




COMPOSITIO MATHEMATICA

On straightening for Segal spaces

Joost Nuiten 

Compositio Math. **160** (2024), 586–656.

[doi:10.1112/S0010437X23007674](https://doi.org/10.1112/S0010437X23007674)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865





On straightening for Segal spaces

Joost Nuiten

ABSTRACT

The straightening–unstraightening correspondence of Grothendieck–Lurie provides an equivalence between cocartesian fibrations between $(\infty, 1)$ -categories and diagrams of $(\infty, 1)$ -categories. We provide an alternative proof of this correspondence, as well as an extension of straightening–unstraightening to all higher categorical dimensions. This is based on an explicit combinatorial result relating two types of fibrations between double categories, which can be applied inductively to construct the straightening of a cocartesian fibration between higher categories.

Contents

1	Introduction	587
	Outline	590
	Conventions and notation	591
2	Fibrations of double categories	591
	2.1 Recollections on double categories	592
	2.2 Fibrations of double categories	594
	2.3 The reflection of a (left, cart)-fibration	598
3	Reflecting fibrations of double categories	602
	3.1 Marked bisimplicial spaces	603
	3.2 Construction of the functors	605
	3.3 Proof of Theorem 3.1	607
4	Recollections on higher categories	613
	4.1 Higher categories via Segal spaces	613
	4.2 Point-set models	616
	4.3 Copresheaves	619
	4.4 More on representability	625
	4.5 Tensoring	628
5	Cocartesian fibrations of higher categories	629
	5.1 Pre-cocartesian fibrations	629
	5.2 d -cocartesian fibrations	632
	5.3 The $(d + 2)$ -category of d -cocartesian fibrations	635
6	Straightening and unstraightening	638
	6.1 Straightening and unstraightening: 1-categorical version	638

Received 9 April 2022, accepted in final form 1 September 2023, published online 23 February 2024.

2020 Mathematics Subject Classification 18N65, 18N10, 18D30 (primary).

Keywords: straightening, cocartesian fibrations, higher categories.

© 2024 The Author(s). This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited. *Compositio Mathematica* is

© Foundation Compositio Mathematica.

6.2 Straightening and unstraightening: uniqueness and further properties .	642
6.3 Straightening and unstraightening: higher functoriality	645
Acknowledgements	649
Conflicts of Interest	650
Appendix A. Technicalities on categories of fibrant objects	650
References	654

1. Introduction

The Grothendieck construction, or *unstraightening*, is a standard procedure in category theory that associates to a diagram of categories $F: \mathcal{C} \rightarrow \text{Cat}$ a map of categories $\pi: \text{Un}(F) \rightarrow \mathcal{C}$ whose fibers are precisely the values of the diagram F . This maneuver does not result in a loss of information: the unstraightening functor

$$\text{Un}: \text{Fun}(\mathcal{C}, \text{Cat}) \longrightarrow \text{Cat}/\mathcal{C}$$

is an embedding, whose image is the subcategory of *cocartesian fibrations* over \mathcal{C} and maps between them that preserve cocartesian arrows [Gro63].

The practical consequences of this result are twofold. On the one hand, the Grothendieck construction also provides an equivalence between cocartesian fibrations over \mathcal{C} and pseudo-functors $F: \mathcal{C} \rightarrow \text{Cat}$. From this point of view, cocartesian fibrations provide a convenient way to encode coherent diagrams of categories, which can be rectified to strict diagrams via the inverse of unstraightening, aptly called *straightening*. Consequently, the language of fibrations starts to play an essential role in the homotopy coherent setting of $(\infty, 1)$ -categories, where a version of straightening and unstraightening is available as well [Lur09a] (see also [RV18]). Indeed, many homotopy coherent diagrams of $(\infty, 1)$ -categories are most naturally described in terms of cocartesian fibrations.

On the other hand, the unstraightening of a diagram $F: \mathcal{C} \rightarrow \text{Cat}$ is convenient to describe ‘lax constructions’ with categories, stemming from the fact that $\text{Un}(F)$ is the oplax colimit of F (see [GHN17]). For example, a lax natural transformation between two diagrams F and G can be described by a map $\text{Un}(F) \rightarrow \text{Un}(G)$ over \mathcal{C} that need not preserve cocartesian arrows (see, e.g., [GHL20, HHLN23b] for a discussion in the ∞ -categorical setting).

The purpose of this text is to establish a version of straightening and unstraightening for (∞, d) -categories. Recall that at the moment, the theory of (∞, d) -categories is only in the first stages of its development, in particular when compared with $(\infty, 1)$ -category theory; it is already not straightforward to compare the various different homotopy-theoretic models for (∞, d) -categories (see, e.g., [Sim12] for a textbook account and [BS21] for a more recent state of affairs). Nonetheless, (∞, d) -categories have been of interest in various areas of mathematics, also outside of the realm of category theory itself. For example, higher categories play a role in (derived) algebraic geometry via the six functor formalism ($d = 2$) [GR17, Mac22] and other constructions involving correspondences [CHS21], whereas in topology they arise in factorization homology [AFR18, Sch14] and notably in topological field theories [CS19, Lur10].

Given the central place of the (un)straightening correspondence in $(\infty, 1)$ -category theory, it seems fair to say that the lack of a higher-categorical version of this correspondence has been one of the main problems in the theory of (∞, d) -categories (henceforth, simply referred to as *d-categories*). For instance, the unstraightening procedure could be useful in the study of lax natural transformations between diagrams of d -categories and in the study of (lax) limits and colimits. Our main results establish such a procedure and show that it is essentially unique.

To this end, we define for each $(d + 1)$ -category \mathcal{C} two certain (non-full) subcategories

$$\text{Cocart}_d(\mathcal{C}) \subseteq \text{Cat}_{d+1}/\mathcal{C}, \quad \text{Cart}_d(\mathcal{C}) \subseteq \text{Cat}_{d+1}/\mathcal{C}$$

whose objects we refer to as *d-cocartesian fibrations* and *d-cartesian fibrations*; the two notions are equivalent under reversing the directions of morphisms in every categorical dimension. The fibers of a *d-cocartesian fibration* are *d-categories*; in particular, a 0-cocartesian fibration between 1-categories is simply a left fibration between 1-categories [Lur09a]. Furthermore, a 1-cocartesian fibration between 1-categories, viewed as 2-categories with only invertible 2-morphisms, is simply a cocartesian fibration in the usual sense.

More generally, a map of $(d + 1)$ -categories $\pi: \mathcal{D} \rightarrow \mathcal{C}$ is roughly said to be a *d-cocartesian fibration* if it induces $(d - 1)$ -cartesian fibrations on mapping *d-categories* and if every 1-morphism in \mathcal{C} admits enough cocartesian lifts (in an enriched sense; see § 5 for more details). For $d = 2$, the notion of a 2-cocartesian fibration between $(2, 2)$ -categories has also been considered (in a different variance) by Hermida [Her99] (see also [Bak11, Buc14]). More recently, 2-cocartesian fibrations between $(\infty, 2)$ -categories have been described in terms of inner cocartesian 2-fibrations between scaled simplicial sets [GHL21].

Our main result is then the following.

THEOREM A (Theorem 6.1). *For each $(d + 1)$ -category \mathcal{C} , there exist equivalences of 1-categories (here, \mathbf{c}_1 takes the 1-category underlying a $(d + 1)$ -category)*

$$\begin{aligned} \text{Un}: \mathbf{c}_1\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) &\xrightarrow[\sim]{\sim} \text{Cocart}_d(\mathcal{C}): \text{St} \\ \text{Un}: \mathbf{c}_1\text{Fun}_{d+1}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_d) &\xrightarrow[\sim]{\sim} \text{Cart}_d(\mathcal{C}): \text{St} \end{aligned}$$

between *d-cocartesian fibrations over \mathcal{C} and $(d + 1)$ -functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$ to the $(d + 1)$ -category of *d-categories*, respectively, between *d-cartesian fibrations and $(d + 1)$ -functors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}_d$.**

Our proof of Theorem A only relies on the straightening equivalence for left (i.e. 0-cocartesian) fibrations, which we recall in § 4.3. In particular, it provides an independent proof of the straightening–unstraightening equivalence for cocartesian fibrations of Lurie [Lur09a]. In fact, we will use an inductive argument to deduce Theorem A from a slight generalization of the 1-categorical straightening equivalence: Theorem B below.

To facilitate this inductive argument, we make use of the model for $(d + 1)$ -categories given by iterated complete Segal spaces [BS21]. Recall that these are certain types of $(d + 1)$ -fold simplicial spaces

$$\mathcal{C}: \Delta^{\times d+1, \text{op}} \rightarrow \mathcal{S}.$$

Here the *d-fold simplicial space* $\mathcal{C}(0) = \mathcal{C}(0, -)$ is constant and corresponds to the space of objects of \mathcal{C} , whereas $\mathcal{C}(1)$ is itself an iterated complete Segal space, modeling the *d-category* of arrows in \mathcal{C} .

This Segal space model has the benefit that a diagram of *d-categories* indexed by a $(d + 1)$ -category \mathcal{C} can be encoded explicitly by a map of $(d + 1)$ -fold simplicial spaces $X \rightarrow \mathcal{C}$ satisfying the following two conditions [Boa18, Ras21]:

- (1) $X(0)$ is a *d-category*;
- (2) the Segal maps $X(n) \rightarrow \mathcal{C}(n) \times_{\mathcal{C}(\{0\})} X(\{0\})$ are equivalences of *d-categories*.

We will refer to such maps as *Segal copresheaves*, because they should be viewed as encoding a homotopy coherent action of the $(d + 1)$ -category \mathcal{C} on the *d-category* $X(0)$. In particular, fixing the last *d* simplicial coordinates yields a left fibration between 1-categories. Work of

Boavida [Boa18] gives a way to straighten each of these left fibrations, resulting in an equivalence between the 1-category of Segal copresheaves over \mathcal{C} and the 1-category of $(d + 1)$ -functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$ (see also [Ras21] and §4.3). This equivalence can be thought of as a rectification procedure relating an external (Segal-style) definition of copresheaves to their internal definition as functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$ (cf., in particular, Theorem 4.32).

WARNING. Let us emphasize that one should *not* think of a Segal copresheaf $X \rightarrow \mathcal{C}$ as some sort of cocartesian fibration: its domain is typically not a $(d + 1)$ -category (unless $d = 0$) and the notion of a Segal copresheaf is therefore not defined internal to the theory of $(d + 1)$ -categories. In particular, note that the fibers of a Segal copresheaf are constant in the *first* simplicial direction, whereas the fibers of a d -cocartesian fibration between $(d + 1)$ -categories are constant in the *last* simplicial direction. Unfortunately, the terminology employed in [Boa18, Ras21] seems to suggest otherwise.

The straightening–unstraightening equivalence from Theorem A is then given by a d -step combinatorial procedure that passes from Segal copresheaves to d -cocartesian fibrations. In each step, we modify the behaviour of the map of $(d + 1)$ -simplicial spaces $X \rightarrow \mathcal{C}$ only in two consecutive simplicial directions, using the following result.

THEOREM B (Theorem 3.1). *Let \mathbb{C} be a double category, i.e. a bisimplicial space satisfying the complete Segal conditions in the two simplicial directions (referred to as horizontal and vertical). Then there is an equivalence of categories*

$$\Psi^\perp : \mathbf{Fib}^{\text{left, cart}}(\mathbb{C}) \xrightleftharpoons{\sim} \mathbf{Fib}^{\text{cocart, right}}(\mathbb{C}) : \Psi^\top$$

between maps of double categories $p : \mathbb{D} \rightarrow \mathbb{C}$ of the following two types:

- (1) p is a left fibration in the horizontal and a cartesian fibration in the vertical direction;
- (2) p is a cocartesian fibration in the horizontal and a right fibration in the vertical direction.

Furthermore, the fibers of $\mathbb{D} \rightarrow \mathbb{C}$ and $\Psi^\perp(\mathbb{D}) \rightarrow \mathbb{C}$ differ simply by exchanging the horizontal and vertical directions.

The definition of the equivalence Ψ^\perp uses rather simple combinatorics, inspired by the classical Grothendieck construction. Indeed, recall that for $F : \mathcal{C} \rightarrow \mathbf{Cat}$, an object in the Grothendieck construction is a tuple (c, x) with $c \in \mathcal{C}$ and $x \in F(c)$, and a morphism is given by the composite of a cocartesian (‘horizontal’) arrow $f : (c, x) \rightarrow (c', f_!x)$, followed by a ‘vertical’ arrow in the fiber $F(c')$. Likewise, $\Psi^\perp(\mathbb{D})$ is a double category with the same objects as \mathbb{D} , but with a horizontal arrow consisting of a horizontal arrow in \mathbb{D} followed by a fiberwise vertical arrow in \mathbb{D} .

Theorem A now follows from an inductive application of Theorem B. The lowest dimensional case, the straightening–unstraightening correspondence between 2-functors $\mathcal{C} \rightarrow \mathbf{Cat}_1$ and 1-cocartesian fibrations over a 2-category \mathcal{C} , is simply a special case of Theorem B: when the double category \mathbb{C} is a 2-category, the domain and codomain of Ψ^\perp consist precisely of Segal copresheaves and 1-cocartesian fibrations, respectively. In particular, when applied to a diagram of $(1, 1)$ -categories $F : \mathcal{C} \rightarrow \mathbf{Cat}_{(1,1)}$ indexed by a $(1, 1)$ -category \mathcal{C} , one obtains the following two-step process. We first associate to F a Segal copresheaf over \mathcal{C} , which corresponds informally to passing from strict to pseudo-functors, after which the functor Ψ^\perp produces the classical Grothendieck construction.

Note that Theorem A is not complete: both $(d + 1)$ -functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$ and d -cocartesian fibrations over \mathcal{C} can naturally be organized into $(d + 1)$ -categories, rather than 1-categories.

In the latter case, the $(d + 1)$ -category $\mathbf{Cocart}_d(\mathcal{C})$ of d -cocartesian fibrations can be realized as a subcategory of the $(d + 2)$ -category $\mathbf{Cat}_{d+1}/\mathcal{C}$. Both of these (large) $(d + 1)$ -categories furthermore depend functorially on \mathcal{C} by restriction and base change, respectively; more precisely, they can each be organized into functors of $(d + 2)$ -categories $\mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$. We then prove the following refinement of Theorem A.

THEOREM C (Theorem 6.20). *There is a unique natural equivalence of $(d + 2)$ -functors $\mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$ given at a $(d + 1)$ -category \mathcal{C} by an equivalence of $(d + 1)$ -categories*

$$\text{Un}: \text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \xrightarrow{\sim} \mathbf{Cocart}_d(\mathcal{C}): \text{St}.$$

The statement of Theorem C hides the following subtlety: it is not, in fact, so obvious how to write down the $(d + 2)$ -functor

$$\mathbf{Cocart}_d(-): \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}; \quad \mathcal{C} \mapsto \mathbf{Cocart}_d(\mathcal{C}).$$

For example, note that the slice $(d + 2)$ -categories $\mathbf{Cat}_{d+1}/\mathcal{C}$ do not depend on \mathcal{C} in a 2-functorial way in general. As one may expect, it will instead be more convenient to organize all $(d + 1)$ -categories of d -cocartesian fibrations into a $(d + 1)$ -cartesian fibration $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$. The above functor then arises as its straightening, using Theorem A (one categorical dimension higher).

Variants of Theorems A and C have appeared in the literature before. The straightening of 0-cocartesian fibrations nowadays has various proofs [Boa18, HM15, Lur09a]. Theorem A for 1-cocartesian fibrations over 2-categories is the content of Lurie’s scaled straightening and unstraightening theorem [GHL20, Lur09b]. Work of Buckley [Buc14] describes the Grothendieck construction for 2-cocartesian fibrations between $(2, 2)$ -categories as an equivalence of 3-categories. Gaitsgory and Rozenblyum prove a strengthening of Theorem C for 2-cocartesian fibrations between $(\infty, 2)$ -categories, which also allows for lax natural transformations [GR17, Chapter 11, Theorem 1.1.8]; this seems to rely on some unproven statements about $(\infty, 2)$ -categories appearing in Chapter 10 of [GR17], notably about the (conjectured) model for $(\infty, 2)$ -categories in terms of lax squares. Ayala *et al.* have recently deduced this lax refinement of straightening for $(\infty, 2)$ -categories from our results and have also established a version for locally 2-cocartesian fibrations [AMR19]. Finally, let us mention that a version of Theorem C has been hypothesized in [Mac22], where it is used to prove a universal property for the $(\infty, 2)$ -category of correspondences in an $(\infty, 1)$ -category with fiber products.

Outline

Let us give a brief overview of the contents of this paper. Sections 2 and 3 treat Theorem B and contain our main combinatorial results: §2 discusses the fibrations of double categories appearing in Theorem B, and defines the functor Ψ^\perp on objects (§2.3). In §3, we then show that Ψ^\perp extends to an equivalence of categories (Theorem 3.1). This essentially already gives a proof of straightening and unstraightening for cocartesian fibrations of 1-categories (see Corollary 4.35).

Section 4 gives some background material on higher categories, mainly using the model of iterated complete Segal spaces. To produce examples of higher categories (e.g. of the $(d + 1)$ -category \mathbf{Cat}_d of d -categories), we also recall the comparison to the point-set model of (strict) enriched categories (§4.2). Section 4.3 reviews the results of Boavida [Boa18], relating Segal copresheaves over \mathcal{C} to functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$. For our purposes, we will need to slightly rephrase

the results from [Boa18] to make them $(\infty, 1)$ -functorial in the base \mathcal{C} ; this requires some technical results about cartesian fibrations in the setting of categories of fibrant objects, discussed in Appendix A.

In § 5, we give the definitions of d -cocartesian fibrations and d -cartesian fibrations. The main result here (Proposition 5.25) asserts that d -cocartesian fibrations can themselves be organized into a $(d + 1)$ -cartesian fibration $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$.

Finally, § 6 contains the proofs of our main results, Theorems A and C. We first prove a version of straightening–unstraightening at the level of 1-categories (Theorem 6.1). Using this, it follows that there is a $(d + 2)$ -functor $\mathbf{Cocart}_d: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$ sending each $(d + 1)$ -category \mathcal{C} to the large $(d + 1)$ -category of d -cocartesian fibrations over \mathcal{C} . We then show that this functor is representable by \mathbf{Cat}_d (Theorem 6.20), providing the existence of the natural equivalence of Theorem C. The uniqueness of this natural equivalence follows from the results in § 6.2.

Conventions and notation

We phrase our results as much as possible in the language of $(\infty, 1)$ -categories (with the notable exception of §§ 4.2 and 4.3, where we need to recourse to point-set models). This is mostly a matter of terminology: effectively, we only start from the $(\infty, 1)$ -category \mathcal{S} of spaces instead of the Kan–Quillen model structure on simplicial sets to avoid having to impose fibrancy conditions at every step. In particular, we avoid any ∞ -categorical machinery that (tacitly) relies on the straightening results of Lurie.

Throughout the text, we will omit all prefixes ‘ ∞ -’. For instance, we abbreviate $(\infty, 1)$ -categories simply to categories and (∞, d) -categories to d -categories, referring to ordinary categories instead as $(1, 1)$ -categories. We write $\mathfrak{c}_k \mathcal{C}$ for the k -core of a higher category, i.e. the k -category obtained by removing non-invertible morphisms in dimensions $> k$.

A *subspace* of a space is a (-1) -truncated map $S \hookrightarrow T$, i.e. an inclusion of path components. Likewise, by a *subcategory* of a d -category we will mean a (-1) -truncated map $i: \mathcal{C} \hookrightarrow \mathcal{D}$; inductively, this means that i induces a subspace inclusion on spaces of objects and a subcategory inclusion on mapping $(d - 1)$ -categories. (This is, in fact, closer to the classical notion of a *replete* subcategory.) In particular, this means that specifying a subcategory of a d -category comes down to specifying a subcategory of its homotopy (d, d) -category.

Categories and d -categories will typically be denoted by \mathcal{C} , whereas d -fold categories (see § 2) will be denoted \mathbb{C} . We have furthermore tried to differentiate these notationally from specific point-set models by enriched categories or relative categories: for example, a $(d + 1)$ -category \mathcal{C} can be presented by an ordinary category \mathbb{C} strictly enriched in the $(1, 1)$ -category \mathbf{Cat}_d of fibrant objects in the d -categorical model structure (cf. § 4.2). In particular, the $(d + 1)$ -category \mathbf{Cat}_d of d -categories is modeled by \mathbf{Cat}_d with its canonical self-enrichment (Definition 4.22); we write $\mathbf{Cat}_d = \mathfrak{c}_1 \mathbf{Cat}_d$ for its 1-core. Likewise, we will denote by \mathbf{CAT}_d and \mathbf{CAT}_d the (higher) categories of large d -categories.

2. Fibrations of double categories

In this section, we will introduce a certain type of fibration between double categories, behaving like a left fibration in one direction and like a cartesian fibration in the other. We will show that such a fibration has a ‘*reflection*’, which is a cocartesian fibration in one direction and a right fibration in the other.

2.1 Recollections on double categories

Recall that a simplicial space $X: \Delta^{op} \rightarrow \mathcal{S}$ is said to be a *complete Segal space* if it satisfies the following two conditions [Rez01].

- (1) *Segal condition*: for every $n \geq 2$, the natural map

$$X(n) \longrightarrow X(\{0 \leq 1\}) \times_{X(\{1\})} X(\{1 \leq \dots \leq n\})$$

is an equivalence. Equivalently, $X(n) \simeq X(1) \times_{X(0)} \dots \times_{X(0)} X(1)$ via the inclusions of the intervals $\{i \leq i + 1\}$ into $[n]$.

- (2) *Completeness*: consider the simplicial set $H = \Delta[3]/\sim$ obtained by collapsing the 1-simplex $\{0 \leq 2\}$ to a point and collapsing the 1-simplex $\{1 \leq 3\}$ to a (different) point. Then the map $X(0) \rightarrow \text{Map}_{\mathcal{S}}(H, X)$ is an equivalence.

Recall that complete Segal spaces are a model for the homotopy theory of $((\infty, 1)$ -)categories; we will therefore also refer to complete Segal spaces simply as *(1-)categories*. We will be interested in analogues of complete Segal spaces for diagrams of spaces indexed by multiple copies of the simplex category.

Notation 2.1. For any $k \geq 0$, we will write $\vec{n}_k = ([n_1], \dots, [n_k]) \in \Delta^{\times k}$ for a generic (but fixed) k -tuple of objects in Δ . Furthermore, we will abbreviate $\vec{0}_k = ([0], \dots, [0])$. For example, if $X: \Delta^{\times d} \rightarrow \mathcal{S}$ is a d -fold simplicial space, then

$$X(\vec{n}_k, -, \vec{0}_{d-k-1}): \Delta^{op} \longrightarrow \mathcal{S}$$

denotes the simplicial space obtained by fixing the first k inputs to be \vec{n}_k and the last $d - k - 1$ inputs to be 0, varying only the $(k + 1)$ th input. Finally, we will write $X(n): \Delta^{\times d-1} \rightarrow \mathcal{S}$ for the $(d - 1)$ -fold simplicial space given by $X(n, -, \dots, -)$.

DEFINITION 2.2. A *d-fold category* \mathbb{C} is a diagram $\mathbb{C}: (\Delta^{op})^{\times d} \rightarrow \mathcal{S}$ such that for each $1 \leq k \leq d$, all the simplicial spaces

$$\mathbb{C}(\vec{n}_{k-1}, -, \vec{n}_{d-k}): \Delta^{op} \longrightarrow \mathcal{S}$$

are complete Segal spaces. When $d = 2$, we will refer to \mathbb{C} as a *double category*. We will denote by $\text{Cat}^{\otimes d} \subseteq \text{Fun}((\Delta^{op})^{\times d}, \mathcal{S})$ the full subcategory spanned by the d -fold categories.

WARNING 2.3. Because we impose completeness conditions in all directions, this notion of a d -fold category is *not quite* the ∞ -categorical analogue of that of a *d-uple category* in the sense of Ehresmann [Ehr63], i.e. a (strict) category object in $(d - 1)$ -uple categories. The latter type of objects (which are sometimes also called *d-fold categories*) corresponds instead to the more general notion of a *d-uple Segal space* [Hau17].

For instance, for a double category in the sense of Definition 2.2, the completeness conditions in the horizontal and vertical direction imply that the horizontal equivalences and vertical equivalences coincide. Many examples of double (and, more generally, d -uple) Segal spaces do not satisfy this condition: as an example, one can take the double Segal space of associative algebras, with horizontal morphisms given by bimodules and vertical morphisms given by algebra homomorphisms [Hau17].

Notation 2.4 (Opposites). Let \mathbb{C} be a d -fold category and $1 \leq k \leq d$. We will write \mathbb{C}^{k-op} for the d -fold category given by taking opposites in the k th variable:

$$\mathbb{C}^{k-op}: \Delta^{\times d} \xrightarrow{\text{id}^{\times k-1} \times \text{op} \times \text{id}^{\times d-k}} \Delta^{\times d} \xrightarrow{\mathbb{C}} \mathcal{S}.$$

When $k = 1$, we will simply write \mathbb{C}^{op} . If \mathbb{C} is a double category, we will furthermore write \mathbb{C}^{rev} for the double category with its coordinates reflected, i.e. $\mathbb{C}^{\text{rev}}(m, n) = \mathbb{C}(n, m)$.

In the case of double categories, we will employ the following conventions: we will refer to the direction of the first copy of Δ^{op} as the *horizontal* direction and to the second copy of Δ^{op} as the *vertical* direction. Furthermore, we will use the notation \longrightarrow for horizontal arrows and $\longrightarrow\ast$ for vertical arrows.

Example 2.5. Let \mathcal{C} and \mathcal{D} be categories. We denote by $\mathcal{C} \boxtimes \mathcal{D}$ the double category given by

$$\mathcal{C} \boxtimes \mathcal{D}([p], [q]) = \text{Map}_{\text{Cat}}([p], \mathcal{C}) \times \text{Map}_{\text{Cat}}([q], \mathcal{D}).$$

In particular, we write $[m, n] = [m] \boxtimes [n]$ for the (generating) double categories that look as follows.

$$\begin{array}{ccccccc} 00 & \longrightarrow & 10 & \longrightarrow & \dots & \longrightarrow & m0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0n & \longrightarrow & 1n & \longrightarrow & \dots & \longrightarrow & mn \end{array}$$

The squares in this picture should not be viewed as commuting squares, but rather as being filled with some type of non-invertible 2-arrow; no confusion should arise since one cannot compose \longrightarrow and $\longrightarrow\ast$.

Example 2.6. Let $n \geq 0$ and consider the *arrow double category* $\text{Ar}[n]$ defined by

$$\text{Map}_{\text{Cat}^{\otimes 2}}([p, q], \text{Ar}[n]) \simeq \text{Map}_{\text{Cat}}([q] \star [p], [n]).$$

where \star denotes the join. In other words, it is the double category that looks as follows.

$$\begin{array}{ccccccc} 00 & \longrightarrow & 01 & \longrightarrow & \dots & \longrightarrow & 0n \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 11 & \longrightarrow & \dots & \longrightarrow & \dots \\ & & & & \downarrow & & \downarrow \\ & & & & \dots & \longrightarrow & \dots \\ & & & & & & \downarrow \\ & & & & & & nn \end{array}$$

There is an evident inclusion $\text{Ar}[n] \longrightarrow [n, n]$, and we denote by $\pi_{\text{hor}}: \text{Ar}[n] \longrightarrow [n, 0]$ and $\pi_{\text{vert}}: \text{Ar}[n] \longrightarrow [0, n]$ the obvious projections.

