SOLUBLE GROUPS WITH FINITE WIELANDT LENGTH

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1. Introduction and main results. The Wielandt subgroup $\omega(G)$ of a group G is defined to be the intersection of all normalizers of subnormal subgroups of G; the terms of the Wielandt series of G are defined, inductively, by putting $\omega_0(G)=1$ and $\omega_{n+1}(G)/\omega_n(G)=\omega(G/\omega_n(G))$. If, for some integer n, $\omega_n(G)=G$, then G is said to have *finite Wielandt length*; the Wielandt length of G being the minimal n such that $\omega_n(G)=G$.

It may well happen that the Wielandt subgroup of a group G is trivial; for instance, this is the case if G is the infinite dihedral group. On the other hand H. Wielandt showed in **[8]** that in a finite group G the socle (that is the subgroup generated by all minimal normal subgroups) is contained in $\omega(G)$. Thus any finite group has finite Wielandt length.

The relation between the Wielandt length and the derived and Fitting length in a finite soluble group was first investigated by A. Camina in **[2]**. Recently R. Bryce and J. Cossey **[1]** improved on Camina's results by obtaining best possible bounds for both the derived and the Fitting length of a finite soluble group in terms of its Wielandt length.

The aim of this paper is to extend these results to infinite groups. To do this we have to restrict ourselves to the class of groups with finite Wielandt length. Extending the notation of Bryce and Cossey **[1]**, we denote by $\mathfrak{W}_n$ the class of groups G such that $\omega_n(G)=G$, and set $\mathfrak{W}=\bigcup_{n\in\mathbb{N}}\mathfrak{W}_n$. Then we show that the bounds found by R. Bryce and J. Cossey hold for all soluble groups in the class $\mathfrak{W}$. To be more precise, we denote by $\mathfrak{A}$, $\mathfrak{N}$ respectively, the class of abelian groups and the class of nilpotent groups, and if $n\in\mathbb{N}$, by $\mathfrak{A}^n$ and $\mathfrak{N}^n$ the class of those groups admitting a normal series of length n, whose factors are abelian or, respectively, nilpotent. Furthermore, we denote by $\mathfrak{A}_2$ the class of all (abelian) groups of exponent 2. Our main results are then the following.

THEOREM 1. *Let G be a soluble group in $\mathfrak{W}_n$; then $G\in\mathfrak{N}^{n+1}$.*

THEOREM 2. *Let G be a soluble group in $\mathfrak{W}_n$.*
(i) *If $n\equiv 0 \pmod 3$ then $G\in\mathfrak{A}^{5n/3}$.*
(ii) *If $n\equiv 1 \pmod 3$ then $G\in\mathfrak{A}^{5(n-1)/3+2}$.*
(iii) *If $n\equiv 2 \pmod 3$ then $G\in\mathfrak{A}^{5(n-2)/3+3}\mathfrak{A}_2$.*

Our proof is modelled on Bryce and Cossey's approach to the finite case, and uses Robinson's classification **[4]** of soluble T-groups (which are the same as the soluble $\mathfrak{W}_1$-groups).

2. Proofs. If G is a group, we denote by $F(G)$ the Fitting subgroup of G; that is the subgroup generated by all normal nilpotent subgroups of G. If G is soluble, then $C_G(F(G))=Z(F(G))$ (see **[5**; 5.4.4**]**).

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We recall that, if A is a group, a power automorphism of A is an automorphism mapping every subgroup of A on itself. It is well known that Paut(A), the set of all power automorphisms of A, is an abelian normal subgroup of Aut(A); moreover if A is abelian, $\mathrm{Paut}(A) \leq Z(\mathrm{Aut}(A))$ (see Cooper [**3**] for the relevant facts about power automorphisms). We also recall that, according to a result of E. Schenkman [**7**], the intersection of all normalizers of subgroups of a group G (sometimes called the "norm" of G) is contained in $Z_2(G)$, the second centre of G. Now, in a nilpotent group, the Wielandt subgroup coincides with the norm; thus a nilpotent group in $\mathfrak{W}_n$ has nilpotency class at most $2n$. We have therefore the following

