorial direct sum decompositions

$$\begin{aligned} \text{Wh } \pi_1(X \times S^1) &= \text{Wh } \pi_1(X) \oplus \tilde{K}_0\pi_1(X) \oplus \text{Nil Term},\\ \text{Wh } \pi_1(B \times S^1) &= \text{Wh } \pi_1(B) \oplus \tilde{K}_0\pi_1(B) \oplus \text{Nil Term}. \end{aligned}$$

Moreover the map

$$(p_1)_*: \text{Wh } \pi_1(X \times S^1) \to \text{Wh } \pi_1(B \times S^1)$$

is just

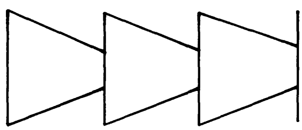
$$p_*: \text{Wh } \pi_1(X) \to \text{Wh } \pi_1(B)$$

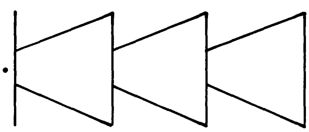
on the Wh $\pi_1(X)$ summand, and

$$p_*: \tilde{K}_0\pi_1(X) \to \tilde{K}_0\pi_1(B)$$

on the $\tilde{K}_0\pi_1(X)$ summand. Let τ be the component of $s_*(\tau(f))$ in $\tilde{K}_0\pi_1(X)$. Then we conclude that $p_*(\tau) = 0$ in $\tilde{K}_0\pi_1(B)$. So all we have to do is prove that $\tau = \sigma(X)$.

For this last part of the proof we will assume some familiarity with [**16**, § 4]. Let $\tilde{A}_n$ be the infinite-cyclic covering of A_n induced by $\varphi: A_n \to S^1$ from the standard covering map $R \to S^1$. Similarly let $\tilde{B}_n$ be the infinite-cyclic covering of B_n induced by $\theta: B_n \to S^1$ from $R \to S^1$. Then $f: A_n \to B_n$ lifts to a proper homotopy equivalence $\tilde{f}: \tilde{A}_n \to \tilde{B}_n$. Also $s: B_n \to X \times S^1$ lifts to a proper homotopy equivalence $\tilde{s}: \tilde{B}_n \to X \times R$. Here is a picture:

$\tilde{A}_n = \cdots$  $\cdots \to +\infty$

$\tilde{B}_n = \cdots$  $\cdots \to +\infty$

By using a mapping cylinder construction we may assume that $\tilde{f}$ is a proper strong deformation retraction of $\tilde{A}_n$ onto $\tilde{B}_n$. This reduction goes just like a corresponding step in the proof of Theorem 1′. Then we have, by [**16**, p. 14],

$$\sigma(\tilde{A}_n, \epsilon_+) = \sigma(\tilde{A}_n, \tilde{B}_n, \epsilon_+) + \sigma(\tilde{B}_n, \epsilon_+),$$

where these are the positive end invariants corresponding to $+\infty$. These are elements of $\tilde{K}_0\pi_1(\tilde{B}_n)$, which is only a slight departure from the notation of [**16**]. Clearly

$$\sigma(\tilde{A}_n, \epsilon_+) = (\tilde{f})_*(\sigma(\tilde{A}_n)) \quad \text{and} \quad \sigma(\tilde{B}_n, \epsilon_+) = 0.$$

Thus

$$\sigma(\tilde{A}_n, \tilde{B}_n, \epsilon_+) = (\tilde{f})_*(\sigma(\tilde{A}_n)) = \sigma(\tilde{B}_n),$$

because σ is an invariant of homotopy type.

There is a natural retraction ρ: Wh $\pi_1(B_n) \to \tilde{K}_0\pi_1(\tilde{B}_n)$ which corresponds to the natural retraction of Wh $\pi_1(X \times S^1)$ to $\tilde{K}_0\pi_1(X)$ which was mentioned above. In fact we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Wh}\pi_1(B_n) \arrow[r, "s_*"] \arrow[d, "\rho"'] & \mathrm{Wh}\pi_1(X \times S^1) \arrow[d] \\
\tilde{K}_0\pi_1(\tilde{B}_n) \arrow[r, "(\tilde{s})_*"] & \tilde{K}_0\pi_1(X \times R) \approx \tilde{K}_0\pi_1(X).
\end{tikzcd}$$

It follows from Proposition 4.7 of [**16**] that $\rho(\tau(f)) = \sigma(\tilde{B}_n)$. Thus τ (the component of $s_*(\tau(f))$ in $\tilde{K}_0\pi_1(X)$) is $(\tilde{s})_*(\sigma(X)) = \sigma(X)$, because σ is a homotopy-type invariant. This completes the proof.

References

1. H. Bass, *Algebraic K-theory* (W. A. Benjamin, New York, 1969).

2. H. Bass, A. Heller, and R. Swan, *The Whitehead group of a polynomial extension*, Publ. I.H.E.S. Paris *22* (1964), 67–79.

3. K. Borsuk, *Sur l'èlimination de phenomènes paradox aux en topologie générale*, Proc. Internat. Congr. Math., Vol. I, Amsterdam (1954), 197–208.

4. T. A. Chapman, *Topological invariance of Whitehead torsion*, Amer. J. Math. *96* (1974), 488–497.

5. ——— *Cell-like mappings*, Lecture Notes in Math. *482* (1973), 230–240.

6. ——— *Homotopy conditions which detect simple homotopy equivalences*, to appear in Pacific J. Math.

7. M. Cohen, *A course in simple-homotopy theory* (Springer-Verlag, New York, 1970).

8. J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).

9. R. D. Edwards, *The topological invariance of simple homotopy type for polyhedra*, preprint.

10. ——— *Siebenmann's variation of West's proof of the* ANR *theorem*, manuscript.

11. Steve Ferry, *The homeomorphism group of a compact Hilbert cube manifold is an* ANR, Annals of Math. *106* (1977), 101–119.

https://doi.org/10.4153/CJM-1980-103-5 Published online by Cambridge University Press

12. ——— *Homotopy, simple homotopy, and compacta*, preprint.

13. Frank Quinn, *Ends of maps, and applications*, preprint.

14. J. Milnor, *On spaces having the homotopy type of a* CW *complex*, Trans. A.M.S. *90* (1959), 272–280.

15. ——— *Whitehead torsion*, Bull. Amer. Math. Soc. *72* (1966), 358–426.

16. L. C. Siebenmann, *A total Whitehead torsion obstruction*, Comment. Math. Helv. *45* (1970), 1–48.

17. E. H. Spanier, *Algebraic topology* (McGraw-Hill, New York, 1966).

18. C. T. C. Wall, *Finiteness conditions for* CW *complexes*, Annals of Math. *81* (1965), 55–69.

19. J. E. West, *Mapping Hilbert cube manifolds to* ANR*s*, Annals of Math. *106* (1977), 1–18.

20. J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math. *72* (1950), 1–57.

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