Example 2.7. Let $n \geq 0$ and consider the double category $\text{Tw}[n]$ defined by

$$\text{Map}_{\text{Cat}^{\otimes 2}}([p, q], \text{Tw}[n]) \simeq \text{Map}_{\text{Cat}}([q]^{\text{op}} \star [p], [n]).$$

In other words, $\text{Tw}[n] \simeq \text{Ar}[n]^{2\text{-op}}$ and can be depicted as follows.

$$\begin{array}{ccccccc} 00 & \longrightarrow & 01 & \longrightarrow & \dots & \longrightarrow & 0n \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 11 & \longrightarrow & \dots & \longrightarrow & \dots \\ & & & & \uparrow & & \uparrow \\ & & & & \dots & \longrightarrow & \dots \\ & & & & & & \uparrow \\ & & & & & & nn \end{array}$$

PROPOSITION 2.8. For any $d \geq 1$, the category $\text{Cat}^{\otimes d}$ is cartesian closed. In fact, the localization

$$\text{Fun}((\Delta^{\text{op}})^{\times d}, \mathcal{S}) \xrightleftharpoons{\quad} \text{Cat}^{\otimes d}$$

of d -fold simplicial spaces is monoidal for the cartesian product.

We will write Y^X for the internal mapping object in d -fold simplicial spaces; the proposition implies that this is also the internal mapping object for d -fold categories.

Proof. It suffices to verify that for each d -fold category \mathbb{C} and $m_1, \dots, m_d \geq 0$, the d -fold simplicial space given by

$$\mathbb{C}^{[m_1, \dots, m_d]}(n_1, \dots, n_d) = \text{Map}([m_1, \dots, m_d] \times [n_1, \dots, n_d], \mathbb{C})$$

remains a d -fold category. Using symmetry and the fact that $[m_1, \dots, m_d] = [m_1, 0, \dots, 0] \times \dots \times [0, \dots, 0, m_d]$, it suffices to verify that $\mathbb{C}^{[m, 0, \dots, 0]}$ is a d -fold category, i.e. satisfies the complete Segal conditions in all simplicial directions.

For the first direction, note that each $\mathbb{C}^{[m, 0, \dots, 0]}(-, \vec{n}_{d-1})$ agrees with $\mathbb{C}(-, \vec{n}_{d-1})^{[m]}$, which is a 1-category by [Rez01, Theorem 7.2]. For the remaining simplicial directions, note that the $(d - 1)$ -fold simplicial space

$$\mathbb{C}^{[m, 0, \dots, 0]}(n, -) : (\Delta^{\text{op}})^{\times d-1} \longrightarrow \mathcal{S}$$

sends each (n_2, \dots, n_d) to the space of maps between simplicial spaces

$$\begin{aligned} \text{Map}_{\mathcal{S}\mathcal{S}}([m] \times [n], \mathbb{C}(-, n_2, \dots, n_d)) &\simeq \text{Map}_{\mathcal{S}\mathcal{S}}\left(\underset{[k] \in \Delta/[m] \times [n]}{\text{colim}} [k], \mathbb{C}(-, n_2, \dots, n_d)\right) \\ &\simeq \lim_{[k] \in (\Delta/[m] \times [n])^{\text{op}}} \mathbb{C}(k, n_2, \dots, n_d). \end{aligned}$$

In other words, $\mathbb{C}^{[m, 0, \dots, 0]}(n, -)$ is naturally equivalent to the limit of a certain diagram consisting of the $(d - 1)$ -fold categories $\mathbb{C}(k, -)$. Since $\text{Cat}^{\otimes d-1} \subseteq \text{Fun}((\Delta^{\text{op}})^{\times d-1}, \mathcal{S})$ is closed under limits, each $\mathbb{C}^{[m, 0, \dots, 0]}(n, -)$ is a $(d - 1)$ -fold category, as desired. \square

2.2 Fibrations of double categories

Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a map of 1-categories. Recall that an arrow $\alpha: d_1 \rightarrow d_2$ in \mathcal{D} is called p -cartesian (or simply cartesian, if p is understood) if for any diagram

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{\sigma} & \mathcal{D} \\ \downarrow & \nearrow & \downarrow \\ [2] & \longrightarrow & \mathcal{C} \end{array}$$

where the restriction of σ to $[1 < 2]$ is given by α , there is a contractible space of lifts, as indicated. Let $\mathcal{D}(1)^{\text{cart}} \subseteq \mathcal{D}(1)$ denote the subspace consisting of cartesian arrows. Then p is said to be a *cartesian fibration* if the map $\mathcal{D}(1)^{\text{cart}} \rightarrow \mathcal{D}(\{1\}) \times_{\mathcal{C}(\{1\})} \mathcal{C}(1)$ induces a surjection on path components [AF20, Lur09a]. It is called a *right fibration* if furthermore every arrow in \mathcal{D} is cartesian. Equivalently, it is a right fibration if the map $\mathcal{D}(1) \rightarrow \mathcal{D}(\{1\}) \times_{\mathcal{C}(\{1\})} \mathcal{C}(1)$ is an equivalence; see, e.g., [Boa18, Proposition 1.7]. Taking opposite categories, one obtains the notion of a *cocartesian* arrow and a *cocartesian* (respectively, *left*) *fibration* (some alternative characterizations also appear in § 5.1).

We will be interested in the following variant of this notion for double categories.

DEFINITION 2.9. A map $\mathbb{D} \rightarrow \mathbb{C}$ of double categories is called a *(left, cart)-fibration* if the following conditions hold.

(1) The square of categories

$$\begin{array}{ccc} \mathbb{D}([1], -) & \longrightarrow & \mathbb{D}(\{0\}, -) \\ \downarrow & & \downarrow \\ \mathbb{C}([1], -) & \longrightarrow & \mathbb{C}(\{0\}, -) \end{array}$$

is cartesian. Equivalently, each $\mathbb{D}(-, n) \rightarrow \mathbb{C}(-, n)$ is a left fibration of categories.

(2) The map $\mathbb{D}(0, -) \rightarrow \mathbb{C}(0, -)$ is a cartesian fibration of categories.

A *strong* map between (left, cart)-fibrations is a commuting square

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & \mathbb{D}' \\ p \downarrow & & \downarrow p' \\ \mathbb{C} & \longrightarrow & \mathbb{C}' \end{array}$$

where f preserves cartesian vertical arrows. We will write $\text{Fib}^{\text{left, cart}} \subseteq \text{Fun}([1], \text{Cat}^{\otimes 2})$ for the subcategory of (left, cart)-fibrations and strong maps between them.

Remark 2.10. By condition (1), the map $\mathbb{D}([1], -) \rightarrow \mathbb{C}([1], -)$ is a cartesian fibration as well and $\mathbb{D}([1], -) \rightarrow \mathbb{D}(\{0\}, -)$ preserves cartesian arrows. The other face map $\mathbb{D}([1], -) \rightarrow \mathbb{D}(\{1\}, -)$ is *not* required to preserve cartesian arrows.

Remark 2.11. The class of (left, cart)-fibrations is stable under composition and base change. Furthermore, a map $\mathbb{D} \rightarrow *$ is a (left, cart)-fibration if and only if \mathbb{D} is constant in the horizontal direction. In particular, the fibers of a (left, cart)-fibration are categories concentrated in the vertical direction (and constant in the horizontal direction).

Informally, a map of double categories $p: \mathbb{D} \rightarrow \mathbb{C}$ is a (left, cart)-fibration if the following conditions hold.

- (a) For every horizontal arrow $\alpha: c_0 \rightarrow c_1$ in \mathbb{C} and $d_0 \in p^{-1}(c_0)$, there exists a unique horizontal arrow $d_0 \rightarrow \alpha_!(d_0)$ lifting α . Furthermore, given any square σ in \mathbb{C} together with a lift of its left vertical arrow

$$\begin{array}{ccc} d_{00} \cdots \rightarrow \alpha_!(d_{00}) & & c_{00} \xrightarrow{\alpha} c_{10} \\ \downarrow \tilde{\sigma} \quad \downarrow & \mapsto & \downarrow \sigma \quad \downarrow \\ d_{01} \cdots \rightarrow \alpha'_!(d_{01}) & & c_{01} \xrightarrow{\alpha'} c_{11} \end{array}$$

there exists a unique dotted lift $\tilde{\sigma}$ as indicated (this lift includes the objects $\alpha_!(d_{00})$ and $\alpha'_!(d_{01})$).

- (b) For every vertical arrow $\beta: c_0 \rightarrow c_1$ in \mathbb{D} and $d_1 \in p^{-1}(c_1)$, there exists a vertically p -cartesian arrow $\beta^*(d_1) \rightarrow d_1$ covering α .

Now let us fix a square σ in \mathbb{C} together with an object $d_{01} \in p^{-1}(c_{01})$. A combination of conditions (a) and (b) then implies that σ admits a unique lift $\tilde{\sigma}$ of the form

$$\begin{array}{ccc} \beta^*(d_{01}) \cdots \rightarrow \alpha_! \beta^*(d_{01}) & & c_{00} \xrightarrow{\alpha} c_{10} \\ \downarrow \tilde{\sigma} \quad \downarrow & \mapsto & \beta \downarrow \quad \sigma \quad \downarrow \beta' \\ d_{01} \cdots \rightarrow \alpha'_!(d_{01}) & & c_{01} \xrightarrow{\alpha'} c_{11} \end{array}$$

which is vertically p -cartesian, in the sense that any other square factors uniquely over $\tilde{\sigma}$ in the vertical direction (Remark 2.10). The resulting vertical arrow $\alpha_1\beta^*(d_{01}) \rightarrow \alpha'_1(d_{01})$ is not necessarily p -cartesian and decomposes as

$$\alpha_1\beta^*(d_{01}) \xrightarrow{h_\sigma} \beta'^*\alpha'_1(d_{01}) \longrightarrow \alpha'_1(d_{01}).$$

The first map h_σ is contained in the fiber $p^{-1}(c_{10})$ and provides a natural comparison between the two ways to change fibers over the square σ , from ‘horizontally after vertically’ to ‘vertically after horizontally’.

To illustrate Definition 2.9, let us consider the following example.

PROPOSITION 2.12. *Let \mathcal{C} and \mathcal{D} be categories and let $p = (p_1, p_2): \mathcal{X} \rightarrow \mathcal{C} \times \mathcal{D}$ be a functor such that:*

- (i) p_1 is a cocartesian fibration and all p_1 -cocartesian arrows in \mathcal{X} map to equivalences in \mathcal{D} ;
- (ii) p_2 is a cartesian fibration and all p_2 -cartesian arrows in \mathcal{X} map to equivalences in \mathcal{C} .

Associated to p is a map of double categories $p': \mathbb{X} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$, where $\mathbb{X}(m, n)$ is the space of diagrams

$$\begin{array}{ccc} [m] \times [n] & \xrightarrow{\sigma} & \mathcal{X} \\ & \searrow \alpha \times \beta & \downarrow p \\ & & \mathcal{C} \times \mathcal{D} \end{array} \tag{2.13}$$

where $\alpha: [m] \rightarrow \mathcal{C}, \beta: [n] \rightarrow \mathcal{D}$ and σ sends each $[m] \times \{j\}$ to p_1 -cocartesian arrows in \mathcal{X} . Then p' is a (left, cart)-fibration of double categories.

Functors $p: \mathcal{X} \rightarrow \mathcal{C} \times \mathcal{D}$ satisfying conditions (i) and (ii) have been studied in detail in [HHLN23a, HHLN23b], where they are called *local orthofibrations*. Let us point out the close analogy between the discussion preceding Proposition 2.12 and the discussion in [HHLN23b, Construction 2.3.5].

Proof. To see condition (1) of Definition 2.9, consider the pullback square of categories

$$\begin{array}{ccc} \text{Fun}'([n], \mathcal{X}) & \xrightarrow{q} & \mathcal{C} \times \text{Map}_{\text{Cat}}([n], \mathcal{D}) \\ \downarrow & & \downarrow \Delta \times \iota \\ \text{Fun}([n], \mathcal{X}) & \longrightarrow & \text{Fun}([n], \mathcal{C}) \times \text{Fun}([n], \mathcal{D}) \end{array}$$

where Δ takes the constant diagram and ι is the inclusion of the space of objects. Since (p_1, p_2) is a local orthofibration, the induced map $(p'_1, p'_2): \text{Fun}([n], \mathcal{X}) \rightarrow \text{Fun}([n], \mathcal{C}) \times \text{Fun}([n], \mathcal{D})$ is a local orthofibration as well: a natural transformation in $\text{Fun}([n], \mathcal{X})$ is p'_1 -cocartesian if and only if it is given pointwise by p_1 -cocartesian maps in \mathcal{X} , and similarly for p'_2 -cartesian natural transformations (this follows, e.g., from Lemma 5.4(3)). In particular, the base change $q: \text{Fun}'([n], \mathcal{X}) \rightarrow \mathcal{C} \times \text{Map}_{\text{Cat}}([n], \mathcal{D})$ is a cocartesian fibration.

Unraveling the definitions, one sees that the space of maps $[m] \rightarrow \text{Fun}'([n], \mathcal{X})$ is equivalent to the space of diagrams $\sigma: [m] \times [n] \rightarrow \mathcal{X}$ of the form (2.13). Furthermore, $[m] \rightarrow \text{Fun}'([n], \mathcal{X})$ sends each arrow to a q -cocartesian arrow if and only if σ sends each $[m] \times \{i\}$ to p_1 -cocartesian

arrows in \mathcal{X} . It follows that there is a factorization

$$\mathbb{X}(-, n) \hookrightarrow \text{Fun}'([n], \mathcal{X}) \xrightarrow{q} \mathcal{C} \times \text{Map}([n], \mathcal{D}) = (\mathcal{C} \boxtimes \mathcal{D})(-, n)$$

where the first map is the inclusion of the wide subcategory spanned by the q -cocartesian arrows. In particular, this shows that each $\mathbb{X}(-, n) \rightarrow (\mathcal{C} \boxtimes \mathcal{D})(-, n)$ is a left fibration, as desired.

For condition (2) of Definition 2.9, it suffices to observe that $\mathbb{X}(0, -) \rightarrow (\mathcal{C} \boxtimes \mathcal{D})(0, -)$ coincides with top horizontal map in the following pullback diagram.

$$\begin{array}{ccc} \mathbb{X}(0, -) & \longrightarrow & \mathfrak{c}_0(\mathcal{C}) \times \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{C} \times \mathcal{D} \end{array}$$

This map is a cartesian fibration by condition (ii) (cf. [HHLN23b, Observation 2.3.2]). □

Remark 2.14. This remark only serves to motivate Definition 2.9, and will not be used in the rest of the text (making it more precise would require a form of straightening).

Elaborating on the discussion following Definition 2.9, one can informally think of a (left, cart)-fibration $p: \mathbb{D} \rightarrow \mathcal{C}$ as a certain type of diagram of categories indexed by \mathcal{C} . Indeed, by Remark 2.11, the fibers of p are all categories (in the vertical direction). Furthermore, each horizontal arrow $\alpha: c_0 \rightarrow c_1$ induces a map of categories $\alpha_!: \mathbb{D}_{c_0} \rightarrow \mathbb{D}_{c_1}$ and each vertical arrow $\beta: c_0 \rightarrow c_1$ induces a map of categories $\beta^*: \mathbb{D}_{c_1} \rightarrow \mathbb{D}_{c_0}$. Finally, each diagram $\sigma: [1, 1] \rightarrow \mathcal{C}$ gives rise to a square of categories and functors commuting up to a natural transformation as follows.

$$\begin{array}{ccc} \mathbb{D}_{c_{00}} & \xrightarrow{\alpha_!} & \mathbb{D}_{c_{10}} \\ \beta^* \uparrow & \searrow h_\sigma & \uparrow \beta'^* \\ \mathbb{D}_{c_{01}} & \xrightarrow{\alpha'_!} & \mathbb{D}_{c_{11}} \end{array}$$

In other words, a (left, cart)-fibration $p: \mathbb{D} \rightarrow \mathcal{C}$ should arise as the unstraightening of a diagram of double categories $\mathbb{C}^{2\text{-op}} \rightarrow \text{Cat}^{\text{oplax}}$, where $\text{Cat}^{\text{oplax}}$ is a certain double category whose objects are categories, horizontal and vertical morphisms are functors and squares are oplax squares as follows.

$$\begin{array}{ccc} \mathcal{D}_{00} & \longrightarrow & \mathcal{D}_{01} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{D}_{10} & \longrightarrow & \mathcal{D}_{11} \end{array}$$

LEMMA 2.15. Consider the following strong map of (left, cart)-fibrations over \mathcal{C} .

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & \mathbb{E} \\ & \searrow p & \swarrow q \\ & \mathcal{C} & \end{array}$$

Then f is an equivalence if and only if for any object $c \in \mathcal{C}$, the induced map between the fibers $\mathbb{D}_c \rightarrow \mathbb{E}_c$ is an equivalence of (vertical) categories.

Proof. Since p and q are left fibrations in the horizontal direction, the map f is an equivalence if and only if the map $f_0: \mathbb{D}(0, -) \rightarrow \mathbb{E}(0, -)$ is an equivalence of cartesian fibrations over $\mathcal{C}(0, -)$;

in turn, the map of cartesian fibrations f_0 is an equivalence if and only if it induces an equivalence on fibers [Lur09a, Corollary 2.4.4.4]. \square

VARIANT 2.16. Taking opposites in the horizontal or vertical direction, and exchanging the vertical and horizontal direction, one obtains seven more types of fibrations. For example, a map $\mathbb{D} \rightarrow \mathbb{C}$ is a (cocart, right)-fibration if $\mathbb{D}(-, 0) \rightarrow \mathbb{C}(-, 0)$ is a cocartesian fibration and each $\mathbb{D}(m, -) \rightarrow \mathbb{C}(m, -)$ is a right fibration. Strong maps between such fibrations are always required to preserve (co)cartesian arrows, either in the horizontal or vertical direction.

For example, one can also associate a (cocart, right)-fibration to a two-variable fibration $(p_1, p_2): \mathcal{X} \rightarrow \mathbb{C} \times \mathbb{D}$ as in Proposition 2.12, by considering maps σ as in (2.13) that send each $\{i\} \times [n]$ to p_2 -cartesian arrows in \mathcal{X} .

Remark 2.17. Similarly to Remark 2.14, one can informally think of a (cocart, right)-fibration $p: \mathbb{D} \rightarrow \mathbb{C}$ as a certain diagram of categories indexed by \mathbb{C} . Indeed, in this case all fibers of p are categories in the *horizontal* direction (and constant in the vertical direction) and again each horizontal arrow $\alpha: c_0 \rightarrow c_1$ induces a map of categories $\alpha!: \mathbb{D}_{c_0} \rightarrow \mathbb{D}_{c_1}$ and each vertical arrow $\beta: c_0 \rightarrow c_1$ induces a functor $\beta^*: \mathbb{D}_{c_1} \rightarrow \mathbb{D}_{c_0}$.

For the behaviour on a square $\sigma: [1, 1] \rightarrow \mathbb{C}$, we can repeat the analysis following Definition 2.9: for each such square and a lift $d_{01} \in p^{-1}(c_{01})$, there exists a lift to a square

$$\begin{array}{ccc} \beta^*(d_{01}) & \longrightarrow & \beta'^* \alpha!(d_{01}) \\ \downarrow & & \downarrow \\ d_{01} & \longrightarrow & \alpha!(d_{01}) \end{array}$$

in \mathbb{D} which is p -cartesian in the vertical direction. The bottom horizontal arrow is p -cocartesian but the top horizontal arrow is not in general, so that one again obtains a natural map $h_\sigma: \alpha! \beta^*(d_{01}) \rightarrow \beta'^* \alpha!(d_{01})$.

In other words, one can think of a (cocart, right)-fibration $p: \mathbb{D} \rightarrow \mathbb{C}$ as the unstraightening of a map of double categories $\mathbb{C}^{2\text{-op}} \rightarrow \mathbb{C}\text{at}^{\text{oplax}}$, where $\mathbb{C}\text{at}^{\text{oplax}}$ is the double category described informally in Remark 2.14.

2.3 The reflection of a (left, cart)-fibration

We will now turn to the main combinatorial construction of this text: we will associate to each (left, cart)-fibration $p: \mathbb{D} \rightarrow \mathbb{C}$ a (cocart, right)-fibration $\Psi^\perp(\mathbb{D}) \rightarrow \mathbb{C}$, which we will refer to as the *reflection* of p . In §3, we will study the functoriality of this construction in more detail.

Before we begin, note that at a heuristic level, Remarks 2.14 and 2.17 show that (left, cart)-fibrations and (cocart, right)-fibrations over a double category \mathbb{C} encode the same kind of data: both should be considered as unstraightened versions of the datum of a map of double categories

$$\mathbb{C}^{2\text{-op}} \longrightarrow \mathbb{C}\text{at}^{\text{oplax}}.$$

One should therefore be able to straighten a (left, cart)-fibration to such a map of double categories and then unstraighten it to obtain a (cocart, right)-fibration. Instead of making this more precise, we will give a purely combinatorial description of (what should be) the composite functor Ψ^\perp , inspired by the explicit formula for the Grothendieck construction and the combinatorics appearing in [HHLN23a]. Informally, the functor Ψ^\perp will first remove the vertical arrows in each

fiber of $p: \mathbb{D} \rightarrow \mathbb{C}$, so that only the cartesian vertical arrows remain, and then replaces them by fiberwise horizontal arrows instead (which can be composed with the cocartesian horizontal arrows already present in \mathbb{D}).

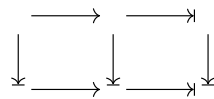
Let us start by considering the following construction of a double category out of a (left, cart)-fibration, which will appear repeatedly in the text.

LEMMA 2.18. *Let $p: \mathbb{D} \rightarrow \mathbb{C}$ be a map of double categories such that each $\mathbb{D}(-, n) \rightarrow \mathbb{C}(-, n)$ is a left fibration. Consider the bisimplicial space X whose space of (m, n) -simplices is given by the space of commuting squares of the following form.*

$$\begin{array}{ccc}
 \text{Ar}[m] \times [0, n] & \longrightarrow & \mathbb{D} \\
 \pi_{\text{hor}} \times \text{id} \downarrow & & \downarrow p \\
 [m, n] & \longrightarrow & \mathbb{C}
 \end{array} \tag{2.19}$$

Then X is a double category.

Remark 2.20. Informally, X is a double category with the same objects as \mathbb{D} , whose squares are given by diagrams in \mathbb{D} of the form



where the right square is a commuting square of vertical arrows in which the top and bottom arrow are sent to degenerate arrows in \mathbb{C} . In other words, we keep the same vertical arrows but replace the horizontal arrows by formal composites $\beta \circ \alpha: \cdot \rightarrow \cdot \rightarrow \cdot$ of a horizontal and a fiberwise vertical arrow. Diagrams of the above form have an evident vertical composition. For the composition of horizontal arrows, consider the following concatenation of two formal composites:

$$\cdot \xrightarrow{\alpha_1} \cdot \xrightarrow{\beta_1} \cdot \xrightarrow{\alpha_2} \cdot \xrightarrow{\beta_2} \cdot$$

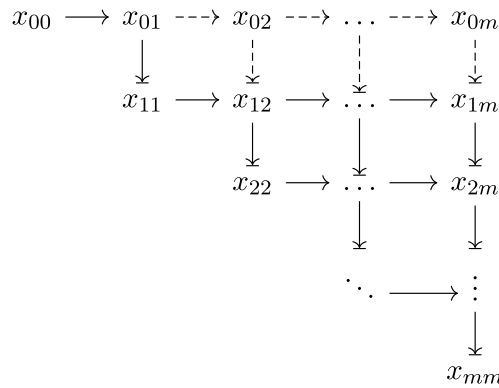
Then β_1 and α_2 fit into an essentially unique $(1, 1)$ -cell of \mathbb{D} that covers the vertically degenerate $(1, 1)$ -cell $p(\alpha_2)$ in \mathbb{C} . The other two sides of this square give a formal composite $\beta'_1 \circ \alpha'_2: \cdot \rightarrow \cdot \rightarrow \cdot$. The composed horizontal arrow in X is then given by the formal composite $(\beta_2 \beta'_1) \circ (\alpha'_2 \alpha_1)$.

Proof. To see that X is a category in the vertical direction, we use that $\text{Cat}^{\otimes 2}$ is cartesian closed (Proposition 2.8): the simplicial space $X(m, -)$ can then be identified with the category $(\mathbb{D}^{\text{Ar}[m]} \times_{\mathbb{C}^{\text{Ar}[m]}} \mathbb{C}^{[m, 0]})(0, -)$.

To see that each $X(-, n)$ is a category, it suffices to treat the case $n = 0$: indeed, the general case then follows by considering the functor $\mathbb{D}^{[0, n]} \rightarrow \mathbb{C}^{[0, n]}$, which is still a left fibration in the horizontal direction. Unraveling the definitions shows that the Segal condition for $X(-, 0)$ translates into the following unique lifting problem.

$$\begin{array}{ccc}
 \text{Ar}[0 \leq 1] \amalg_{\{11\}} \text{Ar}[1 \leq \dots \leq m] & \longrightarrow & \mathbb{D} \\
 \downarrow & \nearrow \text{dashed} & \downarrow \pi \\
 \text{Ar}[0 \leq \dots \leq m] & \longrightarrow & [m, 0] \longrightarrow \mathbb{C}
 \end{array}$$

In other words, given a solid diagram in \mathbb{D} of the form

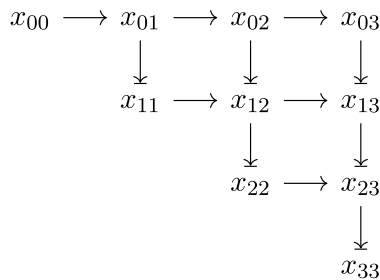


whose image in \mathbb{C} has a fixed dashed extension (i.e. including the top squares), there exists a contractible space of dashed extensions as indicated. This follows immediately from the fact that each $\mathbb{D}(-, n) \rightarrow \mathbb{C}(-, n)$ is a left fibration of categories.

For completeness of $X(-, 0)$, recall the simplicial set $H = \Delta[3]/\sim$ in the definition of completeness from § 2, and note that a map

$$H = \Delta[3]/\sim \longrightarrow X(-, 0)$$

corresponds to a diagram in \mathbb{D} of the form



in which $x_{00} \rightarrow x_{02}$ and $x_{02} \rightarrow x_{22}$ are degenerate and $x_{11} \rightarrow x_{13}$ and $x_{13} \rightarrow x_{33}$ are degenerate, and whose image in \mathbb{C} consists only of degenerate arrows (since \mathbb{C} is complete). We need to show that the above diagram is degenerate both in the horizontal and the vertical direction. Using that each $\mathbb{D}(-, n) \rightarrow \mathbb{C}(-, n)$ is a left fibration, one sees that the above diagram is degenerate in the horizontal direction. Furthermore, the rightmost column is degenerate in the vertical direction, since both $x_{03} \rightarrow x_{23}$ and $x_{13} \rightarrow x_{33}$ are degenerate and $\mathbb{D}(0, -)$ is a category. \square

DEFINITION 2.21. Let $p: \mathbb{D} \rightarrow \mathbb{C}$ be a (left, cart)-fibration and let X be the double category from Lemma 2.18. We will denote by $\Psi^\perp(\mathbb{D}) \subseteq X$ the bisimplicial subspace whose (m, n) -simplices correspond to diagrams (2.19) with the following property: the top map $\text{Ar}[m] \times [0, n] \rightarrow \mathbb{D}$ sends each $\{ii\} \times [0, n]$ into the subspace of cartesian vertical arrows of \mathbb{D} . We will refer to the evident projection $\Psi^\perp(p): \Psi^\perp(\mathbb{D}) \rightarrow \mathbb{C}$ as the *reflection* of p .

In terms of the heuristic description of X from Remark 2.20, one can identify $\Psi^\perp(\mathbb{D}) \subseteq X$ with the largest double subcategory of X with the property that any vertical morphism $[0, 1] \rightarrow \Psi^\perp(\mathbb{D})$ corresponds to a p -cartesian vertical arrow of \mathbb{D} . For the reason behind the terminology, see Proposition 3.15.

PROPOSITION 2.22. Let $p: \mathbb{D} \rightarrow \mathbb{C}$ be a (left, cart)-fibration. Then the reflection $\Psi^\perp(\mathbb{D}) \rightarrow \mathbb{C}$ is a (cocart, right)-fibration between double categories.