LEMMA 1. *Let G be a group in $\mathfrak{W}_n$, $n \in \mathbb{N}$. Then $F(G)$ is nilpotent of class at most $2n$.*

Proof. This is because every finite subset of $F(G)$ lies in a normal nilpotent subgroup of G, which by the observation above, has class at most $2n$. ■

Finally, we recall that a soluble $\mathfrak{W}_1$-group is metabelian (see [**4**] or also [**5**; 13.4.2]), and so $\omega(G)$ is metabelian for every soluble group G.

LEMMA 2. *Let G be a soluble group, $W = \omega(G)$ and $N = F(W)$. Then $W/N \leq Z(G/N)$.*

Proof. Let $A = W'$; then A is abelian and so $A \leq N$. Moreover, W acts by conjugation as a group of power automorphisms on A. Hence $[G, W] \leq C_G(A) \cap W = C_W(A)$. By Robinson [**4**; Lemma 2.2.2], $C_W(A) = N$. Thus $[G, W] \leq N$, as wanted. ■

Proof of Theorem 1. Proceed by induction on n. If $n = 1$, then G is metabelian. Let $n > 1$ and let $W = \omega(G)$. By inductive hypothesis, $G/W \in \mathfrak{N}^n$. Set $N = F(W)$ and $R/W = F(G/W)$. By Lemma 2, $W/N \leq Z(G/N)$ whence, in particular, R/N is nilpotent. Thus $G/N \in \mathfrak{N}^n$ and so $G \in \mathfrak{N}^{n+1}$. ■

In order to avoid repetitions in the next proofs, we state as a lemma a standard and well known argument (see [**6**]).

LEMMA 3. *Let Z be a central subgroup of the group A, such that A/Z is abelian.*
(a) *If A/Z is a π-group, for a set π of primes, then so is A'.*
(b) *If A/Z is periodic, then so is A'.*
(c) *If A/Z is divisible and periodic, then A is abelian.*

Proof. (a) Let $g, h \in A$. Then, for some π-number n, $g^n \in Z$ since A/Z is a π-group. Moreover A is nilpotent of class 2, whence $[g, h]^n = [g^n, h] = 1$. Thus A' is an abelian group which is generated by elements whose order is a π-number, and so A' is a π-group.
(b) Apply the same argument as in part (a).
(c) Let $g, h \in A$, $r \in \mathbb{N}$. Since A/Z is divisible, there exists $g_1 \in A$ such that $g_1^r = gz$, where $z \in Z$. Now

$$[g_1, h]^r = [g_1^r, h] = [gz, h] = [g, h].$$

Since A' is an abelian group, generated by the commutators $[g, h]$, $g, h \in A$, it follows, together with (b), that A' is a periodic divisible abelian group. By a well known property of divisible subgroups of abelian groups, we have $Z = A' \times B$, B a suitable subgroup of Z. Now, A/B is a nilpotent periodic group with a normal series $A/B \geq Z/B \geq 1$, whose factors are divisible; by a result of Cernikov (see [**6**, Theorem 9.23]) A/B is abelian. Hence $B \geq A'$ and so $A' = 1$. ∎

We observe that in (a) and (b) of the previous Lemma, it is enough to assume that A/Z is locally finite instead of abelian (see [**6**; Vol. I, p. 102]), but we do not need this.

As observed before, if G is a soluble group, $\omega(G)$ is a soluble T-group. Following Robinson [**4**], we split the class of soluble T-groups into four mutually disjoint subclasses, namely:

(1) the class of abelian groups;

(2) the class of periodic non abelian soluble T-groups;

(3) the class of T-groups of type I, that is all non abelian soluble T-groups G in which $C_G(G')$ is not periodic;

(4) the class of T-groups of type II, that is all soluble non periodic T-groups G in which $C_G(G')$ is periodic.

The following Lemma may be compared to Lemma 4.2 in [**1**].