Proof. We verify each of the conditions, starting with the fact that $\Psi^\perp(\mathbb{D})$ is a double category.

Segal conditions. Recall the double category X from Lemma 2.18. Since each $\Psi^\perp(\mathbb{D})(-, n) \subseteq X(-, n)$ is a full subcategory of the category $X(-, n)$, it follows that $\Psi^\perp(\mathbb{D})(-, n)$ is a category (i.e. complete Segal space) itself. In the other direction, note that there is the following pullback square of simplicial spaces.

$$\begin{array}{ccc} \Psi^\perp(\mathbb{D})([m], -) & \hookrightarrow & X([m], -) \\ \downarrow & & \downarrow \\ \prod_{i=0}^m \Psi^\perp(\mathbb{D})(\{i\}, -) & \hookrightarrow & \prod_{i=0}^m X(\{i\}, -) \end{array}$$

Since $X([m], -)$ is a category by Lemma 2.18, it suffices to verify that $\Psi^\perp(\mathbb{D})(0, -) \hookrightarrow X(0, -)$ is a subcategory. But this can simply be identified with the inclusion of the wide subcategory of $\mathbb{D}(0, -)$ whose morphisms are cartesian morphisms.

Vertical right fibration. Next, let us verify that each $\Psi^\perp(\mathbb{D})(m, -) \rightarrow \mathbb{C}(m, -)$ is a right fibration. By the horizontal Segal condition, it suffices to treat the case $m = 0$ and $m = 1$. Unraveling the definitions, we have to verify that there are contractible spaces of diagonal lifts

$$\begin{array}{ccc} \{1\} & \longrightarrow & \mathbb{D} \\ \downarrow & \nearrow & \downarrow \pi \\ [0, 1] & \longrightarrow & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \text{Ar}[1] \times \{1\} & \longrightarrow & \mathbb{D} \\ \downarrow & \nearrow & \downarrow \pi \\ \text{Ar}[1] \times [0, 1] & \longrightarrow & [1, 1] \longrightarrow \mathbb{C} \end{array}$$

with the following property: the vertical arrow in $[0, 1]$, respectively the vertical arrows in $\{00, 11\} \times [0, 1]$, are sent to cartesian vertical arrows in \mathbb{D} . For the left square, there is a unique such lift since $\mathbb{D}(0, -) \rightarrow \mathbb{C}(0, -)$ is a cartesian fibration. For the right square, we have to verify that for any solid diagram

$$\begin{array}{ccc} & x_{00,0} \dashrightarrow x_{01,0} & \\ x_{00,1} \xrightarrow{\kappa_\emptyset} & x_{01,1} & \xrightarrow{\quad} \mathbb{D} \\ & \downarrow & \downarrow \pi \\ & x_{11,1} & \dashrightarrow x_{11,0} \\ & \downarrow \kappa_\emptyset & \\ c_{0,0} \xrightarrow{\quad} & c_{1,0} & \xrightarrow{\quad} \mathbb{C} \\ c_{0,1} \xrightarrow{\quad} & c_{1,1} & \end{array}$$

there exist unique objects $x_{00,0}, x_{01,0}, x_{11,0}$ together with a dashed extension as indicated, such that the two arrows $x_{00,0} \twoheadrightarrow x_{00,1}$ and $x_{11,0} \twoheadrightarrow x_{11,1}$ are cartesian. To see this, first take these two arrows to be the (unique) cartesian lifts of their image in \mathbb{C} . Then the top square can be filled uniquely using that $\mathbb{D} \rightarrow \mathbb{C}$ is a left fibration in the horizontal direction. The right square (living entirely in the vertical direction) can then be filled uniquely since $x_{11,0} \twoheadrightarrow x_{11,1}$ was cartesian.

Horizontal cocartesian fibration. Note that the space $\Psi^\perp(\mathbb{D})(1, 0)$ of horizontal arrows in $\Psi^\perp(\mathbb{D})$ is the space of diagrams in \mathbb{D} of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ & & \downarrow \\ & & z \end{array}$$

covering a horizontal arrow in \mathbb{C} . Let us say that a horizontal arrow in $\Psi^\perp(\mathbb{D})$ is *marked* if the corresponding vertical map $y \twoheadrightarrow z$ is an equivalence in \mathbb{D} . We claim that all marked horizontal arrows in $\Psi^\perp(\mathbb{D})$ are $\Psi^\perp(p)$ -cocartesian. In other words, we have to find a unique diagonal lift for every diagram of categories

$$\begin{array}{ccc} \Lambda^0[2] & \longrightarrow & \Psi^\perp(\mathbb{D})(-, 0) \\ \downarrow & \nearrow & \downarrow \Psi^\perp(p) \\ [2] & \longrightarrow & \mathbb{C}(-, 0) \end{array}$$

where the top map sends $0 \longrightarrow 1$ to a marked arrow. Unraveling the definitions, this corresponds to a solid diagram in \mathbb{D} covering a horizontal 2-simplex in \mathbb{C}

$$\begin{array}{ccccc} & & \text{---} & & \\ & \text{---} & & \text{---} & \\ x_{00} & \longrightarrow & x_{01} & \dashrightarrow & x_{02} \\ & & \sim \downarrow & & \downarrow \\ & & x_{11} & \dashrightarrow & x_{12} \\ & & & & \downarrow \\ & & & & x_{22} \end{array}$$

where the map $x_{01} \twoheadrightarrow x_{11}$ is an equivalence, of which one has to find a unique dashed extension as indicated. To find the desired unique extension, using that $\mathbb{D} \longrightarrow \mathbb{C}$ is a left fibration in the horizontal direction, it follows that there is a unique way to extend the square making the top triangle (living purely in the horizontal direction) commute. Since $x_{01} \twoheadrightarrow x_{11}$ was an equivalence, the resulting vertical arrow $x_{02} \twoheadrightarrow x_{12}$ is an equivalence as well. It then follows that there is a unique vertical map $x_{12} \twoheadrightarrow x_{22}$ with a filling of the right triangle (purely in the vertical direction). We conclude that all marked arrows in $\Psi^\perp(\mathbb{D})$ are cocartesian. Finally, for every horizontal arrow $f: c \longrightarrow c'$ in \mathbb{C} and a lift $x \in \Psi^\perp(\mathbb{D})_c$, there exist a marked lift $\tilde{f}: x \longrightarrow x'$, using that $\mathbb{D}(-, 0) \longrightarrow \mathbb{C}(-, 0)$ was a left fibration: indeed, one can take \tilde{f} to correspond to the diagram $x \longrightarrow f_!x \twoheadrightarrow f_!x$ in \mathbb{D} , where the vertical morphism is the identity. \square

3. Reflecting fibrations of double categories

This section provides a more detailed analysis of the reflection of (left, cart)-fibrations between double categories (Definition 2.21). More precisely, our goal will be to prove the following theorem, which is our main technical result.

THEOREM 3.1. *There is an equivalence of cartesian fibrations*

$$\begin{array}{ccc} \Psi^\perp : \text{Fib}^{\text{left, cart}} & \xrightleftharpoons{\sim} & \text{Fib}^{\text{cocart, right}} : \Psi^\top \\ \text{codom} \searrow & & \swarrow \text{codom} \\ & \text{Cat}^{\otimes 2} & \end{array}$$

between the categories of (left, cart)-fibrations and (cocart, right)-fibrations. In particular, for any double category \mathbb{C} this restricts to an equivalence $\text{Fib}^{\text{left, cart}}(\mathbb{C}) \simeq \text{Fib}^{\text{cocart, right}}(\mathbb{C})$.

This should not be unexpected: in light of Remarks 2.14 and 2.17, both (left, cart)-fibrations and (cocart, right)-fibrations should be equivalent to functors of double categories $\mathbb{C}^{2\text{-op}} \rightarrow \text{Cat}^{\text{oplax}}$ under some form of straightening.

At the level of objects, the functor Ψ^\perp is given by Definition 2.21. The functor Ψ^\top is defined analogously, taking opposites and reversing the roles of the horizontal and vertical directions. To extend Ψ^\perp and Ψ^\top to functors, it will be convenient to describe the ∞ -categories of (left, cart)-fibrations and (cocart, right)-fibrations in terms of marked bisimplicial spaces (§ 3.1). This allows one to describe the functor Ψ^\perp very explicitly in terms of combinatorial data (§ 3.2). Finally, we will provide explicit natural equivalences exhibiting Ψ^\perp and Ψ^\top as mutual inverses (§ 3.3). We will do this by explicitly writing down a zigzag of equivalences connecting their composition to the identity.

Remark 3.2. In model categorical terms, both functors Ψ^\perp and Ψ^\top arise as right Quillen functors between the arrow categories of marked bisimplicial spaces. In particular, all of our arguments can be reformulated without difficulties into such model-categorical terms.

3.1 Marked bisimplicial spaces

For technical reasons, it will be convenient to give a description of the category $\text{Fib}^{\text{left, cart}}$ using markings, analogous to the model for cartesian fibrations using marked simplicial sets in [Lur09a].

Notation 3.3. Let $\Delta_{\downarrow}^{\times 2}$ denote the category obtained from $\Delta^{\times 2}$ by freely adding a factorization of the vertical degeneracy map $[0, 1] \rightarrow [0, 0]$ as follows.

$$\begin{array}{ccc}
 [0, 1] & \xrightarrow{\quad} & [0, 0] \\
 & \searrow & \nearrow \\
 & [0, 1]^\# &
 \end{array}$$

A presheaf $X: (\Delta_{\downarrow}^{\times 2})^{\text{op}} \rightarrow \mathcal{S}$ is said to be a *vertically marked bisimplicial space* if the map $X([0, 1]^\#) \rightarrow X(0, 1)$ is a subspace inclusion. We will write $\text{bis}\mathcal{S}_{\downarrow} \subseteq \text{Fun}(\Delta_{\downarrow}^{\times 2, \text{op}}, \mathcal{S})$ for the full subcategory spanned by the vertically marked bisimplicial spaces, and typically use the same symbol for a vertically marked bisimplicial space and its underlying bisimplicial space.

Remark 3.4. The forgetful functor $\text{bis}\mathcal{S}_{\downarrow} \rightarrow \text{bis}\mathcal{S}$ induces a subspace inclusion

$$\text{Map}_{\text{bis}\mathcal{S}_{\downarrow}}(X, Y) \hookrightarrow \text{Map}_{\text{bis}\mathcal{S}}(X, Y)$$

whose image consists of the path components of maps of bisimplicial spaces $f: X \rightarrow Y$ with the property that $X(0, 1) \rightarrow Y(0, 1)$ preserves the subspace of marked vertical arrows.

DEFINITION 3.5. A *vertically marked double category* is a vertically marked bisimplicial space whose underlying bisimplicial space is a double category. We will write $\text{Cat}_{\downarrow}^{\otimes 2} \subseteq \text{bis}\mathcal{S}_{\downarrow}$ for the full subcategory of vertically marked double categories.

Likewise, we write $\text{Cat}_{\rightarrow}^{\otimes 2} \subseteq \text{bis}\mathcal{S}_{\rightarrow}$ for the full subcategory of *horizontally marked bisimplicial spaces* spanned by the *horizontally marked double categories*.

Notation 3.6. As usual, the forgetful functor $\text{Cat}_{\downarrow}^{\otimes 2} \rightarrow \text{Cat}^{\otimes 2}$ admits a fully faithful right adjoint, sending a double category \mathbb{C} to the vertically marked double category $\mathbb{C}^\#$ where all vertical arrows are marked. We will also write $\mathbb{C}^\#$ for \mathbb{C} with all *horizontal* arrows marked; it will always be clear from the context whether we mark horizontal or vertical arrows.

DEFINITION 3.7. Consider the following full subcategories of $\text{Fun}([1], \text{bis}\mathcal{S}_\downarrow)$.

- (1) Let $\text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow)$ be spanned by the maps $Y \rightarrow X$ where X is maximally marked, i.e. every $[0, 1]$ -simplex of X is marked.
- (2) Let $\text{Fun}^\sharp([1], \text{Cat}_\downarrow^{\otimes 2})$ be spanned by the maps $\mathbb{D} \rightarrow \mathbb{C}^\sharp$ between marked double categories, with \mathbb{C}^\sharp maximally marked.
- (3) Let $\text{Fib}^{\text{left, cart}}$ be spanned by the maps $p: \mathbb{D}^\sharp \rightarrow \mathbb{C}^\sharp$ between marked double categories, whose underlying map of double categories is a (left, cart)-fibration and where a vertical arrow in \mathbb{D}^\sharp is marked if and only if it is p -cartesian.

The codomain projections (forgetting the maximal marking) then define the following diagram of cartesian fibrations and maps preserving cartesian arrows.

$$\begin{array}{ccccc}
 \text{Fib}^{\text{left, cart}} & \hookrightarrow & \text{Fun}^\sharp([1], \text{Cat}_\downarrow^{\otimes 2}) & \hookrightarrow & \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow) \\
 \text{codom} \downarrow & & \downarrow \text{codom} & & \downarrow \text{codom} \\
 \text{Cat}^{\otimes 2} & \xrightarrow{=} & \text{Cat}^{\otimes 2} & \hookrightarrow & \text{bis}\mathcal{S}
 \end{array}$$

In a similar way, one can define fully faithful inclusions as follows.

$$\begin{array}{ccccc}
 \text{Fib}^{\text{cocart, right}} & \hookrightarrow & \text{Fun}^\sharp([1], \text{Cat}_{\rightarrow}^{\otimes 2}) & \hookrightarrow & \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_{\rightarrow}) \\
 \text{codom} \downarrow & & \downarrow \text{codom} & & \downarrow \text{codom} \\
 \text{Cat}^{\otimes 2} & \xrightarrow{=} & \text{Cat}^{\otimes 2} & \hookrightarrow & \text{bis}\mathcal{S}
 \end{array}$$

LEMMA 3.8. Forgetting the marking defines a functor $\text{Fib}^{\text{left, cart}} \rightarrow \text{Fun}([1], \text{Cat}^{\otimes 2})$ which is an equivalence onto the subcategory whose objects are (left, cart)-fibrations and whose morphisms are strong morphisms (Definition 2.9).

Proof. By Remark 3.4, the forgetful functor $\text{Fun}([1], \text{Cat}_\downarrow^{\otimes 2}) \rightarrow \text{Fun}([1], \text{Cat}^{\otimes 2})$ induces inclusions of path components on mapping spaces, whose images consist of those commuting squares of double categories such that all maps preserve marked arrows. The result follows immediately from this. \square

Remark 3.9. The full subcategory $\text{Fib}^{\text{left, cart}} \hookrightarrow \text{Fun}([1], \text{bis}\mathcal{S}_\downarrow)$ can be obtained as a left Bousfield localization (and, in particular, is a presentable category). One can also present $\text{Fib}^{\text{left, cart}}$ in terms of model $(1, 1)$ -categories, as a left Bousfield localization of the injective model structure on functors $\Delta_\downarrow^{\times 2, \text{op}} \rightarrow \text{sSet}$ (where sSet carries the Kan–Quillen model structure). All arguments appearing below can be carried out in this setting as well.

CONSTRUCTION 3.10. Let us fix the following data.

- (a) For each $m, n \geq 0$, a map of vertically marked double categories $S[m, n] \rightarrow [m, n]^\sharp$, natural in the sense that it defines a functor fitting into the following commuting triangle.

$$\begin{array}{ccc}
 & & \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow) \\
 & \nearrow S & \downarrow \text{codom} \\
 \Delta^{\times 2} & \hookrightarrow & \text{bis}\mathcal{S}
 \end{array}$$

- (b) In the vertically marked double category $S[1, 0]$, an *additional* set of path components of marked vertical edges. We will write $S[1, 0]^+$ for the resulting vertically marked double category.

Associated to these data is the following natural map of cartesian fibrations.

$$\begin{array}{ccc}
 \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow) & \xrightarrow{\Psi_S} & \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\rightarrow) \\
 \searrow \text{codom} & & \swarrow \text{codom} \\
 & \text{bis}\mathcal{S} &
 \end{array} \tag{3.11}$$

To construct this diagram, observe that the codomain projection admits a left adjoint $L: \text{bis}\mathcal{S} \rightarrow \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow)$, sending \mathbb{C} to $\emptyset \rightarrow \mathbb{C}^\sharp$; the same holds in the horizontally marked case. Now consider the following commuting squares.

$$\begin{array}{ccc}
 [1]^{\text{op}} \times \Delta^{\times 2} \xrightarrow{\sigma} \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow) & & \text{Fun}([1], \text{PSh}(\Delta^{\times 2})) \xrightarrow{\Phi_S} \text{Fun}^\sharp([1], \text{bis}\mathcal{S}_\downarrow) \\
 \uparrow \int & & \uparrow L \\
 \{1\} \times \Delta^{\times 2} \xleftarrow{h} \text{bis}\mathcal{S} & & \text{bis}\mathcal{S} \xrightarrow{=} \text{bis}\mathcal{S}
 \end{array}$$

In the left square, h is the Yoneda embedding and σ is the functor sending

$$(1, [m, n]) \mapsto (\emptyset \rightarrow [m, n]^\sharp) \quad (0, [m, n]) \mapsto (S[m, n] \rightarrow [m, n]^\sharp).$$

Furthermore, σ sends $(0, [1, 0]^\sharp)$ to $S[1, 0]^+ \rightarrow [1, 0]^\sharp$ and $(1, [1, 0]^\sharp)$ to $\emptyset \rightarrow [1, 0]^\sharp$. This induces the commuting square of left adjoint functors on the right, using that h and σ induce unique colimit-preserving functors out of the corresponding presheaf categories [Dug01], [Lur09a, Theorem 5.1.5.6].

Let us write Ψ_S for the right adjoint to Φ_S . Explicitly, let $p: Y \rightarrow X^\sharp$ be a map of vertically marked bisimplicial spaces whose target is maximally marked. Then $\Psi_S(p): \Psi_S(Y) \rightarrow X^\sharp$ can be described as follows: the underlying bisimplicial space of $\Psi_S(Y)$ has (m, n) -simplices determined by the following.

$$\begin{array}{ccc}
 \Psi_S(Y) & & S[m, n] \cdots \cdots \rightarrow Y \\
 \downarrow \Psi_S(p) & \iff & \downarrow p \\
 [m, n] \xrightarrow{\alpha} X & & [m, n]^\sharp \xrightarrow{\alpha} X^\sharp
 \end{array} \tag{3.12}$$

Furthermore, the space of marked $[1, 0]$ -simplices in $\Psi_S(Y)$ is the subspace consisting of those maps $S[1, 0] \rightarrow Y$ that also preserve the additional marked arrows from data (b). In particular, this implies that Ψ_S sends vertically marked bisimplicial spaces into horizontally marked bisimplicial spaces (i.e. marked edges form a subspace). We therefore obtain the desired commuting diagram (3.11). Furthermore, it follows immediately from (3.12) that Ψ_K preserves cartesian morphisms: for every $Y \rightarrow X^\sharp$ and $X' \rightarrow X$, we have an equivalence

$$\Psi_S(Y \times_{X^\sharp} X'^\sharp) \xrightarrow{\sim} \Psi_S(Y) \times_{X^\sharp} X'^\sharp.$$

3.2 Construction of the functors

We will use Construction 3.10 to define the functor Ψ^\perp , as follows.

CONSTRUCTION 3.13. For any $[m, n] \in \Delta^{\times 2}$, consider the following natural map of vertically marked double categories:

$$K[m, n] = \text{Ar}[m] \times [0, n] \xrightarrow{\pi_{\text{hor}} \times \text{id}} [m, 0] \times [0, n]^\sharp = [m, n]^\sharp.$$

Here $K[m, n]$ comes equipped with the marking where for each $ii \in \text{Ar}[m]$ (cf. Example 2.6), all vertical arrows in $\{ii\} \times [0, n]$ are marked. Furthermore, $K[1, 0]^+$ is the double category $\text{Ar}[1]$, where, in addition, the vertical arrow $01 \rightarrow 11$ is marked.

Using this, Construction 3.10 produces the following functor over $\text{bis}\mathbb{S}$:

$$\Psi_K : \text{Fun}^\sharp([1], \text{bis}\mathbb{S}_\downarrow) \longrightarrow \text{Fun}^\sharp([1], \text{bis}\mathbb{S}_\rightarrow).$$

PROPOSITION 3.14. *Let $p: \mathbb{D}^\natural \rightarrow \mathbb{C}^\sharp$ be a (left, cart)-fibration with its natural marking by the p -cartesian vertical arrows. Then $\Psi_K(p): \Psi_K(\mathbb{D}) \rightarrow \mathbb{C}^\sharp$ coincides with the (cocart, right)-fibration $\Psi^\perp(p)$ from Definition 2.21, with its natural horizontal marking. Consequently, Ψ_K restricts to a map of cartesian fibrations that we will denote as follows.*

$$\begin{array}{ccc} \text{Fib}^{\text{left, cart}} & \xrightarrow{\Psi^\perp} & \text{Fib}^{\text{cocart, right}} \\ \searrow \text{codom} & & \swarrow \text{codom} \\ & \text{Cat}^{\otimes 2} & \end{array}$$

Proof. Unraveling the definition, one sees that the map of bisimplicial spaces $\Psi_K(\mathbb{D}) \rightarrow \mathbb{C}$ is precisely given by the reflection $\Psi^\perp(p): \Psi^\perp(\mathbb{D}) \rightarrow \mathbb{C}$ from Definition 2.21; in particular, it is a (cocart, right)-fibration by Proposition 2.22.

It remains to identify the marked horizontal arrows in $\Psi_K(\mathbb{D}^\natural)$; these correspond to diagrams $x \rightarrow y \twoheadrightarrow z$ in \mathbb{D} with the property that $y \twoheadrightarrow z$ is a marked arrow covering an equivalence in \mathbb{C} . This just means that $y \twoheadrightarrow z$ is an equivalence, so that the marked horizontal arrows coincide with the marked horizontal arrows introduced in the proof of Proposition 2.22. We have already seen there that these marked arrows are precisely the cocartesian horizontal arrows. \square

In particular, by restricting to the fibers over a double category \mathbb{C} we obtain a functor $\Psi^\perp: \text{Fib}^{\text{left, cart}}(\mathbb{C}) \rightarrow \text{Fib}^{\text{cocart, right}}(\mathbb{C})$ that we will refer to as the *reflection functor*. To motivate the terminology, let us investigate more precisely the behaviour of Ψ^\perp when \mathbb{C} is a space.

PROPOSITION 3.15. *Let S be a space and let Cat/S be the category of categories over S . Consider the endofunctor of Cat/S given by the composite*

$$\text{Cat}/S \xrightarrow[\sim]{[0] \boxtimes (-)} \text{Fib}^{\text{left, cart}}(S) \xrightarrow{\Psi^\perp} \text{Fib}^{\text{right, cocart}}(S) \xleftarrow[\sim]{(-) \boxtimes [0]} \text{Cat}/S$$

where the left equivalence sends a category \mathcal{C} to the (left, cart)-fibration $[0] \boxtimes \mathcal{C} \rightarrow [0] \boxtimes S = S$ and the right equivalence to the (cocart, right)-fibration $\mathcal{C} \boxtimes [0] \rightarrow S \boxtimes [0] = S$ (see Example 2.5 and Remark 2.11). Then the above endofunctor of Cat/S is homotopic to the identity.

In particular, at the level of fibers the reflection functor Ψ^\perp simply exchanges the horizontal and vertical direction.

Proof. Consider the natural map $K[m, n] = \text{Ar}[m] \times [0, n] \rightarrow \text{Ar}[m] \xrightarrow{\pi_{\text{ver}}} [0, m]$, where the second map is the vertical projection (see Example 2.6). This map sends the marked vertical arrows in $K[m, n]$ from Construction 3.13 to degenerate edges. For each $\mathbb{D} \in \text{Fib}^{\text{left, cart}}$, we therefore obtain a natural transformation of double categories, given in degree (m, n) by

$$\text{Map}_{\text{Cat}^{\otimes 2}}([0, m], \mathbb{D}) \longrightarrow \text{Map}_{\text{Cat}^{\otimes 2}}(K[m, n], \mathbb{D}) = \Psi_K(\mathbb{D})(m, n).$$

Taking $\mathbb{D} = [0] \boxtimes \mathcal{C}$, this produces the desired map $\mathcal{C} \boxtimes [0] \rightarrow \Psi^\perp([0] \boxtimes \mathcal{C})$. To see that this is an equivalence, note that the domain and codomain are both constant in the vertical direction and complete Segal spaces in the horizontal direction. In degree $(0, 0)$, the above is an equivalence since $K[0, 0] \rightarrow [0, 0]$ is an isomorphism. In degree $(1, 0)$, it suffices to note that the map $\pi_{\text{ver}}: \text{Ar}[1] \rightarrow [0, 1]$ induces an equivalence on mapping spaces into a double category of the form $[0] \boxtimes \mathcal{C}$. \square

One can obtain variants of the functor Ψ^\perp for different kinds of fibrations, by taking opposites in the horizontal or vertical direction and exchanging horizontal and vertical directions.

VARIANT 3.16. Let us introduce the following two variants of the functor Ψ^\perp .

- (1) There is a functor $\Psi^\top : \text{Fib}^{\text{cocart, right}} \longrightarrow \text{Fib}^{\text{left, cart}}$, obtained by conjugating Ψ^\perp with the functor sending $\mathbb{C} \mapsto (\mathbb{C}^{\text{rev}})^{(1,2)\text{-op}}$. Unraveling the definitions, one sees that $\Psi^\top = \Psi_{K'}$ is defined as in Construction 3.10 (with the roles of horizontal and vertical reversed), but this time using the maps

$$K'[m, n] = [m, 0] \times \text{Ar}[n] \xrightarrow{\text{id} \times \pi_{\text{ver}}} [m, 0] \times [0, n] = [m, n].$$

Here we mark the horizontal arrows in $K'[m, n]$ contained in $[m, 0] \times \{ii\}$ for each $i \in [n]$. Furthermore, $K'([0, 1]^\sharp)$ is given by $\text{Ar}[1]$ where we mark all horizontal arrows.

- (2) There is another functor $\Psi^\dagger : \text{Fib}^{\text{cocart, left}} \longrightarrow \text{Fib}^{\text{left, cocart}}$, obtained by conjugating Ψ^\perp with the functor sending $\mathbb{C} \mapsto (\mathbb{C}^{\text{rev}})^{1\text{-op}}$. Unraveling the definitions, one sees that $\Psi^\dagger = \Psi_L$ is defined as in Construction 3.13, but the role of the vertically marked double category $K[m, n]$ is now played by

$$L[m, n] = [m, 0] \times \text{Tw}([n]^{\text{op}}) \xrightarrow{\text{id} \times \pi_{\text{ver}}} [m, 0] \times [0, n] = [m, n].$$

In $L[m, n]$, we mark the horizontal arrows in each $[m, 0] \times \{ii\}$. Furthermore, $L([1, 0]^\sharp)$ is given by $\text{Tw}([1]^{\text{op}})$ with all horizontal arrows marked.

Remark 3.17. Proposition 3.15 holds for the functor Ψ^\top as well: indeed, the functor $(-)^{\text{rev}}$ commutes with the functor $\mathbb{C} \mapsto (\mathbb{C}^{\text{rev}})^{(1,2)\text{-op}}$. In contrast, it does not apply to the functor Ψ^\dagger : instead, the resulting endofunctor of Cat takes opposite categories.

3.3 Proof of Theorem 3.1

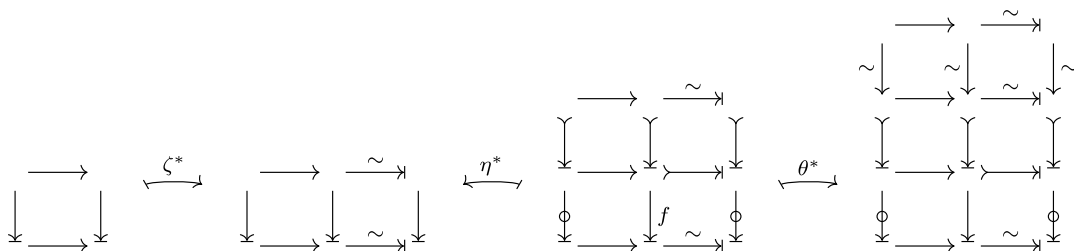
Consider the functors from Proposition 3.14 and Variant 3.16:

$$\Psi^\perp : \text{Fib}^{\text{left, cart}} \longrightarrow \text{Fib}^{\text{cocart, right}}, \quad \Psi^\top : \text{Fib}^{\text{cocart, right}} \longrightarrow \text{Fib}^{\text{left, cart}}.$$

We will show that these are mutually inverse, up to homotopy. More precisely, we will prove that for every (left, cart)-fibration $\mathbb{D} \longrightarrow \mathbb{C}$, there is the following natural zigzag of equivalences between (left, cart)-fibrations over \mathbb{C} .

$$\mathbb{D} \xrightarrow{\zeta^*} \Psi_A(\mathbb{D}) \xleftarrow{\eta^*} \Psi_B(\mathbb{D}) \xrightarrow{\theta^*} \Psi^\top(\Psi^\perp(\mathbb{D}))$$

To motivate the precise combinatorial constructions that will follow, let us already disclose that a typical (1, 1)-cell in each of these four double categories will correspond to a diagram in \mathbb{D} of the following form.



Here $\dashv\rightarrow$ denotes a p -cartesian vertical arrow in \mathbb{D} and $\dashv\rightarrow$ denotes a map that is sent to a degenerate arrow in \mathbb{C} . The rightmost diagram can be obtained by explicitly unraveling the definition of the composite functor $\Psi^\top \circ \Psi^\perp$. The maps ζ^* and θ^* will then be the evident equivalences which simply add degenerate squares on the right, respectively, on the top.

The map η^* takes vertical compositions and has an inverse given informally as follows (cf. the proof of Proposition 2.22). First, one decomposes the left and right vertical maps into fiberwise maps followed by p -cartesian maps. The middle column is then uniquely determined by the left column and the right half of the diagram is determined uniquely because the right vertical map was p -cartesian. Note that the map f need not be p -cartesian; this is why we need to add a degenerate square using ζ^* before we can decompose the left and right vertical arrow.

Let us now start by defining the endofunctors $\Psi_A, \Psi_B: \text{Fib}^{\text{left, cart}} \rightarrow \text{Fib}^{\text{left, cart}}$ more precisely, using (the vertically marked version of) Construction 3.10.

CONSTRUCTION 3.18. For each $[m, n] \in \mathbf{\Delta}^{\times 2}$, let

$$A[m, n] = \text{Ar}[m] \times [0, n] \xrightarrow{\pi_{\text{hor}} \times \text{id}} [m, n].$$

In $A[m, n]$, we mark all vertical arrows contained in $\text{Ar}[m] \times \{i\}$, for all $i \in [n]$; in particular, note that this marking is different from Construction 3.13. In addition, define $A([0, 1])^+ = [0, 1]^\sharp$.

Likewise, for each $[m, n] \in \mathbf{\Delta}^{\times 2}$, let us denote

$$B[m, n] = \text{Ar}[m] \times ([0] \boxtimes \text{Fun}([1], [n])) \xrightarrow{\pi_{\text{hor}} \times \text{dom}} [m, 0] \times [0, n] = [m, n].$$

In $B[m, n] = \text{Ar}[m] \times ([0] \boxtimes \text{Fun}([1], [n]))$, we mark the following vertical arrows:

- for each object $aa \in \text{Ar}[m]$ with $0 \leq a \leq m$, we mark all edges in $\{aa\} \boxtimes \text{Fun}([1], [n])$ of the form $ik \rightarrow jk$;
- for each $ii \in \text{Fun}([1], [n])$ with $0 \leq i \leq n$, we mark all vertical arrows in $\text{Ar}[m] \times \{ii\}$.

In addition, define $B([0, 1])^+$ to be $[0] \boxtimes \text{Fun}([1], [1])$, where all vertical arrows are marked.

Note that these vertically marked double categories are related by natural transformations as follows.

$$\begin{array}{ccc} [m, n] & \xleftarrow{\zeta} & A[m, n] & \xrightarrow{\eta} & B[m, n] \\ & \searrow = & \downarrow \pi_{\text{hor}} \times \text{id} & \swarrow \pi_{\text{hor}} \times \text{dom} & \\ & & [m, n]^\sharp & & \end{array}$$

Here $\zeta: \text{Ar}[m] \times [0, n] \rightarrow [m, n]$ just arises from the horizontal projection $\text{Ar}[m] \rightarrow [m, 0]$. This sends the vertical arrows within each $\text{Ar}[m] \times \{i\}$ to degenerate arrows and, hence, preserves markings. The map $\eta: \text{Ar}[m] \times [0, n] \rightarrow \text{Ar}[m] \times ([0] \boxtimes \text{Fun}([1], [n]))$ arises from the map $\text{cst}: [n] \rightarrow \text{Fun}([1], [n])$ sending each $i \in [n]$ to the constant arrow $i \leq i$.

Furthermore, the maps ζ and η induce natural maps $[0, 1]^\sharp \leftarrow A[0, 1]^+ \rightarrow B[0, 1]^+$ preserving the additional marked vertical arrows. Using the naturality of Construction 3.10, we obtain a zigzag of natural transformations between functors $\text{Fun}^\sharp([1], \text{bisS}_\downarrow) \rightarrow \text{Fun}^\sharp([1], \text{bisS}_\downarrow)$ of the form

$$\text{id} \xrightarrow{\zeta^*} \Psi_A \xleftarrow{\eta^*} \Psi_B.$$

PROPOSITION 3.19. Let $p: \mathbb{D}^\sharp \rightarrow \mathbb{C}^\sharp$ be a (left, cart)-fibration. Then the natural transformations ζ^* and η^* above are equivalences of marked bisimplicial spaces. In particular, $\Psi_A(\mathbb{D}^\sharp) \rightarrow \mathbb{C}^\sharp$ and $\Psi_B(\mathbb{D}^\sharp) \rightarrow \mathbb{C}^\sharp$ are (left, cart)-fibrations of double categories.

Proof. Recall from the general description (3.12) that the space of (m, n) -simplices in $\Psi_A(\mathbb{D}^\sharp)$ is given by the space of diagrams of vertically marked double categories

$$\begin{array}{ccc} A[m, n] & \longrightarrow & \mathbb{D}^\sharp \\ \downarrow & & \downarrow p \\ [m, n]^\sharp & \longrightarrow & \mathbb{C}^\sharp \end{array}$$

and likewise for $\Psi_B(\mathbb{D}^\sharp)$. The maps ζ^* and η^* simply arise by restriction along the natural maps $\zeta: A[m, n] \rightarrow [m, n]$ and $\eta: A[m, n] \rightarrow B[m, n]$.

ζ^* is an equivalence. Note that $\Psi_A(\mathbb{D}^\sharp) \hookrightarrow X$ is a bisimplicial subspace of the bisimplicial space X from Lemma 2.18. Indeed, the space of maps $[m, n] \rightarrow \Psi_A(\mathbb{D}^\sharp)$ is the subspace of those diagrams (2.19) such that $\text{Ar}[m] \times [0, n] \rightarrow \mathbb{D}$ sends the vertical arrows in each $\text{Ar}[m] \times \{i\}$ with $0 \leq i \leq n$ to equivalences.

By Lemma 2.18, X is double category and one sees that each simplicial subspace $\Psi_A(\mathbb{D}^\sharp)(m, -) \hookrightarrow X(m, -)$ is the inclusion of a full subcategory. In the horizontal direction, the proof of Lemma 2.18 shows that the simplicial subspace $\Psi_A(\mathbb{D}^\sharp)(-, n) \hookrightarrow X(-, n)$ satisfies the Segal conditions and is an equivalence on spaces of objects. This implies that $\Psi_A(\mathbb{D}^\sharp)(-, n)$ is a category, so that $\Psi_A(\mathbb{D}^\sharp)$ is a double category.

Let us now show that the map of marked double categories $\zeta^*: \mathbb{D} \rightarrow \Psi_A(\mathbb{D}^\sharp)$ is an equivalence. Since ζ induces an isomorphism $A[0, 1]^+ \xrightarrow{\sim} [0, 1]^\sharp$, ζ^* is an equivalence on marked vertical arrows. It then remains to verify that the map on $(1, 1)$ -simplices is an equivalence. Unraveling the definitions, a $(1, 1)$ -simplex in $\Psi_A(\mathbb{D}^\sharp)$ is a diagram in \mathbb{D}

$$\begin{array}{ccccc} x_0 & \longrightarrow & y_0 & \xrightarrow{\sim} & z_0 \\ \downarrow & & \downarrow & & \downarrow \\ x_1 & \longrightarrow & y_1 & \xrightarrow{\sim} & z_1 \end{array}$$

where the right square covers a vertical arrow in \mathbb{C} . The map $\zeta^*: \mathbb{D}(1, 1) \rightarrow \Psi_A(\mathbb{D})(1, 1)$ is simply the inclusion of the diagrams of the above form where the right square is degenerate; this is an equivalence by completeness of \mathbb{D} .

η^* is an equivalence. Let us again consider the double category X from Lemma 2.18. For each fixed n , consider the mapping double category $X^{[0] \boxtimes \text{Fun}([1], [n])}$ and note that there is a diagram of simplicial spaces

$$\begin{array}{ccc} \Psi_B(\mathbb{D}^\sharp)(-, n) & \hookrightarrow & P \hookrightarrow X^{[0] \boxtimes \text{Fun}([1], [n])}(-, 0) \\ & & \downarrow \qquad \qquad \downarrow \\ & & \prod_{i=0}^n \Psi_A(\mathbb{D}^\sharp)(-, 0) \hookrightarrow \prod_{i=0}^n X^{\{ii\}}(-, 0) \end{array}$$

where P denotes the pullback. Since the bottom map is the inclusion of a wide subcategory (as we have seen above), P is a wide subcategory of $X^{[0] \boxtimes \text{Fun}([1], [n])}(-, 0)$. Unraveling the definitions, we then see that the inclusion $\Psi_B(\mathbb{D}^\sharp)(-, n) \hookrightarrow P$ is the inclusion of a full subcategory: it is the inclusion of the full subcategory on those diagrams $[0] \boxtimes \text{Fun}([1], [n]) \rightarrow X$ such that each vertical arrow $ik \rightarrow jk$ in $\text{Fun}([1], [n])$ is sent to a cartesian arrow in $X(0, -) \simeq \mathbb{D}(0, -)$.

It follows that $\Psi_B(\mathbb{D}^\sharp)$ satisfies the (complete) Segal conditions in the horizontal direction. It therefore suffices to verify that $\eta^*: \Psi_B(\mathbb{D}^\sharp)(1, -) \rightarrow \Psi_A(\mathbb{D})(1, -)$ is an equivalence of simplicial

spaces. Unraveling the definitions, a $(1, n)$ -simplex in $\Psi_B(\mathbb{D}^\sharp)$ is given by the following diagram.

Here each of the slices labeled by x, y and z is a diagram in $\mathbb{D}(0, -)$ parametrized by $\text{Fun}([1], [n])$, while the maps $x_{ij} \rightarrow y_{ij} \rightarrow z_{ij}$ define a map $\text{Ar}[1] \rightarrow \mathbb{D}$. Furthermore, the condition that $B[1, n] = \text{Ar}[1] \times ([0] \boxtimes \text{Fun}([1], [n])) \rightarrow \mathbb{D}$ preserves marked arrows translates into the following conditions.

- (a) The maps $y_{ii} \rightarrow z_{ii}$ are equivalences for all $0 \leq i \leq n$, as indicated.
- (b) All downward maps in the x -slice and the z -slice are p -cartesian, indicated by $\square \circlearrowright$.

The map $\eta^*: \Psi_B(\mathbb{D}^\sharp)(1, n) \rightarrow \Psi_A(\mathbb{D}^\sharp)(1, n)$ restricts the above diagram to the diagonal face, spanned by all x_{ii}, y_{ii} and z_{ii} . We have to prove that there is a unique way to extend the above diagram (with properties (a) and (b)) from this diagonal face (covering the map $[m, n] \rightarrow \mathbb{C}$). Let us first look at the x - and z -slices, both of which correspond to diagrams as follows.

$$\begin{array}{ccc} \text{Fun}([1], [n]) & \xrightarrow{x, z} & \mathbb{D}(0, -) \\ \text{dom} \downarrow & & \downarrow p \\ [n] & \longrightarrow & \mathbb{C}(0, -) \end{array}$$

Let $\text{cst}: [n] \hookrightarrow \text{Fun}([1], [n])$ be the inclusion of the diagonal. For every $ij \in \text{Fun}([1], [n])$, the comma category ij/cst admits an initial object, corresponding to the map $ij \rightarrow jj$. Using this, one sees that condition (b) is equivalent to the assertion that the two diagrams $x, z: \text{Fun}([1], [n]) \rightarrow \mathbb{D}$ are relative right Kan extensions of their restriction to the diagonal.

Consequently, starting from the diagonal face there is a unique way to fill the x - and z -slices in (3.20) so that condition (b) holds [Lur09a, Proposition 4.3.2.1]. Using that $\mathbb{D} \rightarrow \mathbb{C}$ is a left fibration in the horizontal direction, there is then a unique extension to a horizontal transformation between the x - and y -slices ($x \Rightarrow y$): $[1] \boxtimes \text{Fun}([1], [n]) \rightarrow \mathbb{D}$. Finally, the natural transformation $y_{ii} \rightarrow z_{ii}$ on the diagonal then extends uniquely to a natural transformation $y_{ij} \rightarrow z_{ij}$, since the target is the relative right Kan extension of its restriction to the diagonal. We conclude that $\eta^*: \Psi_B(\mathbb{D}^\sharp) \rightarrow \Psi_A(\mathbb{D}^\sharp)$ is an equivalence on the underlying bisimplicial spaces. Finally, it identifies the marking since $A[0, 1]^+ \simeq B[0, 1]^+ \simeq [0, 1]^\sharp$. \square

Our next goal is to describe the composite functor $\Psi^\top \circ \Psi^\perp$ in terms of Construction 3.10.

CONSTRUCTION 3.21. For each $m, n \geq 0$, let $I[m, n]$ denote the $(1, 1)$ -category whose objects are quadruples of maps of linear orders

$$[a] \xrightarrow{\alpha} [m], \quad [b] \star [a] \xrightarrow{\beta} [n], \quad [t] \star [s] \xrightarrow{\gamma} [a], \quad [t] \xrightarrow{\delta} [b]. \tag{3.22}$$

Given two such quadruples, a map between them is given by a quadruple of maps

$$[a] \longrightarrow [a'], \quad [b] \longrightarrow [b'], \quad [s] \longrightarrow [s'], \quad [t] \longrightarrow [t']$$

that intertwines $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$. Note that $I[m, n]$ depends functorially on $[m, n] \in \mathbf{\Delta}^{\times 2}$, by postcomposing α and β .

Now consider the natural map of bisimplicial spaces (in fact, sets)

$$\pi : T[m, n] = \operatorname{colim}_{I[m, n]} [s, t] \longrightarrow [m, n]$$

defined as follows: to each $[s, t] \longrightarrow T[m, n]$ determined by a tuple $(\alpha, \beta, \gamma, \delta)$, we associate the map $\sigma \boxtimes \tau_0 : [s, t] \longrightarrow [m, n]$ determined by

$$\sigma : [s] \longrightarrow [t] \star [s] \xrightarrow{\gamma} [a] \xrightarrow{\alpha} [m], \quad \tau_0 : [t] \xrightarrow{\delta} [b] \longrightarrow [b] \star [a] \xrightarrow{\beta} [n].$$

We view $T[m, n]$ as a marked bisimplicial space by marking those $(0, 1)$ -simplices corresponding to a quadruple $(\alpha, \beta, \gamma, \delta)$ (with $s = 0, t = 1$) such that either:

- (a) the map $\gamma : [1] \star [0] \longrightarrow [a]$ factors over a 0-simplex of $[a]$;
- (b) the map $\beta : [b] \star [a] \longrightarrow [n]$ factors over a 0-simplex of $[n]$.

For every map $[m, n] \longrightarrow [m', n']$, the induced map $T[m, n] \longrightarrow T[m', n']$ preserves such marked arrows.

Finally, let $T[0, 1]^+$ denote $T[0, 1]$, where *all* vertical arrows are marked.

LEMMA 3.23. *There is a natural equivalence $\Psi_T \simeq \Psi_{K'} \circ \Psi_K$.*

Proof. Recall from Construction 3.10 that $\Psi_{K'}$ arises as the right adjoint to a functor $\Phi_{K'}$. This functor $\Phi_{K'} : \operatorname{Fun}^\sharp([1], \operatorname{bis}\mathcal{S}_1) \longrightarrow \operatorname{Fun}^\sharp([1], \operatorname{bis}\mathcal{S}_\rightarrow)$ is uniquely determined by the fact that it preserves colimits and sends $\emptyset \longrightarrow [m, n]^\sharp$ to $\emptyset \longrightarrow [m, n]^\sharp$ and

$$\begin{array}{ccc} [m, n] & \xrightarrow{\quad} & K'[m, n] & & [0, 1]^\sharp & \xrightarrow{\quad} & K'[1, 0]^+ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [m, n]^\sharp & & [m, n]^\sharp & & [0, 1]^\sharp & & [0, 1]^\sharp \end{array}$$

Here $K'[m, n]$ was described in Variant 3.16. Using Example 2.6, we can write $K'[m, n]$ as a (canonical) colimit of bisimplices

$$K'[m, n] = \operatorname{colim}_{\substack{\alpha : [a] \rightarrow [m] \\ \beta : [b] \star [a] \rightarrow [n]}} [a, b].$$

Unraveling the definitions, the marked $(1, 0)$ -simplices of $K'[m, n]$ correspond precisely to tuples of $\alpha : [1] \longrightarrow [m]$ and $\beta : [0] \star [1] \longrightarrow [n]$ where β is degenerate. Furthermore, all $(1, 0)$ -simplices in $K'[0, 1]$ are marked.

We can apply the same reasoning to the left adjoint $\Phi_K \circ \Phi_{K'}$: this functor preserves colimits and sends $\emptyset \longrightarrow [m, n]^\sharp$ to $\emptyset \longrightarrow [m, n]^\sharp$. It is therefore uniquely determined by its values on

$[m, n] \longrightarrow [m, n]^\sharp$ and $[0, 1]^\sharp \longrightarrow [0, 1]^\sharp$. The value on $[m, n]$ is given by

$$\Phi_K(K'[m, n]) = \operatorname{colim}_{\substack{\alpha: [a] \rightarrow [m] \\ \beta: [b] \star [a] \rightarrow [n]}} K[a, b] \simeq \operatorname{colim}_{\substack{\alpha: [a] \rightarrow [m] \\ \beta: [b] \star [a] \rightarrow [n]}} \left(\operatorname{colim}_{\substack{\gamma: [t] \star [s] \rightarrow [a] \\ \delta: [t] \rightarrow [b]}} [s, t] \right).$$

In other words, it is precisely given by $T[m, n]$; unraveling the definitions shows that the marking coincides with that of Construction 3.21. Likewise, $\Phi_K(K'[0, 1]^+)$ can be identified with $T[0, 1]$ where all vertical arrows are marked. It follows that $\Phi_K \circ \Phi_{K'} \simeq \Phi_T$, so that by adjunction $\Psi_T \simeq \Psi_{K'} \circ \Psi_K$. \square

Proof (of Theorem 3.1). By Proposition 3.14 (and Variant 3.16), we have functors

$$\Psi^\perp : \operatorname{Fib}^{\text{left, cart}} \longrightarrow \operatorname{Fib}^{\text{cocart, right}} \quad \text{and} \quad \Psi^\top : \operatorname{Fib}^{\text{cocart, right}} \longrightarrow \operatorname{Fib}^{\text{left, cart}}.$$

To see that they are mutually inverse, it suffices to prove that $\Psi^\top \circ \Psi^\perp \simeq \operatorname{id}$. Indeed, in this case, conjugating by the functor $\mathbb{C} \mapsto (\mathbb{C}^{\text{rev}})^{(1,2)\text{-op}}$ provides the desired equivalence $\Psi^\perp \circ \Psi^\top \simeq \operatorname{id}$. We construct the desired natural equivalence as a zigzag

$$\operatorname{id} \xrightarrow[\simeq]{\zeta^*} \Psi_A \xleftarrow[\simeq]{\eta^*} \Psi_B \xrightarrow{\theta^*} \Psi_T \simeq \Psi^\top \circ \Psi^\perp.$$

Here the first two equivalences follow from Proposition 3.19 and the last identification follows from Lemma 3.23. The remaining natural transformation θ^* is induced by a natural map $\theta: T[m, n] \longrightarrow B[m, n]$ of vertically marked bisimplicial spaces over $[m, n]$. This map is defined as follows: for every (s, t) -simplex of $T[m, n]$, determined by a quadruple $(\alpha, \beta, \gamma, \delta)$ as in (3.22), consider the map

$$\tau_1 : [t] \longrightarrow [t] \star [s] \xrightarrow{\gamma} [a] \longrightarrow [b] \star [a] \xrightarrow{\beta} [n].$$

Clearly $\tau_0(i) \leq \tau_1(i)$ for all i , where τ_0 is defined in Construction 3.21. We therefore obtain a map $(\tau_0 \leq \tau_1) : [s, t] \longrightarrow [0] \boxtimes \operatorname{Fun}([1], [n])$. Likewise, note that $\alpha \circ \gamma$ defines a map $\alpha \circ \gamma : [s, t] \longrightarrow \operatorname{Ar}[m]$ (cf. Example 2.6). Combining these two, we obtain a natural map

$$(\alpha \circ \gamma, \tau_0 \leq \tau_1) : [s, t] \longrightarrow \operatorname{Ar}[m] \times \left([0] \boxtimes \operatorname{Fun}([1], [n]) \right) = B[m, n]$$

for every (s, t) -simplex in $T[m, n]$. This defines the desired natural map of bisimplicial spaces $\theta: T[m, n] \longrightarrow B[m, n]$. Note that θ sends a marked $(0, 1)$ -simplex in $T[m, n]$ (Construction 3.21) to a marked $(0, 1)$ -simplex in $B[m, n]$ (Construction 3.18). Finally, note that in both $T[0, 1]^\sharp$ and $B[0, 1]^\sharp$, all vertical arrows are marked, so θ induces a natural map $T[0, 1]^\sharp \longrightarrow B[0, 1]^\sharp$ as well.

For any (left, cart)-fibration $p: \mathbb{D}^\sharp \longrightarrow \mathbb{C}^\sharp$, restriction along θ therefore defines the following natural map of marked bisimplicial spaces.

$$\begin{array}{ccc} \Psi_B(\mathbb{D}^\sharp) & \xrightarrow{\theta^*} & \Psi_T(\mathbb{D}^\sharp) \\ & \searrow & \swarrow \\ & \mathbb{C}^\sharp & \end{array}$$

Since both downwards-pointing maps are (left, cart)-fibrations (with marked arrows being the cartesian arrows), it suffices to verify that θ^* induces equivalences between spaces of vertices and vertical arrows. On vertices, note that $T[0, 0] \simeq B[0, 0] \simeq [0, 0]$, so that the spaces of objects of $\Psi_B(\mathbb{D}^\sharp)$ and $\Psi_T(\mathbb{D}^\sharp)$ are both equivalent to the space of objects of \mathbb{D} .

Unraveling the definitions, a vertical arrow in $\Psi_T(\mathbb{D}^\natural) \simeq \Psi^\top(\Psi^\perp(\mathbb{D}^\natural))$ covering a vertical arrow $c_0 \twoheadrightarrow c_1$ in \mathbb{C}^\sharp is given by a diagram in \mathbb{D} of the form

$$d_0 \longrightarrow d'_0 \twoheadrightarrow d''_0 \twoheadrightarrow d_1.$$

Here the first two maps live in the fiber over c and the vertical arrow $d''_0 \twoheadrightarrow d_1$ is a cartesian lift of $c_0 \twoheadrightarrow c_1$. Since $\mathbb{D} \rightarrow \mathbb{C}$ was a horizontal left fibration, this means that the arrow $d_0 \rightarrow d'_0$ is an equivalence.

On the other hand, a vertical arrow in $\Psi_B(\mathbb{D}^\natural)$ is a sequence $d'_0 \twoheadrightarrow d''_0 \twoheadrightarrow d_1$ of vertical maps where the first map lives in the fiber over c and $d''_0 \twoheadrightarrow d_1$ is cartesian. The map θ^* can be identified with the map sending this sequence to $d_0 = d'_0 \twoheadrightarrow d''_0 \twoheadrightarrow d_1$; it is then an equivalence by the completeness of \mathbb{D} . □

4. Recollections on higher categories

In the remainder of the text, we will use Theorem 3.1 to establish a version of straightening and unstraightening for cocartesian fibrations of d -categories. To facilitate this, we will start in this section by recalling some preliminary results on d -categories; our default model for d -categories will be in terms of iterated Segal spaces [BS21]. Section 4.2 provides a reminder on point-set models for d -categories; its main purpose is to allow us to construct explicit examples of higher categories (such as the $(d + 1)$ -category of d -categories) from enriched categories.

Sections 4.3 and 4.4 describe the theory of copresheaves (of d -categories) on $(d + 1)$ -categories; this is mostly a recollection of the work of Boavida [Boa18]. For our purposes, it will be convenient to reformulate the results of [Boa18] as Proposition 4.40: this provides a natural equivalence between the *external* definition of copresheaves over a $(d + 1)$ -category \mathcal{C} , in terms of certain fibrations of $(d + 1)$ -fold simplicial spaces $X \rightarrow \mathcal{C}$, and the *internal* definition of copresheaves, as maps of $(d + 1)$ -categories $\mathcal{C} \rightarrow \mathbf{Cat}_d$.

4.1 Higher categories via Segal spaces

Recall the iterated Segal space model for d -categories [BS21].

DEFINITION 4.1. A d -category \mathcal{C} is a d -fold category with the property that for each \vec{n}_{k-1} in $\Delta^{\times k-1}$, the $(d - k)$ -fold category $\mathcal{C}(\vec{n}_{k-1}, 0, -, \dots, -)$ is a space, i.e. constant. We will write $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}^{\otimes d}$ for the full subcategory spanned by the d -categories.

Remark 4.2. The above definition of a d -category is *a priori* more restrictive than the usual one from [BS21]: we ask all $\mathcal{C}(\vec{n}_{k-1}, -, \vec{n}_{d-k})$ to be complete, whereas this is usually only required if $\vec{n}_{d-k} = \vec{0}_{d-k}$. It follows from [JS17, Lemma 2.8] that the two definitions coincide, essentially by an Eckmann–Hilton-type argument.

If \mathcal{C} is a d -category, recall that the (constant $(d - 1)$ -fold simplicial) space $\mathcal{C}(0)$ describes the *space of objects* of \mathcal{C} . Given two objects $x_0, x_1 \in \mathcal{C}$, we will write

$$\mathrm{Map}_{\mathcal{C}}(x_0, x_1) = \mathcal{C}(1) \times_{\mathcal{C}(\{0\}) \times \mathcal{C}(\{1\})} \{(x_0, x_1)\}$$

for the *mapping $(d - 1)$ -category* between them. For any triple of objects x_0, x_1, x_2 , there is a composition map, determined by the following commuting square.

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x_1, x_2) \times \mathrm{Map}_{\mathcal{C}}(x_0, x_1) & \xrightarrow{\circ} & \mathrm{Map}_{\mathcal{C}}(x_0, x_2) \\ \sim \uparrow & & \uparrow = \\ \mathcal{C}(2) \times_{\mathcal{C}(\{0\}) \times \mathcal{C}(\{1\}) \times \mathcal{C}(\{2\})} \{(x_0, x_1, x_2)\} & \xrightarrow{d_1} & \mathcal{C}(1) \times_{\mathcal{C}(\{0\}) \times \mathcal{C}(\{1\})} \{(x_0, x_2)\} \end{array}$$

Here the left vertical map is an equivalence by the Segal conditions.