Lemma 4. *Let Z be a central subgroup of the group H, and suppose that H/Z is a soluble T-group.*

(a) *If H/Z is of type II, then H is metabelian.*

(b) *If Z is a 2-group of finite exponent, H is metabelian.*

(c) *There exists $C \trianglelefteq H$, C metabelian, such that H/C is an elementary abelian 2-group and $[H, \mathrm{Aut}(H)] \leq C$.*

Proof. (a) Let H/Z be a T-group of type II, and put $D/Z = (H/Z)'$. By Robinson [**4**; 4.3.1], D/Z is a periodic abelian divisible group. Since $D' \leq Z$ and $Z \leq Z(H)$, it follows from Lemma 3(c) that D is abelian, and so H is metabelian.

(b) Let Z be a 2-group of finite exponent. If H/Z is abelian or of type II, H is metabelian. Let H/Z be a T-group of type I and set $D/Z = (H/Z)'$. Then (see Robinson [**4**; 3.1.1]) D/Z admits a 2-divisible subgroup T/Z of index at most 2, and $D/Z = \langle T/Z, xZ \rangle$, with $x^2 \in Z$. By the same argument used in the proof of Lemma 3(c), T' is 2-divisible; since $T' \leq Z$ and Z has finite exponent, we get $T' = 1$. Hence T is abelian; moreover $\langle x, Z \rangle$ is abelian and $[T, x] = 1$; in fact, if $g \in T$ and $\exp(Z) = 2^r$, then, by the 2-divisibility of T/Z, there exist $g_1 \in T$ and $z \in Z$ such that $g_1^{2^r} = gz$, whence

$$1 = [g_1, x]^{2^r} = [g_1^{2^r}, x] = [gz, x] = [g, x].$$

Therefore $D = \langle T, x \rangle$ is abelian, and so H is metabelian.

Finally, let H/Z be a non abelian periodic T-group; then H is periodic. Let $X = O_{2'}(H)$ be the maximal normal subgroup of H without elements of order 2. Since $X \cap Z = 1$, it is enough to prove that H/X is metabelian; thus we may assume $X = 1$. In this case $N/Z = F(H/Z)$ is a 2-group. By Robinson [**4**; 4.2.2], H/Z is a 2-group, whence (Robinson [**4**; 4.2.1]) either H/Z is a Dedekind 2-group or $D/Z = (H/Z)' = \langle T/Z, xZ \rangle$, where T/Z is divisible and $x^2 \in Z$. In the first case, H is a nilpotent 2-group of class at most 3 and therefore it is metabelian. In the second case, T is abelian by Lemma 3(c), and so, arguing as in the case in which H/Z is of type I, we have that D is also abelian, proving that H is metabelian.

(c) By part (a), we may assume that H/Z is of type I or it is periodic and non abelian (for otherwise we take $C=H$).

If H/Z is of type I, then if $C=F(H)$, C satisfies the required conditions (see Robinson [**4**; 3.1.1]).

Thus, let H/Z be a non abelian periodic T-group. Write $D/Z=(H/Z)'$ and, for every prime p, let D_p/Z be the p-component of D/Z; let further π be the set of those primes p, such that D_p is not abelian (possibly $\pi=\emptyset$). We observe that, by Robinson [**4**; *4.2.1 and 4.2.2], if D_2/Z is not trivial*, then it is the extention of a (possibly trivial) divisible 2-group T/Z, by an element xZ, with $x^2 \in Z$. By Lemma 3(c), T is abelian. As in the proof of part (b), $\langle Z, x\rangle$ is abelian and, if $g \in T$, there exist $g_1 \in T$, $z \in Z$ such that $g_1^2 = gz$; whence $[g,x]=[gz,x]=[g_1^2,x]=[g_1,x]^2=[g_1,x^2]=1$. Thus $D_2=\langle T,x\rangle$ is abelian, and so $2 \notin \pi$.