To efficiently deal with the additional constancy conditions appearing in Definition 4.1, let us introduce the following notation.

Notation 4.3. For a category \mathcal{J} , let us denote by $\lambda: \mathbf{\Delta} \times \mathcal{J} \longrightarrow \mathbf{\Delta}_{\mathcal{J}}$ the $((\infty, 1)$ -categorical) localization of $\mathbf{\Delta} \times \mathcal{J}$ at the morphisms in $\{0\} \times \mathcal{J}$. Recall that restriction and Kan extension determine adjoint pairs

$$\lambda_!: \text{Fun}((\mathbf{\Delta} \times \mathcal{J})^{\text{op}}, \mathcal{S}) \xrightarrow{\quad} \text{Fun}(\mathbf{\Delta}_{\mathcal{J}}^{\text{op}}, \mathcal{S}) : \lambda^* \qquad \lambda^*: \text{Fun}(\mathbf{\Delta}_{\mathcal{J}}^{\text{op}}, \mathcal{S}) \xleftarrow{\quad} \text{Fun}((\mathbf{\Delta} \times \mathcal{J})^{\text{op}}, \mathcal{S}) : \lambda_*$$

that exhibit the category of presheaves on $\mathbf{\Delta}_{\mathcal{J}}$ both as a localization and a colocalization of the category of presheaves on $\mathbf{\Delta} \times \mathcal{J}$.

LEMMA 4.4. *Let \mathcal{J} be a $(1, 1)$ -category with a terminal object t and let $\lambda: \mathbf{\Delta} \times \mathcal{J} \longrightarrow \mathbf{\Delta}_{\mathcal{J}}$ denote the localization at the morphisms in $\{0\} \times \mathcal{J}$. Then the following assertions hold:*

- (1) $\mathbf{\Delta}_{\mathcal{J}}$ is a $(1, 1)$ -category with a terminal object;
- (2) for any $X: (\mathbf{\Delta} \times \mathcal{J})^{\text{op}} \longrightarrow \mathcal{S}$, consider the natural map of presheaves on $\mathbf{\Delta} \times \mathcal{J}$:

$$X \times_{i_* i^* X} i_*(X(0, t)) \longrightarrow X$$

where i_* denote the right Kan extension along $i: \{0\} \times \mathcal{J} \hookrightarrow \mathbf{\Delta} \times \mathcal{J}$ and $i^* X \longrightarrow X(0, t)$ denotes the natural map from the presheaf $X(0, -)$ on \mathcal{J} to the constant presheaf with value $X(0, t)$; this map exhibits the counit of the adjoint pair (λ^*, λ_*) .

In fact, in this case $\mathbf{\Delta}_{\mathcal{J}}$ can be identified with a (non-full) subcategory of $\mathbf{\Delta} \wr \mathcal{J}$, given by the image of the functor $\delta: \mathbf{\Delta} \times \mathcal{J} \longrightarrow \mathbf{\Delta} \wr \mathcal{J}$ from [Ber07].

Proof. For each $([m], i) \in \mathbf{\Delta} \times \mathcal{J}$, let $h_{[m], i} = \text{Map}(-, ([m], i))$ be the representable presheaf. For assertion (1), consider the following (homotopy) pushout of (space-valued) presheaves.

$$\begin{array}{ccc} \coprod_{[0] \rightarrow [m]} h_{[0], i} & \longrightarrow & h_{[m], i} \\ \downarrow & & \downarrow \\ \coprod_{[0] \rightarrow [m]} h_{[0], t} & \longrightarrow & F \end{array}$$

Here the top map is given pointwise by the inclusion of the subset of maps $([n], j) \longrightarrow ([m], i)$ such that the map $[n] \longrightarrow [m]$ factors over $[0]$. Using this, it follows immediately that the presheaf F takes values in sets and sends maps in $\{0\} \times \mathcal{J}$ to isomorphisms. Consequently, $F \simeq \lambda^* G$ for some set-valued presheaf G on $\mathbf{\Delta}_{\mathcal{J}}$.

On the other hand, the left Kan extension functor $\lambda_!$ sends the left vertical map to an equivalence, so that we find $h_{\lambda([m], i)} \simeq \lambda_! h_{[m], i} \simeq \lambda_! F$. We therefore find that the representable presheaf $h_{\lambda([m], i)}$ is equivalent to the set-valued presheaf $\lambda_! \lambda^* G \simeq G$, so that $\mathbf{\Delta}_{\mathcal{J}}$ is a $(1, 1)$ -category. In the special case where $[m] = 0$, we see that F is the terminal presheaf, so that the image of $([0], i)$ defines a terminal object in $\mathbf{\Delta}_{\mathcal{J}}$.

For assertion (2), let us abbreviate $Y = X \times_{i_* i^* X} i_*(X(0, t))$. Since i is fully faithful, $i^* i_* \simeq \text{id}$ and the restriction of Y along $i: \{0\} \times \mathcal{J} \longrightarrow \mathbf{\Delta} \times \mathcal{J}$ coincides with the constant presheaf with value $X(0, t)$. This means that Y is contained in the essential image of λ^* or, equivalently, that Y is a $\lambda^* \lambda_*$ -colocal object.

It therefore remains to verify that the projection $Y \longrightarrow X$ is a colocal equivalence, i.e. that it is sent to an equivalence by λ_* . Since $Y \longrightarrow X$ is the base change of the map $i_*(X(0, t)) \longrightarrow i_* i^* X$, it suffices to verify that $\lambda_* i_*(X(0, t)) \longrightarrow \lambda_* i_* i^* X$ is an equivalence. By part (1), the functor λi decomposes as $\mathcal{J} \longrightarrow * \longrightarrow \mathbf{\Delta}_{\mathcal{J}}$, where the second functor is the inclusion of the terminal

object. The result then follows from the fact that the natural map

$$X(0, t) \simeq \lim_{j \circ p} X(0, t) \longrightarrow \lim_{j \circ p} i^* X$$

is an equivalence since t was the terminal object of \mathcal{J} . □

Notation 4.5. We inductively define Δ_d by $\Delta_0 = *$ and $\Delta_d = \Delta_{\Delta_{d-1}}$, so that there is the following composition of localization functors of $(1, 1)$ -categories:

$$\delta_d: \Delta^{\times d} \longrightarrow \Delta^{\times d-2} \times \Delta_2 \longrightarrow \cdots \longrightarrow \Delta \times \Delta_{d-1} \longrightarrow \Delta_d. \tag{4.6}$$

By construction, restriction along δ_d determines a fully faithful inclusion

$$\delta_d^*: \text{Fun}(\Delta_d^{\text{op}}, \mathcal{S}) \hookrightarrow \text{Fun}(\Delta^{\times d, \text{op}}, \mathcal{S})$$

whose essential image consists of d -fold simplicial spaces X such that each $(d - k)$ -fold simplicial space $X(\vec{n}_{k-1}, 0, -, \dots, -)$ is constant. We will systematically omit δ_d^* and simply identify presheaves on Δ_d with d -fold simplicial spaces using this inclusion. Likewise, we can identify presheaves on Δ_d with simplicial object in presheaves on Δ_{d-1} which are constant in simplicial degree 0, etc.

In these terms, we have the following pullback square of fully faithful functors.

$$\begin{array}{ccc} \text{Cat}_d & \hookrightarrow & \text{Cat}^{\otimes d} \\ \downarrow & & \downarrow \\ \text{Fun}(\Delta_d^{\text{op}}, \mathcal{S}) & \xrightarrow{\delta_d^*} & \text{Fun}(\Delta^{\times d, \text{op}}, \mathcal{S}) \end{array}$$

Since the bottom and the right functors are right adjoint functors between presentable categories, the inclusion $\text{Cat}_d \rightarrow \text{Cat}^{\otimes d}$ admits a left adjoint. It also admits a right adjoint, by the following observation.

LEMMA 4.7. *Let $r_d: \text{Fun}(\Delta^{\times d, \text{op}}, \mathcal{S}) \rightarrow \text{Fun}(\Delta_d, \mathcal{S})$ be the right adjoint to δ_d^* , taking right Kan extension along $\delta_d: \Delta^{\times d, \text{op}} \rightarrow \Delta_d^{\text{op}}$. Then r_d maps the full subcategory $\text{Cat}^{\otimes d}$ to Cat_d .*

Proof. We proceed by induction, the case $d = 1$ being clear. Now let us take $d > 1$ and assume that r_{d-1} sends $(d - 1)$ -fold categories to d -categories. Suppose that $X: \Delta^{\times d, \text{op}} \rightarrow \mathcal{S}$ is a d -fold category and factor the map δ_d as

$$\Delta^{\times d} \xrightarrow{\text{id} \times \delta_{d-1}} \Delta \times \Delta_{d-1} \xrightarrow{\lambda} \Delta_{\Delta_{d-1}} = \Delta_d.$$

The right Kan extension along the functor $\text{id} \times \delta_{d-1}$ sends X to the simplicial object $X': \Delta^{\text{op}} \rightarrow \text{Fun}(\Delta_{d-1}^{\text{op}}, \mathcal{S})$ given by $X'(n) = r_{d-1}(X(n))$. Each $X'(n)$ is then a $(d - 1)$ -category by inductive hypothesis and the simplicial object X' satisfies the complete Segal conditions since r_{d-1} is a right adjoint.

Next, we compute $r_d(X) = \lambda_*(X')$ using Lemma 4.4(2). Denoting by $i: \{0\} \times \Delta_{d-1} \hookrightarrow \Delta \times \Delta_{d-1}$ the inclusion, we have that

$$r_d(X) = X' \times_{i_* i^* X'} i_*(X'(\vec{0}_d)),$$

at least after precomposing with the localization $\lambda: \Delta \times \Delta_{d-1} \rightarrow \Delta_d$. The value of this fiber product at $[m] \in \Delta$ is given by the presheaf on Δ_{d-1} of the form

$$r_d(X)(m, -) = X'(m, -) \times_{X'(0, -)^{\times m+1}} X'(0, \vec{0}_{d-1})^{\times m+1}.$$

Here the map $X'(m, -) \rightarrow X'(0, -)^{\times m+1}$ arises from the vertex inclusions $\{k\} \hookrightarrow [m]$ for all $k \in [m]$. This implies that $r_d(X)$ satisfies the Segal condition in the first simplicial direction (as the

spine inclusions induce bijections on objects). Furthermore, each $r_d(X)(m)$ is a $(d - 1)$ -category, since all terms in the above pullback diagram are $(d - 1)$ -categories.

It remains to verify the completeness condition. By Remark 4.2, it suffices to verify the completeness condition after setting the last variables equal to $\vec{0}_{d-1}$. But in that case the above formula shows that $r_d(X)(-, \vec{0}_{d-1}) \simeq X'(-, \vec{0}_{d-1})$, for which we already verified that it was a complete Segal space. \square

Remark 4.8. In fact, one sees that for any d -fold category \mathbb{C} , the counit map $r_d(\mathbb{C}) \hookrightarrow \mathbb{C}$ is a d -fold simplicial subspace, given in degree (m_1, \dots, m_d) by the path components x satisfying (inductively) the following two conditions:

- (1) x is contained in $r_{d-1}(\mathbb{C}(m_1)) \hookrightarrow \mathbb{C}(m_1)$;
- (2) for each $i = 0, \dots, m_1$, the image of x in $\mathbb{C}(\{i\}, m_2, \dots, m_d)$ is contained in the path components $\mathbb{C}(\{i\}, 0, \dots, 0) \subseteq \mathbb{C}(\{i\}, m_2, \dots, m_d)$.

COROLLARY 4.9. *The category Cat_d is cartesian closed. For any two d -categories \mathcal{C} and \mathcal{D} , the internal mapping object $\text{Fun}_d(\mathcal{C}, \mathcal{D})$ in Cat_d is the simplicial subspace of the internal mapping object $\mathcal{D}^{\mathcal{C}}$ in d -fold simplicial spaces given by*

$$\text{Fun}_d(\mathcal{C}, \mathcal{D}) = r_d(\mathcal{D}^{\mathcal{C}}) \subseteq \mathcal{D}^{\mathcal{C}}.$$

Example 4.10. Let \mathcal{A} be a $(d - 1)$ -category. We define $[1]_{\mathcal{A}}$ to be the d -category associated to the d -fold category $[1] \boxtimes \mathcal{A}$ by the left adjoint of Lemma 4.7. In other words, maps of d -categories $[1]_{\mathcal{A}} \rightarrow \mathcal{C}$ are equivalent to maps of $(d - 1)$ -categories $\mathcal{A} \rightarrow \mathcal{C}(1)$. Explicitly, $[1]_{\mathcal{A}}$ is the d -fold simplicial space $[1] \boxtimes \mathcal{A} \amalg_{\{0,1\} \boxtimes \mathcal{A}} \{0, 1\}$, i.e. it is a d -category with two objects $0, 1$ with trivial endomorphisms and $\text{Map}_{[1]_{\mathcal{A}}}(0, 1) = \mathcal{A}$.

If \mathcal{C} is a d -category, then the d -category

$$\text{Fun}_d([1]_{\mathcal{A}}, \mathcal{C}) \simeq r_d(\mathcal{C}^{[1] \boxtimes \mathcal{A}})$$

can be described as follows. Its space of objects is given by the space of maps $\alpha: [1]_{\mathcal{A}} \rightarrow \mathcal{C}$, i.e. of tuples $\alpha_0, \alpha_1 \in \mathcal{C}$ and $\alpha: \mathcal{A} \rightarrow \text{Map}_{\mathcal{C}}(\alpha_0, \alpha_1)$. Unraveling the definitions, one sees that the $(d - 1)$ -category of maps between two such $\alpha, \beta: [1]_{\mathcal{A}} \rightarrow \mathcal{C}$ is given by the following pullback.

$$\begin{array}{ccc} \text{Map}_{\text{Fun}_d([1]_{\mathcal{A}}, \mathcal{C})}(\alpha, \beta) & \longrightarrow & \text{Map}_{\mathcal{C}}(\alpha_0, \beta_0) \\ \downarrow & & \downarrow \beta_* \\ \text{Map}_{\mathcal{C}}(\alpha_1, \beta_1) & \xrightarrow{\alpha^*} & \text{Fun}_{d-1}(\mathcal{A}, \text{Map}_{\mathcal{C}}(\alpha_0, \beta_1)) \end{array}$$

Remark 4.11. There is a fully faithful functor $\iota: \text{Cat}_{d-1} \hookrightarrow \text{Cat}_d$, sending a $(d - 1)$ -category \mathcal{C} to the d -fold simplicial space $\iota(\mathcal{C})(m_1, \dots, m_d) = \mathcal{C}(m_1, \dots, m_{d-1})$ which is constant in the last variable. This functor has a left adjoint $|-|$ sending a d -fold category \mathcal{C} to the $(d - 1)$ -fold category

$$|\mathcal{C}|(m_1, \dots, m_{d-1}) = \text{colim}_{m_d \in \Delta^{\text{op}}} \mathcal{C}(m_1, \dots, m_d)$$

and a right adjoint \mathfrak{c}_{d-1} given by $\mathfrak{c}_{d-1}(\mathcal{C})(m_1, \dots, m_{d-1}) = \mathcal{C}(m_1, \dots, m_{d-1}, 0)$, which we will refer to as the $(d - 1)$ -core.

4.2 Point-set models

To produce examples of d -categories, it will be more useful to think of d -categories as categories enriched in $(d - 1)$ -categories. This is probably best done using the theory of enriched ∞ -categories [GH15, Hin20]. We will take a more rigid, model-categorical approach instead

(which has the advantage of not relying on Lurie’s (un)straightening equivalence). Recall from Notation 4.5 how we view presheaves on Δ_d as d -fold simplicial spaces using the localizations $\Delta^{\times d} \rightarrow \Delta \times \Delta_{d-1} \rightarrow \Delta_d$.

DEFINITION 4.12. A functor $X: \Delta_d \rightarrow \mathcal{S}$ is said to be a d -categorical algebra if it satisfies the following conditions:

- (1) each $X(n): \Delta_{d-1} \rightarrow \mathcal{S}$ is a $(d - 1)$ -category (and $X(0)$ is a space);
- (2) X satisfies the Segal condition, i.e. $X(n) \rightarrow X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$ is an equivalence.

Remark 4.13. The category of d -categories can be identified with the full subcategory of the category of d -categorical algebras, on those d -categorical algebras that are furthermore complete [GH15, Lur09b]. The inclusion of d -categories into d -categorical algebras admits a left adjoint L_{DK} , which localizes at the class of Dwyer–Kan equivalences, i.e. maps $Y \rightarrow X$ with the following two properties (see [GH15, Lur09b]).

- (1) *Fully faithful*: $Y(1) \rightarrow X(1) \times_{X(0) \times X(0)} Y(0)^{\times 2}$ is an equivalence of $(d - 1)$ -categories.
- (2) *Essential surjectivity*: the induced functor on homotopy categories $ho(Y) \rightarrow ho(X)$ is essentially surjective.

In particular, for each d -categorical algebra X , the map $X \rightarrow L_{DK}(X)$ is a Dwyer–Kan equivalence. In fact, the explicit construction shows that $X \rightarrow L_{DK}(X)$ is not just essentially surjective, but that $X(0) \rightarrow L_{DK}(X)(0)$ is already surjective on π_0 (see [Lur09b, Proposition 1.2.27]).

DEFINITION 4.14. Recall that there is a localization functor $sSet \rightarrow \mathcal{S}$ inverting the weak equivalences for the Kan–Quillen model structure. We will say that a functor $X: \Delta_d^{op} \rightarrow sSet$ is a d -category model (respectively, d -categorical algebra model) if the composed functor $\Delta_d^{op} \rightarrow sSet \rightarrow \mathcal{S}$ is a d -category (respectively, d -categorical algebra).

CONSTRUCTION 4.15. The d -categorical model structure on $sSh(\Delta_d) = Fun(\Delta_d^{op}, sSet)$ is the left proper, combinatorial model structure characterized uniquely by the following properties [Lur09b, Proposition 1.5.4].

- (1) The cofibrations are the monomorphisms and, in particular, every object is cofibrant.
- (2) An object X is fibrant if and only if it is fibrant in the injective model structure and it is a d -category model.
- (3) A weak equivalence between d -category models is a levelwise weak equivalence of simplicial sets (in the Kan–Quillen model structure).

We will write Cat_d for the full subcategory of $sSh(\Delta_d)$ on the injectively fibrant d -category models.

Remark 4.16. The d -categorical model structure is a left Bousfield localization of the injective model structure on $sSh(\Delta_d)$. Since the latter is a model for the ∞ -category of functors $\Delta_d^{op} \rightarrow \mathcal{S}$ (see [Lur09a, Proposition A.3.4.13]), it follows immediately that Cat_d is equivalent to the localization of Cat_d at the d -categorical (equivalently, pointwise) weak equivalences.

Remark 4.17. The injective model structure on $sSh(\Delta_d)$ is a monoidal model structure for the cartesian product [Rez10, Proposition 2.9], which presents the cartesian monoidal structure on $Fun(\Delta_d^{op}, \mathcal{S})$. Corollary 4.9 shows that $Fun(\Delta_d^{op}, \mathcal{S}) \rightleftarrows Cat_d$ is a monoidal localization for the cartesian product; this implies that the d -categorical model structure on $sSh(\Delta_d)$ is monoidal as well. In particular, this implies that the category Cat_d is cartesian closed.

CONSTRUCTION 4.18. Let $\text{CatAlg}_d \subseteq \text{sPSh}(\Delta_d) \subseteq \text{Fun}(\Delta^{\text{op}}, \text{sPSh}(\Delta_{d-1}))$ denote the full subcategory spanned by the functors $X: \Delta^{\text{op}} \rightarrow \text{Cat}_{d-1}$ that are d -categorical algebra models, i.e. explicitly:

- each $X(n): \Delta_{d-1}^{\text{op}} \rightarrow \text{sSet}$ is an injectively fibrant $(d-1)$ -category model and $X(0)$ is constant on a Kan complex;
- X satisfies the Segal conditions in the first simplicial variable.

We will say that a map $Y \rightarrow X$ is a Dwyer–Kan equivalence if the corresponding map of categorical algebras is a Dwyer–Kan equivalence as in Remark 4.13. Furthermore, we will say that a map $Y \rightarrow X$ is an *isofibration* if:

- (1) each $Y(n) \rightarrow X(n)$ is a fibration in Cat_{d-1} , i.e. an injective fibration of simplicial presheaves on Δ_{d-1} ;
- (2) the induced map on homotopy categories $\text{ho}(Y) \rightarrow \text{ho}(X)$ is an isofibration.

With these classes of maps, CatAlg_d becomes a category of fibrant objects as in Appendix A. A map $Y \rightarrow X$ is an *acyclic fibration* if and only if each $Y(n) \rightarrow X(n)$ is an injective fibration, $Y(0) \rightarrow X(0)$ is surjective on π_0 and $Y(1) \rightarrow X(1) \times_{X(0) \times 2} Y(0)^{\times 2}$ is a weak equivalence. To factor a map $Y \rightarrow X$ in CatAlg_d into a weak equivalence followed by a fibration, one can simply factor it into a weak equivalence followed by a fibration with respect to the d -categorical model structure on $\text{sPSh}(\Delta_d)$; such fibrations are isofibrations by the right lifting property against $\{0\} \rightarrow H$, where H is the simplicial set from § 2.

CONSTRUCTION 4.19. Let us write $\text{Cat}(\text{Cat}_{d-1})$ for the $(1, 1)$ -category of categories (strictly) enriched over the cartesian monoidal category Cat_{d-1} . The *nerve* provides a fully faithful functor $N: \text{Cat}(\text{Cat}_{d-1}) \hookrightarrow \text{CatAlg}_d$, which sends an enriched category C to the simplicial diagram in Cat_{d-1} given by

$$N(C)(n) = \coprod_{c_0, \dots, c_n} \text{Map}_C(c_{n-1}, c_n) \times \dots \times \text{Map}_C(c_0, c_1).$$

In particular, $N(C)(0)$ is simply the set of objects of C . We will say that a map of enriched categories is a Dwyer–Kan equivalence, respectively, an isofibration, if its image under the nerve functor is such. This makes $\text{Cat}(\text{Cat}_{d-1})$ a category of fibrant objects.

More precisely, the category of categories enriched in $\text{sPSh}(\Delta_{d-1})$ carries a model structure [Lur09a, Proposition A.3.2.4], since the $(d-1)$ -categorical model structure on $\text{sPSh}(\Delta_{d-1})$ is excellent in the sense of [Lur09a, Definition A.3.2.16]. It follows from [Lur09a, Theorem A.3.2.24] that $\text{Cat}(\text{Cat}_{d-1})$ (with the above classes of Dwyer–Kan equivalences and isofibrations) is the category of fibrant objects associated to this model category.

PROPOSITION 4.20. *The three categories of fibrant objects considered above fit into a diagram of fully faithful inclusions*

$$\text{Cat}(\text{Cat}_{d-1}) \xrightarrow{N} \text{CatAlg}_d \longleftarrow \text{Cat}_d$$

that preserve fibrations and weak equivalences. The induced functors on localizations are equivalences, so that each of these relative categories is a model for Cat_d .

Proof. The right inclusion $\text{Cat}_d \rightarrow \text{CatAlg}_d$ has a homotopy inverse L_{DK} , given by taking a fibrant replacement of a d -categorical algebra model in the d -categorical model structure. The resulting objects are then related by a Dwyer–Kan equivalence.

Furthermore, the composite functor $L_{DK} \circ N: \text{Cat}(\mathbf{Cat}_{d-1}) \rightarrow \mathbf{Cat}_d$ can be identified with the composite

$$\text{Cat}(\text{sPSh}(\mathbf{\Delta}_{d-1})) \xrightarrow{N} \text{Seg}_{\text{sPSh}(\mathbf{\Delta}_{d-1})} \xrightarrow{\text{UnPre}} \text{sPSh}(\mathbf{\Delta}_d) \xrightarrow{(-)^{\text{fib}}} \mathbf{Cat}_d$$

where $\text{Seg}_{\text{sPSh}(\mathbf{\Delta}_{d-1})}$ is the category of Segal precategories from [Lur09b, §2]. Here the first functor N is a right Quillen equivalence preserving weak equivalences [Lur09b, Theorem 2.2.16, Remark 2.2.19]. The functor UnPre is a left Quillen equivalence preserving weak equivalences by [Lur09b, Propositions 2.3.1 and 2.3.9 and Lemma 2.3.14]: this uses the fact that the inclusion $\text{sPSh}(\mathbf{\Delta}_d) \hookrightarrow \text{Fun}(\mathbf{\Delta}^{\text{op}}, \text{sPSh}(\mathbf{\Delta}_{d-1}))$ is a Quillen equivalence between the d -categorical model structure and the ‘complete Segal model structure’ of [Lur09b], which detects cofibrations and weak equivalences. The last functor takes a fibrant replacement, so it follows that $L_{DK} \circ N$ induces an equivalence on localizations. \square

Remark 4.21. Let us write $\widehat{N}: \text{Cat}(\mathbf{Cat}_{d-1}) \rightarrow \mathbf{Cat}_d$ for the composite functor $L_{DK} \circ N$. By Remark 4.13, $\widehat{N}(\mathbf{C})$ is a d -category which comes with a Dwyer–Kan equivalence $N(\mathbf{C}) \rightarrow \widehat{N}(\mathbf{C})$ from the nerve of \mathbf{C} ; in particular, every object of $\widehat{N}(\mathbf{C})$ is homotopic to an object from \mathbf{C} and for any two objects in \mathbf{C} , $\text{Map}_{\mathbf{C}}(c_0, c_1)$ is a model for the $(d - 1)$ -category of maps between their images in $\widehat{N}(\mathbf{C})$.

DEFINITION 4.22. The (large) $(d + 1)$ -category \mathbf{Cat}_d of d -categories is given by $\widehat{N}(\mathbf{Cat}_d)$, where \mathbf{Cat}_d is enriched over itself as in Remark 4.17.

Remark 4.23. Note that the 1-category underlying the $(d + 1)$ -category \mathbf{Cat}_d arises from the simplicially enriched category of fibrant–cofibrant objects in the d -categorical model structure; in particular, it is equivalent to Cat_d (see [Lur09a, Lemma A.3.6.17]).

It follows from [Hau15] that our definition of \mathbf{Cat}_d (via point-set models) coincides with the definition of \mathbf{Cat}_d as the d -category-enriched ∞ -category associated to Cat_d , viewed as enriched over itself using the cartesian closed structure.

4.3 Copesheaves

If \mathcal{C} is a $(d + 1)$ -category, we define the $(d + 1)$ -category of *copesheaves* (of d -categories) of \mathcal{C} to be the functor $(d + 1)$ -category $\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d)$, i.e. as the internal mapping object in the category of (large) $(d + 1)$ -categories. This $(d + 1)$ -category can be presented explicitly in terms of the enriched categories model for $(d + 1)$ -categories.

PROPOSITION 4.24 [Lur09a, Proposition A.3.4.13]. *Let \mathbf{C} be a small \mathbf{Cat}_d -enriched category and denote by $\text{Fun}_{\text{enr}}(\mathbf{C}, \mathbf{Cat}_d)^\circ$ the \mathbf{Cat}_d -enriched category of enriched functors $\mathbf{C} \rightarrow \text{sPSh}(\mathbf{\Delta}_d)$ that are fibrant–cofibrant in the projective model structure (for the d -categorical model structure on simplicial presheaves). Then the map*

$$\text{ev}: \widehat{N}\left(\text{Fun}_{\text{enr}}(\mathbf{C}, \mathbf{Cat}_d)^\circ\right) \times \widehat{N}(\mathbf{C}) \longrightarrow \widehat{N}(\mathbf{Cat}_d)$$

exhibits $\widehat{N}(\text{Fun}_{\text{enr}}(\mathbf{C}, \mathbf{Cat}_d)^\circ)$ as the internal mapping object in the category of (large) $(d + 1)$ -categories.

In particular, using the point-set model of \mathbf{Cat}_d -enriched categories, a copesheaf $\mathcal{C} \rightarrow \mathbf{Cat}_d$ can be modeled by an enriched functor $\mathbf{C} \rightarrow \mathbf{Cat}_d$. In addition to this ‘internal’ definition of copesheaves, one can also describe copesheaves externally in terms of Segal objects, following [Boa18] (see also [Ras21] for an in-depth discussion).

DEFINITION 4.25. Let \mathcal{C} be a $(d + 1)$ -categorical algebra. A map of $(d + 1)$ -fold simplicial spaces $X \rightarrow \mathcal{C}$ is said to be a *Segal copresheaf* if the following two conditions hold.

- (1) The d -fold simplicial space $X(0)$ is a d -category.
- (2) For each $[n] \in \Delta$, there is the following pullback square of d -categories.

$$\begin{array}{ccc} X(n) & \longrightarrow & X(\{0\}) \\ \downarrow & & \downarrow \\ \mathcal{C}(n) & \longrightarrow & \mathcal{C}(\{0\}) \end{array}$$

Let us write $\text{coPSh}_d^{\text{Seg}} \subseteq \text{Fun}([1] \times \Delta^{\times d+1, \text{op}}, \mathcal{S})$ for the full subcategory spanned by the Segal copresheaves $X \rightarrow \mathcal{C}$ where \mathcal{C} is a $(d + 1)$ -category. The codomain projection $\pi: \text{coPSh}_d^{\text{Seg}} \rightarrow \text{Cat}_{d+1}$ is a cartesian fibration, since the pullback of a Segal copresheaf $X \rightarrow \mathcal{C}$ along a map of $(d + 1)$ -categories $\mathcal{C}' \rightarrow \mathcal{C}$ is again a Segal copresheaf. We will write $\text{coPSh}_d^{\text{Seg}}(\mathcal{C})$ for the fiber over \mathcal{C} .

WARNING 4.26. When $d = 0$, the domain of a Segal copresheaf over a 1-category is itself a 1-category [Boa18, Corollary 1.19]; in this case, Segal copresheaves can be identified with left fibrations (or, in the terminology of § 5, *0-cocartesian fibrations*). Note that for $d \geq 1$, the domain of a Segal copresheaf $X \rightarrow \mathcal{C}$ over a $(d + 1)$ -category is typically *not* itself a $(d + 1)$ -category: if the domain were a $(d + 1)$ -category, then all the fibers of the fibration would be spaces.

In the remainder of this section, we will recall the work of Boavida [Boa18], which shows that the category of Segal copresheaves $\text{coPSh}_d^{\text{Seg}}(\mathcal{C})$ coincides with the 1-category underlying $\text{Fun}_{d+1}(\mathcal{C}, \text{Cat}_d)$. We will make use of point-set models to compare with Proposition 4.24.

CONSTRUCTION 4.27. Consider the functor $\text{Cat}(\text{Cat}_d)^{\text{op}} \rightarrow \text{FibCat}$ sending each Cat_d -enriched category \mathcal{C} to the category $\text{Fun}_{\text{enr}}(\mathcal{C}, \text{Cat}_d)$ of enriched functors. Note that this is the full subcategory of fibrant objects in the projective model structure on $\text{Fun}(\mathcal{C}, \text{sPSh}(\Delta_d))$. We will write

$$\text{EnrFun}_d \longrightarrow \text{Cat}(\text{Cat}_d)$$

for its Grothendieck construction. Explicitly, an object $(\mathcal{C}, F) \in \text{EnrFun}_d$ is a Cat_d -enriched category \mathcal{C} together with an enriched functor $F: \mathcal{C} \rightarrow \text{Cat}_d$.

LEMMA 4.28. *The map $\text{EnrFun}_d \rightarrow \text{Cat}(\text{Cat}_d)$ is a homotopy cartesian fibration between categories of fibrant objects, in the sense of Definition A.5.*

Proof. We verify conditions (a) and (b) of Lemma A.2. For condition (a), note that for each weak equivalence $f: \mathcal{C} \rightarrow \mathcal{D}$, $f^*: \text{Fun}_{\text{enr}}(\mathcal{D}, \text{Cat}_d) \rightarrow \text{Fun}_{\text{enr}}(\mathcal{C}, \text{Cat}_d)$ is induced by a Quillen equivalence [Lur09a, Proposition A.3.3.8]. For condition (b), we use Remark A.3: we can factor any $f: \mathcal{C} \rightarrow \mathcal{D}$ into a trivial cofibration $i: \mathcal{C} \rightarrow \mathcal{C}'$ (in the model structure on enriched categories) followed by a fibration $p: \mathcal{C}' \rightarrow \mathcal{D}$. Then $i_!: \text{Fun}_{\text{enr}}(\mathcal{C}, \text{sPSh}(\Delta_d)) \rightleftarrows \text{Fun}_{\text{enr}}(\mathcal{C}', \text{sPSh}(\Delta_d)): i^*$ is a Quillen equivalence where $i_!$ preserves all weak equivalences. Indeed, this follows from the fact that $i_!$ is left Quillen for the injective model structure as well, as a consequence of [Lur09a, Remark A.2.9.27 and Proposition A.3.3.9]. □

CONSTRUCTION 4.29. Let $X: \Delta^{\text{op}} \rightarrow \text{Cat}_d$ be an object of CatAlg_{d+1} , i.e. X satisfies the Segal conditions and $X(0) \in \text{sPSh}(\Delta_d)$ is constant on a simplicial set. The category $\text{sPSh}(\Delta \times \Delta_d)_X$ can be endowed with the *projective covariant model structure*, in which:

- trivial fibrations are maps $Y \rightarrow Y'$ (over X) such that each $Y(n) \rightarrow Y'(n)$ is an injective trivial fibration in $\text{sPSh}(\mathbf{\Delta}_d)$;
- fibrant objects are $Y \rightarrow X$ such that each $Y(n) \rightarrow X(n)$ is an injective fibration between injectively fibrant $(d - 1)$ -category models, such that $Y(n) \rightarrow X(n) \times_{X(\{0\})}^h Y(\{0\})$ is a weak equivalence.

The identity functor determines a Quillen equivalence to the *injective covariant model structure*, defined similarly but as a left Bousfield localization of the injective model structure on $\text{sPSh}(\mathbf{\Delta} \times \mathbf{\Delta}_d)/X$.

We will write $\text{coPSh}_d^{\text{Seg,proj}}(X)$ for the category of fibrant objects in the projective covariant model structure. This determines a (pseudo-)functor $\text{CatAlg}_{d+1}^{\text{op}} \rightarrow \text{FibCat}$ sending $X \mapsto \text{coPSh}_d^{\text{Seg,proj}}(X)$, where CatAlg_{d+1} is a category of fibrant objects as in Construction 4.18. Let us write $\text{coPSh}_d^{\text{Seg,proj}} \rightarrow \text{CatAlg}_{d+1}$ for its Grothendieck construction.

LEMMA 4.30. *The map $\text{coPSh}_d^{\text{Seg,proj}} \rightarrow \text{CatAlg}_{d+1}$ is a homotopy cartesian fibration between categories of fibrant objects, in the sense of Definition A.5.*

Proof. We verify conditions (a) and (b) of Lemma A.2: for any Dwyer–Kan equivalence $f: X \rightarrow X'$, there is an adjoint pair $f_!: \text{sPSh}(\mathbf{\Delta} \times \mathbf{\Delta}_d)/X \rightleftarrows \text{sPSh}(\mathbf{\Delta} \times \mathbf{\Delta}_d)/X': f^*$. This is both a Quillen pair for the projective covariant and injective covariant model structure; in particular, $f_!$ preserves all weak equivalences. It now follows from [Boa18, Proposition 5.5] that $(f_!, f^*)$ is a Quillen equivalence: indeed, the result from [Boa18] shows that the derived unit and counit are already equivalences pointwise in $\mathbf{\Delta}_d$. This immediately gives condition (a), whereas condition (b) follows from Remark A.3. □

CONSTRUCTION 4.31. Finally, for each injectively fibrant $(d + 1)$ -category model X , let us denote by $\text{coPSh}_d^{\text{Seg,inj}}(X) \subseteq \text{sPSh}(\mathbf{\Delta} \times \mathbf{\Delta}_d)/X$ the full subcategory of fibrant objects in the injective covariant model structure. This determines a (pseudo-)functor $\text{Cat}_{d+1}^{\text{op}} \rightarrow \text{FibCat}$, where Cat_{d+1} is a category of fibrant objects as in Construction 4.15. We will write $\text{coPSh}_d^{\text{Seg,inj}} \rightarrow \text{Cat}_{d+1}$ for its Grothendieck construction.

The same proof as in Lemma 4.30 shows that $\text{coPSh}_d^{\text{Seg,inj}} \rightarrow \text{Cat}_{d+1}$ is a homotopy cartesian fibration between categories of fibrant objects. In fact, unraveling the definitions shows that $\text{coPSh}_d^{\text{Seg,inj}}$ can be identified with the category of fibrant objects for a model structure on $\text{Fun}([1], \text{sPSh}(\mathbf{\Delta} \times \mathbf{\Delta}_d))$: indeed, one can take a left Bousfield localization of the injective model structure whose fibrant objects are injective fibrations $Y \rightarrow X$ that model Segal copresheaves in the sense of Definition 4.25. In particular, the localization of $\text{coPSh}_d^{\text{Seg,inj}} \rightarrow \text{Cat}_{d+1}$ is precisely the functor $\text{coPSh}_d^{\text{Seg}} \rightarrow \text{Cat}_{d+1}$ from Definition 4.25.

THEOREM 4.32 (Rectification of Segal copresheaves, cf. [Boa18, Theorem A]). *For each $d \geq 0$, there is a commuting diagram of categories of fibrant objects*

$$\begin{array}{ccccc}
 \text{EnrFun}_d & \xrightarrow{N} & \text{coPSh}_d^{\text{Seg,proj}} & \longleftarrow & \text{coPSh}_d^{\text{Seg,inj}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Cat}(\text{Cat}_d) & \xrightarrow{N} & \text{CatAlg}_{d+1} & \longleftarrow & \text{Cat}_{d+1}
 \end{array}$$

in which the vertical functors are homotopy cartesian fibrations and the horizontal functors induce equivalences upon localization.

Proof. The bottom horizontal functors induce equivalences on localizations by Proposition 4.20. By Proposition A.7 and [Lur09a, Corollary 2.4.4.4], the top horizontal functors induce equivalences on localizations as soon as the induced functors between fibers have this property. For the right square, this is immediate: the maps on fibers are induced by the Quillen equivalence between the injective and projective covariant model structure.

For the left square, one has to verify that for any enriched category \mathcal{C} , the functor between fibers $N: \text{Fun}_{\text{enr}}(\mathcal{C}, \mathbf{Cat}_d) \rightarrow \text{coPSh}_d^{\text{Seg,proj}}(N(\mathcal{C}))$ induces an equivalence on localizations. This functor sends an enriched functor $F: \mathcal{C} \rightarrow \mathbf{Cat}_d$ to the obvious left fibration as follows:

$$N(F)(n) = \coprod_{c_0, \dots, c_n} \text{Map}_{\mathcal{C}}(c_{n-1}, c_n) \times \cdots \times \text{Map}_{\mathcal{C}}(c_0, c_1) \times F(c_0) \longrightarrow N(\mathcal{C})(n).$$

Note that this arises from a right Quillen functor $N: \text{Fun}_{\text{enr}}(\mathcal{C}, \text{sPSh}(\Delta_d)) \rightarrow \text{sPSh}(\Delta \times \Delta_d)_{/N(\mathcal{C})}$ between the projective model structure on enriched functors and the projective covariant model structure on $\text{sPSh}(\Delta \times \Delta_d)_{/N(\mathcal{C})}$. It follows from [Boa18, Theorem A] that the derived unit and counit maps are pointwise weak equivalences in Δ_d . Consequently, N is a right Quillen equivalence (even when $\text{sPSh}(\Delta_d)$ is endowed with the injective, rather than d -categorical model structure) and the result follows. \square

COROLLARY 4.33 (Cf. [Boa18, Theorem A]). *For each $(d + 1)$ -category \mathcal{C} , there is an equivalence of 1-categories*

$$\mathfrak{c}_1\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \xrightarrow{\sim} \text{coPSh}_d^{\text{Seg}}(\mathcal{C}).$$

Proof. It follows from Theorem 4.32 that the functor $\text{coPSh}_d^{\text{Seg}} \rightarrow \mathbf{Cat}_{d+1}$ is modeled by $\pi: \text{EnrFun}_d \rightarrow \mathbf{Cat}(\mathbf{Cat}_d)$. Proposition A.7 now implies that for any enriched category \mathcal{C} , the localization of the fiber $\pi^{-1}(\mathcal{C})$ is a model for the fiber $\text{coPSh}_d^{\text{Seg}}(\mathcal{C})$ over the corresponding $(d + 1)$ -category. The fiber $\pi^{-1}(\mathcal{C})$ is precisely the category of enriched functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$, whose localization models $\mathfrak{c}_1\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d)$ by Proposition 4.24. \square

Remark 4.34. For $d = 0$, this gives a version of (un)straightening for left fibrations over 1-categories. For $d > 0$, viewing \mathcal{C} as a presheaf on $\Delta \times \Delta_d$, the above result essentially corresponds to (un)straightening for left fibrations, applied pointwise in Δ_d .

COROLLARY 4.35. *Let \mathcal{C} be a 1-category, viewed as a 2-category. Then there is an equivalence*

$$\text{Fun}(\mathcal{C}, \mathbf{Cat}_1) \xrightarrow{\sim} \text{coPSh}_1^{\text{Seg}}(\mathcal{C}) \xrightarrow[\sim]{\Psi^\perp} \text{Cocart}_1(\mathcal{C})$$

to the category of cocartesian fibrations over \mathcal{C} and maps preserving cocartesian morphisms.

Proof. We can combine Corollary 4.33 and Theorem 3.1, using that there are obvious equivalences $\text{coPSh}_1^{\text{Seg}}(\mathcal{C}) \simeq \text{Fib}^{\text{left, cart}}(\mathcal{C} \boxtimes [0])$ and $\text{Cocart}_1(\mathcal{C}) \simeq \text{Fib}^{\text{cocart, right}}(\mathcal{C} \boxtimes [0])$. \square

Example 4.36. Let \mathcal{C} be a $(d + 1)$ -category and let $\text{Tw}(\mathcal{C}): \Delta^{\text{op}} \rightarrow \mathbf{Cat}_d$ be the functor given by $\text{Tw}(\mathcal{C})(n) = \mathcal{C}([n]^{\text{op}} \star [n])$. Restriction along the inclusions $[n]^{\text{op}} \hookrightarrow [n]^{\text{op}} \star [n] \hookrightarrow [n]$ induces the following natural map of $(d + 1)$ -fold simplicial spaces:

$$\text{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}.$$

The Segal conditions on \mathcal{C} imply that this is a Segal copresheaf. By Corollary 4.33, this corresponds to a $(d + 1)$ -functor that we will denote by

$$\text{Map}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{Cat}_d.$$

This functor is easily described using point-set models: if \mathcal{C} is an enriched category, then the above map of simplicial spaces is modeled by the object $\text{Tw}(N(\mathcal{C})) \rightarrow N(\mathcal{C})^{\text{op}} \times N(\mathcal{C})$

in $\text{coPSh}^{\text{Seg,proj}}(\mathbf{N}(\mathbf{C}))$. Unraveling the definitions, this left fibration is exactly the image under $N: \text{Fun}_{\text{enr}}(\mathbf{C}^{\text{op}} \times \mathbf{C}, \text{Cat}_d) \rightarrow \text{coPSh}^{\text{Seg,proj}}(\mathbf{N}(\mathbf{C}))$ of the (strict) mapping space functor $\text{Map}_{\mathbf{C}}(-, -): \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \text{Cat}_d$.

Example 4.37. Consider a map in $\text{coPSh}_d^{\text{Seg}}$ corresponding to the following commuting square.

$$\begin{array}{ccc} * & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ * & \xrightarrow{c} & \mathcal{C} \end{array} \tag{4.38}$$

We will say that x is a *representation* of X if it defines a cocartesian arrow in $\text{coPSh}_d^{\text{Seg}}$. Note that if we model \mathcal{C} by an enriched category \mathbf{C} , then $c^*: \text{Fun}_{\text{enr}}(\mathbf{C}, \text{Cat}_d) \rightarrow \text{Cat}_d$ admits a left adjoint $c_!$ preserving trivial fibrations. It follows from Proposition A.7(4) that the map $* \rightarrow \text{Map}_{\mathbf{C}}(c, -)$ in EnrFun_d defines such a cocartesian arrow in $\text{coPSh}^{\text{Seg}}$. Consequently, (4.38) defines a representation of X if and only if it induces an equivalence $\text{Map}_{\mathcal{C}}(c, -) \simeq X$.

We will use the language of Example 4.37 to partially address the functoriality of Corollary 4.33.

Notation 4.39. For a regular uncountable cardinal κ , let us write $\text{coPSh}_d^{\text{Seg}}(\kappa)$ for the category of Segal copresheaves $X \rightarrow \mathcal{C}$ whose fibers are essentially κ -small. The codomain projection $\text{coPSh}_d^{\text{Seg}}(\kappa) \rightarrow \text{Cat}_{d+1}$ is a cartesian fibration. Taking the subcategory with only cartesian morphisms, we obtain a right fibration $\pi: \text{coPSh}_d^{\text{Seg}}(\kappa)^{\text{cart}} \rightarrow \text{Cat}_{d+1}$.

Note that there is a canonical element $u \in \text{coPSh}_d^{\text{Seg}}(\kappa)^{\text{cart}}$, corresponding under the equivalence of Corollary 4.33 to the identity functor $\mathbf{Cat}_d(\kappa) \rightarrow \mathbf{Cat}_d(\kappa)$ on the small $(d + 1)$ -category of κ -small d -categories.

PROPOSITION 4.40. *The element $u: * \rightarrow \text{coPSh}_d^{\text{Seg}}(\kappa)$ defines a representation of the right fibration $\text{coPSh}_d^{\text{Seg}}(\kappa)^{\text{cart}} \rightarrow \text{Cat}_{d+1}$. In other words, for every $(d + 1)$ -category \mathcal{C} , there is a 1-functorial equivalence between the space of $(d + 1)$ -functors $\mathcal{C} \rightarrow \mathbf{Cat}_d(\kappa)$ and the space of Segal copresheaves $X \rightarrow \mathcal{C}$ with κ -small fibers.*

Remark 4.41. For later purposes, we will mostly be interested in this result in the large setting, with κ being the supremum of all small cardinals: in that case, it provides a 1-functorial equivalence between the spaces of $(d + 1)$ -functors $\mathcal{C} \rightarrow \mathbf{Cat}_d$ and Segal copresheaves $X \rightarrow \mathcal{C}$ with small fibers over a large $(d + 1)$ -category \mathcal{C} .

The proof uses the following well-known criterion for being a representation (cf. [Boa18, Lemma 1.31], [KV14, Lemma 2.3.6] and [Ras21, Theorem 5.25]).

LEMMA 4.42. *Let \mathcal{C} be a $(d + 1)$ -category and consider an arrow in $\text{coPSh}_d^{\text{Seg}}$ of the form (4.38). Then x defines a representation of $p: X \rightarrow \mathcal{C}$ if and only if it determines an initial object in each 1-category $X(-, \vec{n}_d)$.*

Proof. Assuming x defines a representation of $X \rightarrow \mathcal{C}$, we can model \mathcal{C} by an enriched category and identify x with the canonical map $\text{id}_c: * \rightarrow N(\text{Map}_{\mathbf{C}}(c, -))$. One easily verifies that id_c defines an initial object in each degree \vec{n}_d .

Conversely, suppose that x defines an initial object in each $X(-, \vec{n}_d)$. Note that there is a fully faithful inclusion $\text{coPSh}_d^{\text{Seg}} \hookrightarrow \text{Fun}(\Delta_d, \text{coPSh}_0^{\text{Seg}})$ which admits a left adjoint and preserves cartesian arrows. This implies that (4.38) defines a cocartesian arrow in $\text{coPSh}_d^{\text{Seg}}$ as soon as it does in $\text{Fun}(\Delta_d, \text{coPSh}_0^{\text{Seg}})$. It therefore suffices to verify that each $* \rightarrow X(-, \vec{n}_d)$ defines a

cocartesian arrow in the category of Segal copresheaves over 1-categories. This reduces everything to the case $d = 0$, where we can consider the following diagram.

$$\begin{array}{ccccc} * & \xrightarrow{x} & X & \xrightarrow{\text{id}} & X \\ \downarrow & & \downarrow \text{id} & & \downarrow p \\ * & \xrightarrow{x} & X & \xrightarrow{p} & \mathcal{C} \end{array}$$

One easily verifies that the right square defines a cocartesian arrow in the category $\text{coPSh}_0^{\text{Seg}}$ of left fibrations. On the other hand, since $x \in X$ is an initial object, the left square represents the terminal copresheaf on X , i.e. it also defines a cocartesian arrow in $\text{coPSh}_0^{\text{Seg}}$. The composite is then a cocartesian arrow as well, as desired. \square

Proof of Proposition 4.40. By (the opposite of) Lemma 4.42, it suffices to show that u defines a terminal object in $\text{coPSh}_d^{\text{Seg}}(\kappa)^{\text{cart}}$.

Let us write $\text{EnrFun}_d(\kappa) \subseteq \text{EnrFun}_d$ for the subcategory of tuples (\mathcal{C}, F) consisting of a small Cat_d -enriched category and an enriched functor $F: \mathcal{C} \rightarrow \text{Cat}_d$ whose values are homotopy equivalent to κ -small objects. Lemma 4.28 and Theorem 4.32 imply that $\text{EnrFun}_d(\kappa) \rightarrow \text{Cat}(\text{Cat}_d)$ is a homotopically cartesian fibration between categories of fibrant objects, whose localization models $\text{coPSh}_d^{\text{Seg}}(\kappa) \rightarrow \text{Cat}_{d+1}$. The object $u \in \text{coPSh}_d^{\text{Seg}}(\kappa)$ simply corresponds to the tuple $u = (\text{Cat}_d(\kappa), \text{id})$ consisting of the enriched category of κ -small d -category models, together with the identity (or, more precisely, with the fully faithful inclusion $\text{Cat}_d(\kappa) \hookrightarrow \text{Cat}_d$).

Proposition A.7 now implies that $\text{coPSh}_d^{\text{Seg}}(\kappa)^{\text{cart}}$ arises as the localization of the $(1, 1)$ -category $\mathcal{H} = \text{EnrFun}_d(\kappa)^{\text{hocart}}$ of such tuples (\mathcal{C}, F) , with homotopy cartesian maps between them (see Appendix A for more details). We have to show that $\text{Map}_{\mathcal{H}[W^{-1}]}(-, u)$ is the terminal (very large) presheaf. To this end, let $\psi: \mathcal{H} \rightarrow \mathcal{H}[W^{-1}]$ denote the localization and consider the following adjoint pair between categories of very large presheaves:

$$\psi_! : \text{Fun}(\mathcal{H}, \mathcal{S}^{\text{very large}}) \xrightleftharpoons{\psi^*} \text{Fun}(\mathcal{H}[W^{-1}], \mathcal{S}^{\text{very large}}) : \psi^*$$

Here ψ^* is fully faithful with essential image consisting of presheaves sending W to equivalences. We have to show that $\psi_!(u) \simeq *$ (where we omit the Yoneda embedding from the notation).

To see this, consider for each small cardinal $\lambda \geq \kappa$ the full Cat_d -enriched subcategory $\mathcal{U}_{\lambda, n} \subseteq \text{Fun}([n], \text{Cat}_d)$ spanned by sequences of trivial fibrations $A_0 \twoheadrightarrow \cdots \twoheadrightarrow A_n$ where each A_i is λ -small and homotopy equivalent to a κ -small object. Each $\mathcal{U}_{\lambda, n}$ comes with a canonical enriched functor $p_0: \mathcal{U}_{\lambda, n} \rightarrow \text{Cat}_d$ sending $(A_0 \twoheadrightarrow \cdots \twoheadrightarrow A_n)$ to A_0 . Let us write $u_{\lambda, n} = (\mathcal{U}_{\lambda, n}, p_0)$ for the corresponding object in $\text{EnrFun}_d(\kappa)$ and note that $u_{\kappa, 0}$ coincides with $(\text{Cat}_d(\kappa), \text{id})$. For each $\lambda \leq \lambda'$, the evident map $u_{\lambda, n} \rightarrow u_{\lambda', n}$ is a weak equivalence and each $u_{\lambda, \bullet}: \Delta^{\text{op}} \rightarrow \text{EnrFun}_d(\kappa)$ is a simplicial object where each structure map is a weak equivalence; in particular, this defines a simplicial diagram in \mathcal{H} . Taking the (large) filtered colimit over all small cardinals λ , we then obtain a map of presheaves

$$\text{Map}_{\mathcal{H}}(-, u) \longrightarrow \text{colim}_{\lambda} \text{colim}_{\Delta^{\text{op}}} \text{Map}_{\mathcal{H}}(-, u_{\lambda, \bullet}) \tag{4.43}$$

which induces an equivalence upon applying $\psi_!$. It therefore remains to verify that the target is the terminal presheaf (so that its image under $\psi_!$ is terminal as well).

Now fix an object $(\mathcal{C}, F) \in \mathcal{H}$ and for each λ consider the space

$$X_{\lambda} = \text{colim}_{\Delta^{\text{op}}} \text{Map}_{\mathcal{H}}((\mathcal{C}, F), u_{\lambda, \bullet}).$$

It suffices to verify that this space is contractible for large λ ; taking the (filtered) colimit over λ then shows that the target of (4.43) is terminal.

Unraveling the definitions, X_λ is the classifying space of the $(1, 1)$ -category whose objects are weak equivalences $F \rightarrow F'$ of functors $\mathbf{C} \rightarrow \mathbf{Cat}_d$, where F' takes λ -small values, and where a morphism is a (pointwise) trivial fibration $F' \rightarrow F''$ of functors under F . For large enough λ (in particular, so that F takes λ -small values), the under-category $\mathcal{M}_\lambda = F/\text{Fun}_{\text{enr}}(\mathbf{C}, \mathbf{Cat}_d(\lambda))$ has the structure of a category of fibrant objects in the sense of Brown. The space X_λ is then a path component in the classifying space of the wide subcategory $\text{wf}\mathcal{M}_\lambda \subseteq \mathcal{M}_\lambda$ spanned by the trivial fibrations. By [LNS17, Lemmas 14 and 15], the classifying space of $\text{wf}\mathcal{M}_\lambda$ is equivalent to that of the wide subcategory $\text{w}\mathcal{M}_\lambda \subseteq \mathcal{M}_\lambda$ spanned by the weak equivalences. Restricting to the correct path component, this implies that for large λ , X_λ is equivalent to the classifying space of the $(1, 1)$ -category whose objects are weak equivalences $F \rightarrow F'$ of enriched functors with λ -small values, with weak equivalences $F' \rightarrow F''$ as morphisms. This latter category has an initial object (F itself) and is, hence, contractible. \square

VARIANT 4.44. Associated to the cartesian fibration $\text{coPSh}_d^{\text{Seg}}(\kappa) \rightarrow \text{Cat}_{d+1}$ is the following cartesian fibration of 1-categories:

$$\mathcal{X} = \text{Fun}([1], \text{coPSh}_d^{\text{Seg}}(\kappa)) \times_{\text{Fun}([1], \text{Cat}_{d+1})} \text{Cat}_{d+1} \longrightarrow \text{Cat}_{d+1}.$$

The same argument as Theorem 4.32 shows that $p: \mathcal{X} \rightarrow \text{Cat}_{d+1}$ can be obtained as the localization of a homotopy cartesian fibration $\mathbf{X} \rightarrow \text{Cat}(\mathbf{Cat}_d)$ between categories of fibrant objects; here \mathbf{X} is the Grothendieck construction of the functor sending a \mathbf{Cat}_d -enriched category \mathbf{C} to the full subcategory of $\text{Fun}_{\text{enr}}([1] \times \mathbf{C}, \mathbf{Cat}_d)$ on the homotopically κ -small objects. Repeating the argument from Proposition 4.40 then shows that the underlying right fibration $\mathcal{X}^{\text{cart}} \rightarrow \text{Cat}_{d+1}$ is representable by $\text{Fun}_{d+1}([1], \mathbf{Cat}_d(\kappa))$.