For each $p \in \pi$, let $C_p = C_H(D_p/Z)$ and $C = \bigcap_{p\in\pi} C_p$. Let $y \in H$; by Robinson [**4**; 4.2.2], y induces by conjugation on D_p/Z a power automorphism $a \mapsto a^y = a^\alpha$ (where α is a p-adic unit) of order a divisor of $p-1$. Since D_p is not abelian, there exist $g, h \in D_p$ such that $[g,h] \neq 1$. By Lemma 3(a), $[g,h]$ is a p-element. Moreover there exist $z, z' \in Z$ such that $g^y = g^\alpha z$ and $h^y = h^\alpha z'$; since $[g,h] \in Z$, we get

$$[g,h]=[g,h]^y=[g^y,h^y]=[g^\alpha z, h^\alpha z']=[g^\alpha, h^\alpha]=[g,h]^{\alpha^2}.$$

Because $[g,h]\neq 1$, we have therefore $\alpha^2 \equiv 1 \pmod p$ and so, since the order of H/C_p is a divisor of $p-1$, $y^2 \in C_p$. As H/C_p is a non trivial cyclic group, we get $|H/C_p|=2$. It is easy to see that C_p is characteristic in H if Z is characteristic, e.g. if $Z=Z(H)$. This can be assumed without loss of generality in part (c). So $[H, \mathrm{Aut}(H)] \leq C_p$. It follows that H/C is an elementary abelian 2-group, and $[H, \mathrm{Aut}(H)] \leq C$.

It remains to show that C is metabelian. Let q, r be prime numbers not belonging to π; then D_q and D_r are abelian. If $g \in D_q$ and $h \in D_r$ then, for some $u, v \in \mathbb{N}$, we have $1=[g^{q^u},h]=[g,h]^{q^u}$ and $1=[g,h^{r^v}]=[g,h]^{r^v}$. Thus, if $q \neq r$, $[D_q, D_r]=1$. Therefore $D_{\pi'}=\langle D_q \mid q \notin \pi\rangle$ is an abelian group, and, clearly, $D_{\pi'} \leq C$. Let $D_\pi = \langle D_p \mid p \in \pi\rangle$. Now, $C/D_{\pi'}$ is a nilpotent T-group such that $(C/D_{\pi'})' \leq D/D_{\pi'}$ does not admit 2-elements. Hence $C/D_{\pi'}$ is abelian, and so C is metabelian, concluding the proof of the Lemma. ■

If G is a group, and r a prime, we denote by $O_{r'}(G)$ the subgroup of G generated by all normal r'-subgroups of G (that is periodic subgroups all of whose elements have order an r' number). Then $O_{r'}(G)$ is a normal r'-subgroup of G and $O_{r'}(G/O_{r'}(G))=1$. Furthermore, if $\mathbb{P}$ is the set of all prime numbers, then $\bigcap_{r\in\mathbb{P}} O_{r'}(G)=1$.

LEMMA 5. (a) *If G is a soluble $\mathfrak{W}_2$-group, then $G \in \mathfrak{A}^3\mathfrak{A}_2$.*
(b) *If G is a soluble $\mathfrak{W}_3$-group, then $G \in \mathfrak{A}^5$.*

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Proof. Since the classes $\mathfrak{A}^3\mathfrak{A}_2$ and $\mathfrak{A}^5$ are both quotient and residually closed, we may assume, by what was observed above $O_{p'}(G)=1$ for a suitable prime p. Let $W=\omega(G)$ and $N=F(W)$. Since $\mathfrak{W}_1\subseteq\mathfrak{A}^2$ and $\mathfrak{W}_2\subseteq\mathfrak{A}^4$, we are done in both cases if W is abelian. Let us therefore assume that W is not abelian. Then either N is not periodic (and so W is a T-group of type I) or N is a p-group. Let $K=\omega_2(G)$ (whence $K=G$ in case (a)).

Let W be a T-group of type I; then N is abelian and, by Lemma 2, $W/N\leqslant Z(G/N)$. In particular, by Lemma 4(b), K/N is metabelian, since (see Robinson [**4**; 3.1.1]) $|W/N|=2$. Thus, $G^{(3)}=1$ if $G\in\mathfrak{W}_2$ and $G^{(5)}=1$ if $G\in\mathfrak{W}_3$.