4.4 More on representability

Let \mathcal{C} be a $(d + 1)$ -category and consider the $(d + 1)$ -functor

$$h: \mathcal{C}^{\text{op}} \longrightarrow \text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \tag{4.45}$$

adjoint to the $(d + 1)$ -functor $\text{Map}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Cat}_d$ defined in Example 4.36. If \mathcal{C} arises from an enriched category \mathbf{C} , then Proposition 4.24 provides an explicit model for h : it arises from the usual (strictly) enriched Yoneda embedding, i.e. the following strict functor.

$$h: \mathcal{C}^{\text{op}} \longrightarrow \text{Fun}(\mathbf{C}, \mathbf{Cat}_d)^{\circ}; \quad c \longmapsto \text{Map}_{\mathbf{C}}(c, -)$$

Since this is (strictly) fully faithful, the functor (4.45) is fully faithful as well. An ∞ -categorical proof of this can also be found in [Hin20].

DEFINITION 4.46. Let \mathcal{C} be a $(d + 1)$ -category. A functor $F: \mathcal{C} \rightarrow \mathbf{Cat}_d$ is said to be *representable* if it is contained in the essential image of (4.45).

Remark 4.47. Let $F: \mathcal{C} \rightarrow \mathbf{Cat}_d$ be a functor and let $p: X \rightarrow \mathcal{C}$ be the corresponding Segal copresheaf. From the above presentation in terms of enriched categories, one sees that F is representable if and only if p admits a representation in the sense of Example 4.37. By Lemma 4.42, this is equivalent to the 1-category $X(-, \vec{0}_d)$ admitting an initial object and each $X(-, \vec{0}_d) \rightarrow X(-, \vec{n}_d)$ preserving the initial object. The representing object is the image of this initial object in \mathcal{C} .

Example 4.48. For each object c of a $(d + 1)$ -category \mathcal{C} , consider the $(d + 1)$ -functor

$$\text{ev}_c: \text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \longrightarrow \text{Fun}_{d+1}(*, \mathbf{Cat}_d) = \mathbf{Cat}_d$$

given by restriction along $c: * \rightarrow \mathcal{C}$. Then the functor ev_c is corepresentable by an object of $\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d)$, which, in turn, is given by the corepresentable functor h_c . Indeed, this is a special case of the enriched Yoneda lemma, proven in the ∞ -categorical setting in [Hin20, 6.2.7]. In terms of strict models, one can also deduce this from the strict version of the enriched Yoneda lemma: choosing a strictly enriched model \mathbf{C} for \mathcal{C} , one can model ev_c by the strictly enriched functor $ev_c: \text{Fun}_{\text{enr}}(\mathbf{C}, \mathbf{Cat}_d)^\circ \rightarrow \mathbf{Cat}_d$ evaluating a strictly enriched functor at the object $c \in \mathbf{C}$. By the classical enriched Yoneda lemma, this functor is corepresented by $h_c = \text{Map}_{\mathbf{C}}(c, -)$.

Recall that \mathbf{Cat}_{d+1} has a homotopy $(2, 2)$ -category, obtained by taking the homotopy $(1, 1)$ -categories of all mapping $(d + 1)$ -categories $\text{Fun}_{d+1}(\mathcal{C}, \mathcal{D})$. An *adjunction* between two $(d + 1)$ -categories is simply an adjunction in this homotopy $(2, 2)$ -category [RV16]. We then have the following standard criterion for a $(d + 1)$ -functor $g: \mathcal{C} \rightarrow \mathcal{D}$ admitting a left adjoint.

PROPOSITION 4.49. *Let $g: \mathcal{C} \rightarrow \mathcal{D}$ be a map of $(d + 1)$ -categories. Then g admits a left adjoint if and only if for each object $d \in \mathcal{D}$, the functor $\text{Map}_{\mathcal{D}}(d, g(-)): \mathcal{C} \rightarrow \mathbf{Cat}_d$ is representable.*

For example, the map $\mathcal{C} \rightarrow *$ admits a left adjoint if and only if \mathcal{C} admits an initial object, i.e. an object \emptyset such that $\text{Map}_{\mathcal{C}}(\emptyset, c) \simeq *$ for all $c \in \mathcal{C}$.

Proof. As a preliminary observation, let us recall that any 2-functor preserves adjunctions. We are going to apply this in the following two situations.

- (a) To the 2-functor $\text{ho}_{(2,2)}(\mathbf{Cat}_d)^{\text{op}} \rightarrow \text{Cat}_{(1,1)}$ sending $\mathcal{C} \mapsto \text{ho}_{(1,1)}\text{Fun}_d(\mathcal{C}, \mathbf{Cat}_{d-1})$, and (2) -morphisms to the corresponding restriction functors and natural transformations.
- (b) To the localization functor $\text{Cat}(\text{sPSh}(\Delta_{d-1})) \rightarrow \text{ho}_{(2,2)}(\mathbf{Cat}_d)$ sending each $\text{sPSh}(\Delta_{d-1})$ -enriched category to the corresponding d -category (viewed as an object in the homotopy $(2, 2)$ -category of d -categories). Note that this is indeed a 2-functor, since it is naturally tensored over $\text{Cat}_{(1,1)}$ via the cartesian product.

Now let us start by assuming that g admits a left adjoint f . Applying the 2-functor (a) to this adjoint pair, we obtain the following adjoint pair:

$$g^*: \text{ho}_{(1,1)}\text{Fun}_d(\mathcal{D}, \mathbf{Cat}_{d-1}) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{ho}_{(1,1)}\text{Fun}_d(\mathcal{C}, \mathbf{Cat}_{d-1}): f^*.$$

We can model $f: \mathcal{D} \rightarrow \mathcal{C}$ by a functor of \mathbf{Cat}_{d-1} -enriched categories $f: \mathbf{D} \rightarrow \mathbf{C}$. The functor f^* then arises from the right Quillen functor

$$f^*: \text{Fun}_{\text{enr}}(\mathbf{C}, \text{sPSh}(\Delta_d)) \rightarrow \text{Fun}_{\text{enr}}(\mathbf{D}, \text{sPSh}(\Delta_d)).$$

In particular, the left adjoint g^* (on homotopy $(1, 1)$ -categories) is naturally isomorphic to the left derived functor of $f_!: \text{Fun}_{\text{enr}}(\mathbf{D}, \text{sPSh}(\Delta_d)) \rightarrow \text{Fun}_{\text{enr}}(\mathbf{C}, \text{sPSh}(\Delta_d))$. Since the left derived functor of $f_!$ preserves representable copresheaves, it follows that g^* preserves representable copresheaves as well; this is precisely the assertion that $\text{Map}_{\mathcal{D}}(d, g(-))$ is representable for all $d \in \mathcal{D}$.

For the converse, suppose that each $\text{Map}_{\mathcal{D}}(d, g(-)): \mathcal{C} \rightarrow \mathbf{Cat}_{d-1}$ is representable. To prove that g admits a left adjoint in $\text{ho}_{(2,2)}(\mathbf{Cat}_d)$, it will suffice to find a map $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ in $\text{Cat}(\text{sPSh}(\Delta_{d-1}))$ which admits a (strict) right adjoint and whose image under the functor (b) is equivalent to the *opposite* of $g: \mathcal{C} \rightarrow \mathcal{D}$.

To provide the desired model for g , let us choose any cofibration $\gamma: \mathbf{C}_0 \rightarrow \mathbf{D}_0$ between \mathbf{Cat}_{d-1} -enriched categories that models $g: \mathcal{C} \rightarrow \mathcal{D}$. This induces a Quillen pair of (in particular) simplicial model categories

$$\gamma!: \text{Fun}_{\text{enr}}(\mathbf{C}_0, \text{sPSh}(\Delta_{d-1})) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{Fun}_{\text{enr}}(\mathbf{D}_0, \text{sPSh}(\Delta_{d-1})): \gamma^*$$

given by restriction and left Kan extension. Now let us define

$$\mathbf{C}^{\text{op}} \subseteq \text{Fun}_{\text{enr}}(\mathbf{C}_0, \text{sPSh}(\mathbf{\Delta}_{d-1})) \quad \text{and} \quad \mathbf{D}^{\text{op}} \subseteq \text{Fun}_{\text{enr}}(\mathbf{D}_0, \text{sPSh}(\mathbf{\Delta}_{d-1}))$$

to be the full (enriched) subcategories of all enriched functors that admit a simplicial homotopy equivalence to a corepresentable functor. Since $\gamma_!$ preserves simplicial homotopy equivalences and corepresentable functors, we obtain a commuting square of categories enriched over $\text{sPSh}(\mathbf{\Delta}_{d-1})$

$$\begin{array}{ccc} \mathbf{C}_0^{\text{op}} & \xrightarrow{\gamma^{\text{op}}} & \mathbf{D}_0^{\text{op}} \\ h \downarrow & & \downarrow h \\ \mathbf{C}^{\text{op}} & \xrightarrow{\gamma_!} & \mathbf{D}^{\text{op}} \end{array}$$

where the vertical functors h are the Yoneda embeddings. By construction, both of these vertical functors are Dwyer–Kan equivalences of enriched categories, so that $\gamma_! : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ also maps to g^{op} under the localization functor (b). Let us point that this would not necessarily be the case if one replaced ‘simplicial homotopy equivalence’ by ‘weak equivalence’ in the definition of \mathbf{C}^{op} (unless one imposes further fibrancy and cofibrancy conditions, which need not be preserved by both $\gamma_!$ and γ^*).

We claim that the restriction functor γ^* also restricts to a functor $\gamma^* : \mathbf{D}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$. Indeed, since γ^* preserves simplicial homotopy equivalences, it suffices to verify that γ^* sends each corepresentable functor to an object in \mathbf{C}^{op} . In other words, we have to show that for each $d \in \mathbf{D}_0$, the enriched functor $\text{Map}_{\mathbf{D}_0}(d, \gamma(-))$ admits a simplicial homotopy equivalence to a corepresentable object. This enriched functor is a model for $\text{Map}_{\mathcal{D}}(d, g(-))$ and is, hence, weakly equivalent to a corepresentable object. Since $\text{Map}_{\mathbf{D}_0}(d, \gamma(-))$ is projectively fibrant–cofibrant [Lur09a, Proposition A.3.3.9], it then admits a simplicial homotopy equivalence to a corepresentable as well. We conclude that $\gamma_! : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ admits a strict right adjoint γ^* , so that its image g^{op} in $\text{ho}_{(2,2)}(\mathbf{Cat}_d)$ admits a right adjoint as well, as desired. \square

COROLLARY 4.50. *Let $g : \mathcal{C} \rightarrow \mathcal{D}$ be a map of d -categories which admits a fully faithful left adjoint. If $q : \mathcal{D}' \rightarrow \mathcal{D}$ is any functor, then the base change $g' : \mathcal{C} \times_{\mathcal{D}} \mathcal{D}' \rightarrow \mathcal{D}'$ admits a fully faithful left adjoint as well.*

Proof. For $d' \in \mathcal{D}'$, let $c \in \mathcal{C}$ be a representing object for $\text{Map}_{\mathcal{D}}(g(d'), g(-))$. Then $(c, d') \in \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ provides the desired representing object for $\text{Map}_{\mathcal{D}'}(d', g'(-))$. \square

Remark 4.51. Using a similar argument, one can show the following two assertions.

- (1) Consider a pullback diagram $\Lambda^2[2] \rightarrow \text{Fun}([1], \text{Cat}_d)$ sending each i to a functor $g_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$ admitting a fully faithful left adjoint. If the functors $\mathcal{C}_0 \rightarrow \mathcal{C}_2 \leftarrow \mathcal{C}_1$ preserve the essential images of these left adjoints, then the pullback $g : \mathcal{C}_0 \times_{\mathcal{C}_2} \mathcal{C}_1 \rightarrow \mathcal{D}_0 \times_{\mathcal{D}_2} \mathcal{D}_1$ admits a fully faithful left adjoint as well.
- (2) Consider a tower $\mathbb{N}^{\text{op}} \rightarrow \text{Fun}([1], \text{Cat}_d)$ sending each i to a functor $g_n : \mathcal{C}_n \rightarrow \mathcal{D}_n$ admitting a fully faithful left adjoint. If the functors $\mathcal{C}_{n+1} \rightarrow \mathcal{C}_n$ preserve the essential images of these left adjoints, then the map $\lim_n \mathcal{C}_n \rightarrow \lim_n \mathcal{D}_n$ admits a fully faithful left adjoint.

Indeed, in each of these cases consider an object $(d_i) \in \lim \mathcal{D}_i$ in the limit. For each i , there exists an inverse image $c_i \in g_i^{-1}(d_i)$ such that $\text{Map}_{\mathcal{C}_i}(c_i, -) \rightarrow \text{Map}_{\mathcal{D}_i}(d_i, g_i(-))$ is an equivalence. The fact that the functors $\mathcal{C}_i \rightarrow \mathcal{C}_j$ preserve the essential images of the fully faithful left adjoints implies that we can (inductively) lift this tuple of objects c_i to an object $(c_i) \in \lim \mathcal{C}_i$ in the limit. This object maps to $(d_i) \in \lim \mathcal{D}_i$ and has the desired universal property.

4.5 Tensoring

Let us conclude by recalling that questions about higher categories can often be reduced to questions about their underlying 1-categories when their enriched structure arises from a (co)tensoring.

DEFINITION 4.52. Let \mathcal{C} be a $(d + 1)$ -category and consider the following $(d + 1)$ -functor:

$$\mathbf{Cat}_d^{\text{op}} \times \mathcal{C}^{\text{op}} \longrightarrow \text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d); \quad (K, c) \longmapsto \text{Fun}_d(K, \text{Map}_{\mathcal{C}}(c, -)).$$

Then \mathcal{C} is said to be *tensored* over \mathbf{Cat}_d if this takes values in representable functors. In this case, we will write $\otimes: \mathbf{Cat}_d \times \mathcal{C} \rightarrow \mathcal{C}$ for the induced functor.

Remark 4.53. In other words, $K \otimes c$ is the unique object of \mathcal{C} with a natural equivalence

$$\text{Map}_{\mathcal{C}}(K \otimes c, -) \simeq \text{Fun}_d(K, \text{Map}_{\mathcal{C}}(c, -)).$$

This universal property implies that there are equivalences $K \otimes (L \otimes c) \simeq (K \times L) \otimes c$. Furthermore, note that the identity on $K \otimes c$ induces a natural map of d -categories $K \rightarrow \text{Map}_{\mathcal{C}}(c, K \otimes c)$.

DEFINITION 4.54. Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be a map between two $(d + 1)$ -categories which are tensored over \mathbf{Cat}_d . For each $c \in \mathcal{C}$ and $K \in \mathbf{Cat}_d$, the composite map

$$K \longrightarrow \text{Map}_{\mathcal{C}}(c, K \otimes c) \longrightarrow \text{Map}_{\mathcal{D}}(f(c), f(K \otimes c))$$

determines a natural map $\mu: K \otimes f(c) \rightarrow f(K \otimes c)$. We will say that f is *tensored* over \mathbf{Cat}_d if all of these maps μ are equivalences.

Remark 4.55. Likewise, a $(d + 1)$ -category \mathcal{C} is *cotensored* over \mathbf{Cat}_d if \mathcal{C}^{op} is tensored over \mathbf{Cat}_d ; in this case we will write c^K for the cotensoring of $c \in \mathcal{C}$ with $K \in \mathbf{Cat}_d$. A functor $f: \mathcal{C} \rightarrow \mathcal{D}$ is cotensored if all natural maps $\mu: f(c^K) \rightarrow f(c)^K$ are equivalences.

Example 4.56. The $(d + 1)$ -category \mathbf{Cat}_d is evidently tensored and cotensored over \mathbf{Cat}_d itself. Likewise, every functor $(d + 1)$ -category $\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d)$ is tensored and cotensored over \mathbf{Cat}_d , with (co)tensoring computed pointwise. This follows conceptually from the description of enriched functor ∞ -categories by Hinich [Hin20, §6]. Less elegantly, one can see this by working with the strict model $M^\circ = \text{Fun}_{\text{enr}}(\mathcal{C}, \mathbf{Cat}_d)^\circ$ provided Proposition 4.24. Indeed, for each $K \in \mathbf{Cat}_d$ and a fibrant–cofibrant enriched functor F , there is a natural isomorphism

$$\text{Map}_{\mathbf{Cat}_d}(K, \text{Map}_{M^\circ}(F, -)) \cong \text{Map}_{M^\circ}(F \times \text{cst}(K), -)$$

so that the pointwise product $F \times \text{cst}(K)$ with the constant functor determines a (strict) tensoring on M° . Dually, consider the pointwise fibrant enriched functor $F^K(-) = \text{Map}_{\mathbf{Cat}_d}(K, F(-))$ and let $Q(F^K) \in M^\circ$ be a projectively cofibrant replacement. Then there is a natural equivalence

$$\text{Map}_{M^\circ}(-, Q(F^K)) \xrightarrow{\sim} \text{Map}_{\mathbf{Cat}_d}(K, \text{Map}_{M^\circ}(-, F))$$

which exhibits $Q(F^K)$ as a cotensoring.

LEMMA 4.57. *Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be a tensored functor between $(d + 1)$ -categories that are tensored over \mathbf{Cat}_d . If the underlying functor of 1-categories is an equivalence, then f is an equivalence of $(d + 1)$ -categories as well.*

Proof. Note that for any $K \in \mathbf{Cat}_d$, the map between mapping spaces in \mathbf{Cat}_d

$$\mathrm{Map}_{\mathbf{Cat}_d}(K, \mathrm{Map}_{\mathcal{C}}(c, d)) \longrightarrow \mathrm{Map}_{\mathbf{Cat}_d}(K, \mathrm{Map}_{\mathcal{D}}(f(c), f(d)))$$

can be identified with $\mathrm{Map}_{\mathbf{c}_1(\mathcal{C})}(K \otimes c, d) \longrightarrow \mathrm{Map}_{\mathbf{c}_1(\mathcal{D})}(K \otimes f(c), d)$. The latter is an equivalence since f is fully faithful on the underlying 1-categories, so that f is a fully faithful functor between $(d + 1)$ -categories as well. It is essentially surjective since it is essentially surjective on the underlying 1-categories. \square

PROPOSITION 4.58. *Let \mathcal{C} be a $(d + 1)$ -category which is tensored over \mathbf{Cat}_d and let $F: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}_d$ be a $(d + 1)$ -functor. If F is cotensored and the underlying 1-functor*

$$\mathbf{c}_1(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathbf{Cat}_d \xrightarrow{c_0} \mathcal{S}$$

is representable, then F is representable.

Proof. Let $x_u \in \mathbf{c}_0(F(c_u))$ be the element corresponding (by the Yoneda lemma) to the natural equivalence of 1-functors $\mathrm{Map}_{\mathcal{C}}(-, c_u) \xrightarrow{\sim} \mathbf{c}_0(F)$. Viewing x_u as an element in $F(c_u)$, the enriched Yoneda lemma provides a natural transformation of $(d + 1)$ -functors $\mathrm{Map}_{\mathcal{C}}(-, c_u) \longrightarrow F$. To see that this is an equivalence, note that for any $K \in \mathbf{Cat}_d$ and $d \in \mathcal{C}$, there is the following commuting square.

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(K \otimes d, c_u) & \xrightarrow{x_u} & F(K \otimes d) \\ \sim \downarrow & & \sim \downarrow \mu \\ \mathrm{Fun}_d(K, \mathrm{Map}_{\mathcal{C}}(d, c_u)) & \xrightarrow{\mathrm{Fun}_d(K, x_u)} & \mathrm{Fun}_d(K, F(d)) \end{array}$$

The vertical functors are equivalences because F is cotensored. Since the top map induces an equivalence on underlying spaces, it follows that the bottom map induces an equivalence on the underlying spaces for all $K \in \mathbf{Cat}_d$. By the Yoneda lemma, this implies that the map $x_u: \mathrm{Map}_{\mathcal{C}}(d, c_u) \longrightarrow F(d)$ is an equivalence. \square

Remark 4.59. Lemma 4.57 and Proposition 4.58 do not require a full tensoring over \mathbf{Cat}_d . For instance, being (co)tensored over any full subcategory containing Θ_d (and closed under cartesian products) suffices as well.

5. Cocartesian fibrations of higher categories

In this section we discuss the higher categorical analogues of cocartesian fibrations whose fibers are given by d -categories. We will refer to such fibrations as *d-cocartesian fibrations*. Most importantly, we show (§ 5.3) that *d-cocartesian fibrations* between $(d + 1)$ -categories form the domain of a $(d + 1)$ -cartesian fibration between $(d + 2)$ -categories $\mathbf{Cocart}_d \longrightarrow \mathbf{Cat}_{d+1}$.

5.1 Pre-cocartesian fibrations

Let us start with the following preliminary definition, which reduces to that of a cocartesian fibration when $d = 0$ (see [AF20, RV22]).

DEFINITION 5.1. Let $p: \mathcal{D} \longrightarrow \mathcal{C}$ be a map of $(d + 1)$ -categories. Then p is said to be a *pre-cocartesian fibration* if the map of $(d + 1)$ -categories

$$\mathrm{Fun}_{d+1}([1], \mathcal{D}) \longrightarrow \mathrm{Fun}_{d+1}(\{0\}, \mathcal{D}) \times_{\mathrm{Fun}_{d+1}(\{0\}, \mathcal{C})} \mathrm{Fun}_{d+1}([1], \mathcal{C}). \tag{5.2}$$

is a right adjoint and the unit of the adjunction is an equivalence. A 1-morphism in \mathcal{D} is called *(p-)cocartesian* if it is contained in the essential image of the left adjoint.

Remark 5.3. Dually, p is said to be a *pre-cartesian fibration* if the map

$$\text{Fun}_{d+1}([1], \mathcal{D}) \longrightarrow \text{Fun}_{d+1}(\{1\}, \mathcal{D}) \times_{\text{Fun}_{d+1}(\{1\}, \mathcal{C})} \text{Fun}_{d+1}([1], \mathcal{C})$$

is a left adjoint and the counit of the adjunction is an equivalence. A 1-morphism in \mathcal{D} is called (p -)cartesian if it is contained in the essential image of the right adjoint.

This notion of pre-cocartesian fibration coincides with the 2-categorical notion of an abstract cocartesian fibration from Street [Str80] and Riehl and Verity [RV22] (interpreted in the 2-category of $(d + 1)$ -categories). In particular, we have the following properties.

LEMMA 5.4. *The class of pre-cocartesian fibrations has the following stability properties.*

- (1) *If $q: \mathcal{E} \rightarrow \mathcal{D}$ and $p: \mathcal{D} \rightarrow \mathcal{C}$ are pre-cocartesian fibrations, then $pq: \mathcal{E} \rightarrow \mathcal{C}$ is a pre-cocartesian fibration as well.*
- (2) *If $p: \mathcal{D} \rightarrow \mathcal{C}$ is a pre-cocartesian fibration and $\mathcal{C}' \rightarrow \mathcal{C}$ is any map, then the base change $\mathcal{D}' = \mathcal{D} \times_{\mathcal{C}} \mathcal{C}' \rightarrow \mathcal{C}'$ is a pre-cocartesian fibration.*
- (3) *If $p: \mathcal{D} \rightarrow \mathcal{C}$ is a pre-cocartesian fibration, then $\text{Fun}_{d+1}(\mathcal{B}, \mathcal{D}) \rightarrow \text{Fun}_{d+1}(\mathcal{B}, \mathcal{C})$ is a pre-cocartesian fibration for any $(d + 1)$ -category \mathcal{B} . Furthermore, for any $\mathcal{B}' \rightarrow \mathcal{B}$, the restriction functor $\text{Fun}_{d+1}(\mathcal{B}, \mathcal{D}) \rightarrow \text{Fun}_{d+1}(\mathcal{B}', \mathcal{D})$ preserves cocartesian 1-morphisms.*
- (4) *Consider the subcategory of $\text{Fun}([1], \text{Cat}_{d+1})$ whose objects are pre-cocartesian fibrations and morphisms are squares*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \mathcal{D}' \\ p \downarrow & & \downarrow p' \\ \mathcal{C} & \longrightarrow & \mathcal{C}' \end{array}$$

such that f preserves cocartesian 1-morphisms. This subcategory is closed under limits.

Proof. Property (1) is easily verified. For property (2), note that the map (5.2) for $\mathcal{D}' \rightarrow \mathcal{C}'$ is the base change of the same map for $\mathcal{D} \rightarrow \mathcal{C}$. The result then follows from Corollary 4.50.

For property (3), note that $\text{Fun}_{d+1}([1], \text{Fun}_{d+1}(\mathcal{B}, \mathcal{D})) \cong \text{Fun}_{d+1}(\mathcal{B}, \text{Fun}_{d+1}([1], \mathcal{D}))$. Using this, the result follows from the fact that $\text{Fun}_{d+1}(\mathcal{B}, -)$ preserves limits and adjunctions (with fully faithful left adjoint).

Finally, for property (4) one can directly see that the subcategory is closed under products. It then suffices to verify that it is closed under pullbacks and limits of (countable) towers. This comes down to verifying that (5.2) admitting a fully faithful left adjoint is closed under such limits, which follows from Remark 4.51. Alternatively, one can also deduce this from Corollary 5.7, using an inductive argument to find enough cocartesian lifts in the limits. \square

LEMMA 5.5. *Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a map of $(d + 1)$ -categories and $\alpha: d_0 \rightarrow d_1$ a 1-morphism in \mathcal{D} . Then the following are equivalent.*

- (1) *For every $d_2 \in \mathcal{D}$, the square of mapping d -categories*

$$\begin{array}{ccc} \text{Map}_{\mathcal{D}}(d_1, d_2) & \xrightarrow{\alpha^*} & \text{Map}_{\mathcal{D}}(d_0, d_2) \\ p \downarrow & & \downarrow p \\ \text{Map}_{\mathcal{C}}(pd_1, pd_2) & \xrightarrow{p(\alpha)^*} & \text{Map}_{\mathcal{C}}(pd_0, pd_2) \end{array} \tag{5.6}$$

is cartesian.

(2) For every 1-morphism $\beta: d'_0 \rightarrow d'_1$ in \mathcal{D} , the map between mapping d -categories

$$\text{Map}_{\text{Fun}_{d+1}([1], \mathcal{D})}(\alpha, \beta) \longrightarrow \text{Map}_{\mathcal{D}}(d_0, d'_0) \times_{\text{Map}_{\mathcal{C}}(pd_0, pd'_0)} \text{Map}_{\text{Fun}_{d+1}([1], \mathcal{C})}(p\alpha, p\beta)$$

is an equivalence.