Suppose now N is a p-group. If p is odd, N is abelian by Robinson [**4**; 4.2.1] and, by Lemma 2, $W/N\leq Z(G/N)$. By Lemma 4(c), there exists $C\trianglelefteq K$, with K/C an elementary abelian 2-group, C/N metabelian and $[G,K]\leq C$. If $G\in\mathfrak{W}_2$ this yields at once $G\in\mathfrak{A}^3\mathfrak{A}_2$. If $G\in\mathfrak{W}_3$ then, by Lemma 4(b), G/C is metabelian, and so $G\in\mathfrak{A}^5$.

It remains to consider the case $p=2$. In this case, if N is abelian we argue as in the previous case. Otherwise, N is a non abelian Dedekind 2-group, and this implies (see [**4**]) $N=W$. So K/N is a T-group. Let $N=A\times Q$, where Q is a quaternion group of order 8, and A an elementary abelian 2-group. Write $X=Q'\times A$. Then X is a normal abelian subgroup of K, and N/X is elementary abelian of order 4.

Let $M/X=C_{K/X}(N/X)$; then K/M is isomorphic to a subgroup of $\mathrm{Aut}(N/X)\cong S_3$. Let L/M be the inverse image of A_3 in the embedding $K/M\to S_3$. Thus L is a characteristic subgroup of index at most 2 in K. We show that $L^{(3)}=1$. To do this, it is enough to show that L'' is contained in X.

Now, M/N is a T-group so, by Lemma 4(b), M/X is metabelian. If $L=M$ we are done. Otherwise, $|L/M|=3$ and N/X is a minimal normal subgroup of L/X. Set $R/X=(L/X')$. Minimality of N/X implies that either $R\cap N=X$ or $R\geqslant N$. In the first case, since L/N is metabelian, $L''\leq R\cap N=X$, as wanted. Assume therefore $R\geq N$. Now, certainly $R\leq M$ and $(R/X)'\leq N/X$. Thus R/X is a nilpotent group of class at most two. If $g,h\in R/X$, then $[g,h]\in N/X$ and, since N/X is central and has exponent 2, $1=[g,h]^2=[g^2,h]$. Therefore, if $D/X=(R/X)^2$, we get $D/X\leq Z(R/X)$. Now, $(M/X)'$ is abelian and, clearly, it is contained in R/X. So, if we write $V/X=(D/X)(M/X)'$, we have that V/X is an abelian normal subgroup of L/X. Moreover, R/V is an elementary abelian 2-group. Now, L/V, which is a normal section of K/N, (observe that minimality of N/X yields $D\geq N$, otherwise R/X is abelian and we are done), is a T-group. Therefore, L acts by conjugation as a group of power automorphisms on R/V. Since R/V is abelian of exponent 2, it follows that R/V is contained in $Z(L/V)$. Hence L/V is a nilpotent T-group and thus is a Dedekind group. Since $|L/M|=3$ and M/V is abelian, we conclude that L/V is abelian. Using the fact that V/X is abelian, we now get $L''\leqslant X$, and so $L^{(3)}=1$, as required.

Now, if $G\in\mathfrak{W}_2$, $G=K$ and thus, since $|G:L|\leqslant 2$, $G\in\mathfrak{A}^3\mathfrak{A}_2$. If $G\in\mathfrak{W}_3$, then, because L is a characteristic subgroup of index at most 2 in K, we have $[G,K]\leq L$; by Lemma 4(b), $G''\leq L$ and so $G^{(5)}=1$. ■

Proof of Theorem 2. We proceed by induction of n. If $n=1$, 2, 3, the result follows from Robinson [4], and Lemma 5. If $n>3$, apply Lemma 5 and the fact that, if $A=\omega_3(G)$, then $A\in\mathfrak{W}_3$ (and so $A^{(5)}=1$) and $G/A\in\mathfrak{W}_{n-3}$. ■

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