Proof. Using the description of the mapping objects in $\text{Fun}_{d+1}([1], \mathcal{D})$ from Example 4.10, the map appearing in condition (2) can be identified with

$$\text{Map}_{\mathcal{D}}(d_0, d'_0) \times_{\text{Map}_{\mathcal{D}}(d_0, d'_1)} \text{Map}_{\mathcal{D}}(d_1, d'_1) \longrightarrow \text{Map}_{\mathcal{D}}(d_0, d'_0) \times_{\text{Map}_{\mathcal{C}}(pd_0, pd'_1)} \text{Map}_{\mathcal{C}}(pd_1, pd'_1).$$

In particular, condition (1) is equivalent to condition (2) in the case where β is the identity on d_2 . Conversely, the above map arises as the base change of

$$\text{Map}_{\mathcal{D}}(d_1, d'_1) \longrightarrow \text{Map}_{\mathcal{D}}(d_0, d'_1) \times_{\text{Map}_{\mathcal{C}}(pd_0, pd'_1)} \text{Map}_{\mathcal{C}}(pd_1, pd'_1)$$

along the map $\beta_*: \text{Map}_{\mathcal{D}}(d_0, d'_0) \rightarrow \text{Map}_{\mathcal{D}}(d_0, d'_1)$. It follows that condition (1) implies condition (2). □

COROLLARY 5.7. *A map $p: \mathcal{D} \rightarrow \mathcal{C}$ of $(d + 1)$ -categories is a pre-cocartesian fibration if and only if the following condition holds: for every map $\gamma: c_0 \rightarrow c_1$ in \mathcal{C} and every $d_0 \in p^{-1}(c_0)$, there exists an arrow $\alpha: d_0 \rightarrow d_1$ in \mathcal{D} with $p(\alpha) = \gamma$, satisfying the equivalent conditions of Lemma 5.5.*

In this case, the p -cocartesian morphisms are precisely the morphisms satisfying the equivalent conditions of Lemma 5.5. For the 1-categorical case, see also [AF20].

Proof. Using Proposition 4.49, the existence of lifts α satisfying the second condition of Lemma 5.5 is equivalent to p being a pre-cocartesian fibration. □

Remark 5.8. The condition appearing in Corollary 5.7 can also be reformulated as follows: the map $\mathbf{c}_1(\mathcal{D}) \rightarrow \mathbf{c}_1(\mathcal{C})$ is a cocartesian fibration of 1-categories and each map of 1-categories $\mathcal{D}(-, \vec{0}_d) \rightarrow \mathcal{D}(-, \vec{n}_d)$ preserves cocartesian arrows. Using this, a map $f: \mathcal{D} \rightarrow \mathcal{D}'$ preserves cocartesian 1-morphisms if and only if the map $\mathbf{c}_1(\mathcal{D}) \rightarrow \mathbf{c}_1(\mathcal{D}')$ preserves cocartesian arrows in the usual sense.

COROLLARY 5.9. *Consider a square of $(d + 1)$ -categories*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & T \\ p \downarrow & & \downarrow q \\ \mathcal{C} & \longrightarrow & S \end{array}$$

where T and S are spaces. Then p is a pre-cocartesian fibration if and only if for any $t \in T$, the map on fibers $p_t: \mathcal{D}_t \rightarrow \mathcal{C}_{q(t)}$ is a pre-cocartesian fibration. Furthermore, a morphism in \mathcal{D} is p -cocartesian if and only if it defines a p_t -cocartesian morphism in some fiber \mathcal{D}_t .

Proof. Suppose that $\alpha: d_0 \rightarrow d_1$ is an arrow in \mathcal{D} and let $\tau: t_0 \rightarrow t_1$ be its image in T (which is an equivalence). For any object $d_2 \in \mathcal{D}$, the square of d -categories (5.6) then comes with a natural transformation to the following square of spaces.

$$\begin{array}{ccc} \text{Map}_T(t_1, t_2) & \xrightarrow{\tau^*} & \text{Map}_T(t_0, t_2) \\ \downarrow & & \downarrow \\ \text{Map}_S(qt_1, qt_2) & \xrightarrow{q(\tau)^*} & \text{Map}_S(qt_0, qt_2) \end{array}$$

The horizontal maps in this square are equivalences. The square (5.6) of d -categories (i.e. specific kinds of d -fold simplicial spaces) is then cartesian if and only if for every $\tau': t_1 \rightarrow t_2$ in T , the square of fibers over τ' is a cartesian diagram of d -categories. This means precisely that α defines a cocartesian arrow for $p_t: \mathcal{D}_t \rightarrow \mathcal{C}_{q(t)}$, where \mathcal{D}_t is the fiber of \mathcal{D} over the diagram in T of the form

$$t_0 \xrightarrow{\tau} t_1 \xrightarrow{\tau'} t_2.$$

Note that such a diagram in T is contractible, i.e. it is equivalent to a choice of basepoint $t \in T$. In other words, an arrow in \mathcal{D} is p -cocartesian if and only if it defines a p_t -cocartesian arrow where $p_t: \mathcal{D}_t \rightarrow \mathcal{C}_{q(t)}$ is the map between fibers over $t \in T$. Using this and Corollary 5.7, one readily deduces that p is a pre-cocartesian fibration if and only if each p_t is a pre-cocartesian fibration. □

5.2 d -cocartesian fibrations

We now turn to the definition of d -cocartesian fibrations, which proceeds by induction on d .

DEFINITION 5.10. Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a map of n -categories. We then have the following definitions, by induction on d :

- (a) The map p is said to be a *0-cocartesian fibration* if $\mathcal{D}(1) \rightarrow \mathcal{C}(1) \times_{\mathcal{C}(\{0\})} \mathcal{D}(\{0\})$ is an equivalence. A *strong morphism* between two 0-cocartesian fibrations p and p' is a commuting square of the following form.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \mathcal{D}' \\ p \downarrow & & \downarrow p' \\ \mathcal{C} & \longrightarrow & \mathcal{C}' \end{array} \tag{5.11}$$

Dually, p is said to be a *0-cartesian fibration* if $p^{\text{op}}: \mathcal{D}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ is a 0-cocartesian fibration; strong morphisms between those are commuting squares (5.11).

- (b) The map p is said to be a *d -cocartesian fibration* if it satisfies the following two conditions.
 - (1) It is a *homwise $(d - 1)$ -cartesian fibration*: for every $x_0, x_1 \in \mathcal{D}$, the induced map between mapping objects $\text{Map}_{\mathcal{D}}(x_0, x_1) \rightarrow \text{Map}_{\mathcal{C}}(p(x_0), p(x_1))$ is a $(d - 1)$ -cartesian fibration and the composition map determines a strong map of $(d - 1)$ -cartesian fibrations as follows.

$$\begin{array}{ccc} \text{Map}_{\mathcal{D}}(x_1, x_2) \times \text{Map}_{\mathcal{D}}(x_0, x_1) & \longrightarrow & \text{Map}_{\mathcal{D}}(x_0, x_2) \\ p \downarrow & & \downarrow p \\ \text{Map}_{\mathcal{C}}(p(x_1), p(x_2)) \times \text{Map}_{\mathcal{C}}(p(x_0), p(x_1)) & \longrightarrow & \text{Map}_{\mathcal{C}}(p(x_0), p(x_2)) \end{array} \tag{5.12}$$

- (2) It is a pre-cocartesian fibration (Definition 5.1).
- (c) The map p is said to be a *d -cartesian fibration* if it satisfies the following two conditions.
 - (1') It is a *homwise $(d - 1)$ -cocartesian fibration*: it induces $(d - 1)$ -cocartesian fibration on mapping objects and the composition determines a strong map of $(d - 1)$ -cocartesian fibrations (5.12).
 - (2') It is a pre-cartesian fibration (Remark 5.3).

(d) A *strong morphism* between d -(co)cartesian fibrations is a commuting square (5.11) such that f preserves (co)cartesian 1-morphisms and for any $x_0, x_1 \in \mathcal{D}$, the induced square

$$\begin{array}{ccc} \text{Map}_{\mathcal{D}}(x_0, x_1) & \xrightarrow{f} & \text{Map}_{\mathcal{D}'}(fx_0, fx_1) \\ p \downarrow & & \downarrow p' \\ \text{Map}_{\mathcal{C}}(px_0, px_1) & \longrightarrow & \text{Map}_{\mathcal{C}'}(p'(fx_0), p'(fx_1)) \end{array}$$

is a strong morphism of $(d - 1)$ -(co)cartesian fibrations.

WARNING 5.13. A 0-cocartesian fibration can equivalently be viewed as a Segal copresheaf (as in Definition 4.25) whose domain is an n -category. Note that the Segal copresheaves that arise in this way form a very restrictive class, since their fibers are spaces (cf. Warning 4.26).

Remark 5.14. As usual, there are 2^{d+1} variants of the notion of a d -cocartesian fibration between $(d + 1)$ -categories, by taking opposites in the various dimensions. We will only make use of the two notions considered above, which are related by taking opposites in every dimension.

Remark 5.15. The definition of a d -cocartesian fibration $p: \mathcal{D} \rightarrow \mathcal{C}$ exhibits an alternating pattern: odd-dimensional cells are required to admit p -cocartesian lifts and even-dimensional cells are required to admit p -cartesian lifts. Let us explain heuristically why this should encode a fully covariant diagram of d -categories indexed by \mathcal{C} .

First, to make sure that the fiber $p^{-1}(c_0)$ over an object $c_0 \in \mathcal{C}$ depends covariantly on c_0 , we need to be able to associate to each 1-morphism $\alpha: c_0 \rightarrow c_1$ in \mathcal{C} and each $d_0 \in p^{-1}(c_0)$ an object $\alpha_!d_0 \in p^{-1}(c_1)$. Of course, this is done using the (essentially unique) p -cocartesian arrow $\tilde{\alpha}: d_0 \rightarrow \alpha_!d_0$ covering α .

Next, given two objects $d_0, d_1 \in \mathcal{D}$ with images $c_0, c_1 \in \mathcal{C}$, consider the induced map $p: \text{Map}_{\mathcal{D}}(d_0, d_1) \rightarrow \text{Map}_{\mathcal{C}}(c_0, c_1)$. For each $\alpha: c_0 \rightarrow c_1$ in \mathcal{C} , its fiber is given by

$$p^{-1}(\alpha) \simeq \text{Map}_{p^{-1}(c_1)}(\alpha_!d_0, d_1).$$

Consequently, if $\alpha_!d_0$ depends covariantly on the 1-morphism α , then the fiber $p^{-1}(\alpha)$ will depend contravariantly on α . Indeed, each 2-morphism $h: \alpha \rightarrow \beta$ in \mathcal{C} should give rise to a functor

$$h^*: p^{-1}(\beta) \rightarrow p^{-1}(\alpha); \quad (\beta_!d_0 \xrightarrow{g} d_1) \mapsto (\alpha_!d_0 \xrightarrow{h(d_0)} \beta_!d_0 \xrightarrow{g} d_1) \tag{5.16}$$

that precomposes with the natural transformation induced by h . In terms of the fibration p , this means that each 2-morphism $h: \alpha \rightarrow \beta$ in \mathcal{C} and $g \in p^{-1}(\beta)$ should admit a p -cartesian lift $\tilde{h}: h^*(g) \rightarrow g$.

At the level of 2-morphisms, we then obtain a functor of the form

$$p: \text{Map}_{\text{Map}_{\mathcal{D}}(d_0, d_1)}(f, g) \rightarrow \text{Map}_{\text{Map}_{\mathcal{C}}(c_0, c_1)}(\alpha, \beta)$$

for each $f \in p^{-1}(\alpha)$ and $g \in p^{-1}(\beta)$. For a given 2-cell $h: \alpha \rightarrow \beta$ in \mathcal{C} , postcomposition with the p -cartesian lift \tilde{h} induces an equivalence

$$p^{-1}(h) \simeq \text{Map}_{p^{-1}(\alpha)}(f, h^*g).$$

If the natural transformation $h(d_0): \alpha_!d_0 \rightarrow \beta_!d_0$ depends covariantly on the 2-morphism h , then (5.16) shows that h^* depends covariantly on h as well. Consequently, the fiber $p^{-1}(h)$ depends covariantly on h , so that 3-morphisms should have p -cocartesian lifts. In higher dimensions, one can repeat the discussion from the last two paragraphs to arrive at the alternating list of cocartesian and cartesian conditions from Definition 5.10.

Note that for any map of n -categories $p: \mathcal{D} \rightarrow \mathcal{C}$, the property of being a d -cocartesian fibration does not depend on whether we consider p as a map of n -categories or $(n + 1)$ -categories. Similarly, the difference between d -cocartesian and $(d + 1)$ -cocartesian fibrations is only fiberwise.

LEMMA 5.17. *A map $p: \mathcal{D} \rightarrow \mathcal{C}$ between n -categories is a d -cocartesian fibration if and only if it is a $(d + 1)$ -cocartesian fibration and its fibers are d -categories.*

In particular, a functor between n -categories is a d -cocartesian fibration for some $d > n$ if and only if it is an n -cocartesian fibration.

Proof. An inductive argument reduces this to the case $d = 0$: if p is a 0-cocartesian fibration, then certainly its fibers are spaces and it is a homwise 0-cartesian fibration (since each $\mathcal{D}(p) \rightarrow \mathcal{C}(p)$ is the base change of a map between spaces). It is a 1-cocartesian fibration since every arrow in \mathcal{D} is p -cocartesian and every morphism in \mathcal{C} can be lifted to an arrow in \mathcal{D} (since p is a 0-cocartesian fibration).

Conversely, suppose that $p: \mathcal{D} \rightarrow \mathcal{C}$ is a 1-cocartesian fibration whose fibers are spaces. We have to verify that the map $\mathcal{D}(1) \rightarrow \mathcal{C}(1) \times_{\mathcal{C}(\{0\})} \mathcal{D}(\{0\})$ is an equivalence of $(n - 1)$ -categories. Note that this map is obtained from the map of n -categories (5.2) by taking $(n - 1)$ -cores. In particular, it admits a fully faithful left adjoint whose essential image consists of the p -cocartesian morphisms. It therefore suffices to verify that every 1-morphism in \mathcal{D} is p -cocartesian: this follows immediately from the fact that every 1-morphism α factors as $\alpha = \alpha'' \circ \alpha'$, where α' is cocartesian and α'' is a fiberwise morphism (and, hence, an equivalence). \square

Remark 5.18. It will follow from straightening that the notion of a d -cocartesian fibration essentially stabilizes at the level of $(d + 1)$ -categories: if p is a d -cocartesian fibration between $(d + k)$ -categories for $k \geq 2$, then p arises as the base change of a d -cocartesian fibration $\mathcal{D}' \rightarrow |\mathcal{C}|_{d+1}$, where $|\mathcal{C}|_{d+1}$ is the $(d + 1)$ -category obtained by inverting all morphisms in dimension $> d + 1$.

DEFINITION 5.19 (Lateral k -morphisms). Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a d -cocartesian fibration between n -categories. For any $1 \leq k \leq d$, let $\alpha \in \mathcal{D}(1, \dots, 1, \vec{0}_{n-k})$ be a k -morphism in \mathcal{D} , considered as a 1-morphism $\alpha: f \rightarrow g$ between two $(k - 1)$ -morphisms $f, g: x \rightarrow y$. Let

$$p_{(k-1)}: \text{Map}_{\mathcal{D}}(x, y) \longrightarrow \text{Map}_{\mathcal{C}}(p(x), p(y))$$

be the induced map between $(n + 1 - k)$ -categories of $(k - 1)$ -morphisms between x and y . Then α is said to be (p) -lateral if it is:

- a $p_{(k-1)}$ -cocartesian 1-morphism if k is odd;
- a $p_{(k-1)}$ -cartesian 1-morphism if k is even.

When p is an d -cartesian fibration, we define lateral k -morphisms similarly, but with the cases of k even and odd exchanged. In particular, the meaning depends on whether p is considered as a d -cocartesian or d -cartesian fibration.

Unraveling the definitions, a square (5.11) defines a strong morphism between two d -cocartesian fibrations if and only if the map f preserves lateral k -morphisms for all $k \leq d$.

LEMMA 5.20. *Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a map of n -categories covering a map of spaces $q: T \rightarrow S$, as in Corollary 5.9. Then p is a d -cocartesian fibration if and only if each map on fibers $p_t: \mathcal{D}_t \rightarrow \mathcal{C}_{q(t)}$ is a d -cocartesian fibration. For each $k \leq d$, a k -morphism in \mathcal{D} covering a point $t \in T$ is p -lateral if and only if it is p_t -lateral.*

Proof. This follows by induction, using Corollary 5.9. □

LEMMA 5.21. *A map $p: \mathcal{D} \rightarrow \mathcal{C}$ is a homwise $(d - 1)$ -cartesian fibration if and only if it satisfies the following condition: for every $[m] \in \mathbf{\Delta}$, the map $\mathcal{D}(m) \rightarrow \mathcal{C}(m)$ is a $(d - 1)$ -cartesian fibration and each $[m] \rightarrow [m']$ induces a strong morphism between $(d - 1)$ -cartesian fibrations.*

Proof. Taking sources and targets defines a natural transformation from the map $\mathcal{D}(1) \rightarrow \mathcal{C}(1)$ to the map of spaces $\mathcal{D}(0) \times \mathcal{D}(0) \rightarrow \mathcal{C}(0) \times \mathcal{C}(0)$. By Lemma 5.20, $\mathcal{D}(1) \rightarrow \mathcal{C}(1)$ is then a $(d - 1)$ -cartesian fibration if and only if for each $d_0, d_1 \in \mathcal{D}$, the map on fibers $\text{Map}_{\mathcal{D}}(d_0, d_1) \rightarrow \text{Map}_{\mathcal{C}}(pd_0, pd_1)$ is a $(d - 1)$ -cartesian fibration. In turn, the Segal conditions imply that each $\mathcal{D}(m) \rightarrow \mathcal{C}(m)$ is a $(d - 1)$ -cartesian fibration as soon as $\mathcal{D}(1) \rightarrow \mathcal{C}(1)$ is.

In this case, the simplicial structure maps induce strong morphisms between $(d - 1)$ -cartesian fibrations if and only if $d_1: \mathcal{D}(2) \rightarrow \mathcal{D}(1)$ is a strong morphism. Again, Lemma 5.20 shows that this can be verified fiberwise over each triple of objects $d_0, d_1, d_2 \in \mathcal{D}(0)$, which means precisely that $\text{Map}_{\mathcal{D}}(d_1, d_2) \times \text{Map}_{\mathcal{D}}(d_0, d_1) \rightarrow \text{Map}_{\mathcal{D}}(d_0, d_2)$ defines a strong morphism of $(d - 1)$ -cartesian fibrations. □

LEMMA 5.22. *The following assertions hold.*

- (1) *The class of d -cocartesian fibrations is closed under composition and base change.*
- (2) *If $p: \mathcal{D} \rightarrow \mathcal{C}$ is a d -cocartesian fibration, then $\text{Fun}_n(\mathcal{B}, \mathcal{D}) \rightarrow \text{Fun}_n(\mathcal{B}, \mathcal{C})$ is a d -cocartesian fibration as well and restriction along $\mathcal{B}' \rightarrow \mathcal{B}$ defines a strong morphism.*
- (3) *The subcategory of $\text{Fun}([1], \text{Cat}_n)$ spanned by the d -cocartesian fibrations and strong morphisms between them is closed under limits.*

Proof. All properties follow from Lemma 5.4 by induction, the case $d = 0$ being readily verified. The only assertion requiring extra care is part (2). Using part (3), it suffices to verify this in the case where $\mathcal{B} = [1]_{\mathcal{A}}$ for an $(n - 1)$ -category \mathcal{A} (and likewise for \mathcal{B}'). By Lemma 5.4, $q: \text{Fun}_n([1]_{\mathcal{A}}, \mathcal{D}) \rightarrow \text{Fun}_n([1]_{\mathcal{A}}, \mathcal{C})$ is pre-cocartesian and restriction preserves cocartesian 1-morphisms. To see that it is locally $(d - 1)$ -cartesian, consider the diagram of simplicial $(n - 1)$ -categories, given as follows in simplicial degree m .

$$\begin{array}{ccccc}
 \text{Fun}_{n-1}(\mathcal{A}, \mathcal{D}([1] \times [m])) & \longrightarrow & \text{Fun}_{n-1}(\mathcal{A}, \mathcal{D}(1)^{\times m+1}) & \longleftarrow & \text{c}_0 \text{Fun}_{n-1}(\mathcal{A}, \mathcal{D}(1)^{\times m+1}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Fun}_{n-1}(\mathcal{A}, \mathcal{C}([1] \times [m])) & \longrightarrow & \text{Fun}_{n-1}(\mathcal{A}, \mathcal{C}(1)^{\times m+1}) & \longleftarrow & \text{c}_0 \text{Fun}_{n-1}(\mathcal{A}, \mathcal{C}(1)^{\times m+1})
 \end{array}$$

Using Lemma 5.21, it follows by inductive hypothesis that the vertical maps define simplicial diagrams of $(d - 1)$ -cartesian fibrations and strong morphisms between them. Furthermore, the horizontal maps define strong morphisms between these $(d - 1)$ -cartesian fibrations. Taking pullbacks along the rows, we then obtain another simplicial diagram of $(d - 1)$ -cartesian fibrations and strong morphisms. By Corollary 4.9, this is precisely the map $q: \text{Fun}_n([1]_{\mathcal{A}}, \mathcal{D}) \rightarrow \text{Fun}_n([1]_{\mathcal{A}}, \mathcal{C})$.

We conclude by induction and Lemma 5.21 that q is locally $(d - 1)$ -cartesian. From the above description one readily verifies that restriction along $[1]_{\mathcal{A}'} \rightarrow [1]_{\mathcal{A}}$ preserves lateral k -morphisms for all $1 \leq k \leq d$. □

5.3 The $(d + 2)$ -category of d -cocartesian fibrations

Because of Remark 5.18, we will henceforth *only consider d -cocartesian fibrations between $(d + 1)$ -categories*. In light of Lemma 5.17, this is not really restrictive, since we can always choose d large enough to cover n -cocartesian fibration between m -categories for some given m and n .

DEFINITION 5.23. We will write $\mathbf{Cocart}_d \subseteq \text{Fun}_{d+1}([1], \mathbf{Cat}_{d+1})$ for the sub- $(d + 2)$ -category whose objects are the d -cocartesian fibrations $p: \mathcal{D} \rightarrow \mathcal{C}$ and whose 1-morphisms are the strong morphisms between them (and all higher morphisms between those). Let us write $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ for the codomain projection and $\mathbf{Cocart}_d(\mathcal{C})$ for the fiber over a $(d + 1)$ -category \mathcal{C} .

Furthermore, let us write $\pi: \text{Cocart}_d \rightarrow \text{Cat}_{d+1}$ for the induced functor between 1-cores. The fiber over a $(d + 1)$ -category \mathcal{C} is denoted $\text{Cocart}_d(\mathcal{C})$.

Remark 5.24. One can model $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ by a strict functor between \mathbf{Cat}_{d+1} -enriched categories, where \mathbf{Cocart}_d is modeled by an enriched subcategory of $\text{Fun}_{\text{enr}}([1], \mathbf{Cat}_{d+1})$ whose objects are fibrant models for d -cocartesian fibrations and whose mapping objects are the subobjects consisting of strong morphisms.

PROPOSITION 5.25. *The projection $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ is a $(d + 1)$ -cartesian fibration.*

LEMMA 5.26. *Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a d -cocartesian fibration and let \mathcal{A} be a $(d + 1)$ -category equipped with subspaces $S_k \subseteq \mathcal{A}(1, \dots, 1, \vec{0}_{d+1-k})$ of ‘marked k -morphisms’ for all $1 \leq k \leq d$ (without any further condition). Let $\text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D}) \subseteq \text{Fun}_{d+1}(\mathcal{A}, \mathcal{D})$ denote the full sub- $(d + 1)$ -category spanned by those functors $\mathcal{A} \rightarrow \mathcal{D}$ sending every marked k -morphism in \mathcal{A} to a p -lateral k -morphism in \mathcal{D} . Then the natural map $\text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D}) \rightarrow \text{Fun}_{d+1}(\mathcal{A}, \mathcal{C})$ is a d -cocartesian fibration.*

Proof. By Lemma 5.22, $\text{Fun}_{d+1}(\mathcal{A}, \mathcal{D}) \rightarrow \text{Fun}_{d+1}(\mathcal{A}, \mathcal{C})$ is a d -cocartesian fibration. Since $\text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D}) \subseteq \text{Fun}_{d+1}(\mathcal{A}, \mathcal{D})$ is a full sub- $(d + 1)$ -category, $\text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D}) \rightarrow \text{Fun}_{d+1}(\mathcal{A}, \mathcal{C})$ is a homwise $(d - 1)$ -cartesian fibration as well. It then remains to verify the following assertion: if $\mu: f \rightarrow g$ is a cocartesian 1-morphism in $\text{Fun}_{d+1}(\mathcal{A}, \mathcal{D})$ such that $f \in \text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D})$, then $g \in \text{Fun}_{d+1}^\#(\mathcal{A}, \mathcal{D})$ as well.

Let us first verify that g sends all marked 1-morphisms in \mathcal{A} to cocartesian morphisms in \mathcal{D} : indeed, for every such map $\alpha: a_0 \rightarrow a_1$, there is a commuting square in \mathcal{D} of the following form.

$$\begin{array}{ccc} f(a_0) & \xrightarrow{\mu} & g(a_0) \\ f(\alpha) \downarrow & & \downarrow g(\alpha) \\ f(a_1) & \xrightarrow{\mu} & g(a_1) \end{array}$$

The horizontal arrows are cocartesian since μ was a cocartesian natural transformation (Lemma 5.22). Since $f(\alpha)$ is a cocartesian arrow, one sees that $g(\alpha)$ is a cocartesian arrow as well (cf. the argument in [Lur09a, Proposition 2.4.1.7] and [RV22, Lemma 5.1.5]).

To see that g preserves lateral k -morphisms for $k > 1$, note that for each $a_0, a_1 \in \mathcal{A}$ there is the following commuting square of mapping d -categories.

$$\begin{array}{ccc} \text{Map}_{\mathcal{A}}(a_0, a_1) & \xrightarrow{f} & \text{Map}_{\mathcal{D}}(fa_0, fa_1) \\ g \downarrow & & \downarrow \mu(a_1)_* \\ \text{Map}_{\mathcal{D}}(ga_0, ga_1) & \xrightarrow{\mu(a_0)^*} & \text{Map}_{\mathcal{D}}(fa_0, ga_1) \end{array}$$

Since $p: \mathcal{D} \rightarrow \mathcal{C}$ is a d -cocartesian fibration, the functor $\mu(a_1)_*$ postcomposing with $\mu(a_1)$ is strong, i.e. it preserves lateral k -morphisms. Since f sends marked k -morphisms to lateral

k -morphisms, it follows that $\mu(a_1)_* \circ f$ does as well. On the other hand, since $\mu(a_0)$ is a cocartesian 1-morphism, the map $\mu(a_0)^*$ is an equivalence of d -categories. This implies that g sends marked k -morphisms to lateral k -morphisms as well. \square

Proof of Proposition 5.25. Let us start by verifying that $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ is a homwise d -cocartesian fibration. Let $p_0: \mathcal{D}_0 \rightarrow \mathcal{C}_0$ and $p_1: \mathcal{D}_1 \rightarrow \mathcal{C}_1$ be d -cocartesian fibrations. We have to check that the induced map on mapping $(d + 1)$ -categories

$$\pi: \text{Map}_{\mathbf{Cocart}_d}(p_0, p_1) \longrightarrow \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_1)$$

is a d -cocartesian fibration. To see this, note that one can identify

$$\text{Map}_{\mathbf{Cocart}_d}(p_0, p_1) = \text{Fun}_{d+1}^{\natural}(\mathcal{D}_0, \mathcal{D}_1) \times_{\text{Fun}_{d+1}(\mathcal{D}_0, \mathcal{C}_1)} \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_1) \tag{5.27}$$

where $\text{Fun}_{d+1}^{\natural}(\mathcal{D}_0, \mathcal{D}_1) \subseteq \text{Fun}_{d+1}(\mathcal{D}_0, \mathcal{D}_1)$ is the full sub- $(d + 1)$ -category spanned by those functors $\mathcal{D}_0 \rightarrow \mathcal{D}_1$ preserving lateral k -morphisms. The result then follows from Lemma 5.26 and stability of d -cocartesian fibrations under base change (Lemma 5.22).

Next, given another d -cocartesian fibration $p_2: \mathcal{D}_2 \rightarrow \mathcal{C}_2$, we have to verify that the square induced by composition

$$\begin{array}{ccc} \text{Map}_{\mathbf{Cocart}_d}(p_1, p_2) \times \text{Map}_{\mathbf{Cocart}_d}(p_0, p_1) & \xrightarrow{\circ} & \text{Map}_{\mathbf{Cocart}_d}(p_0, p_2) \\ \downarrow & & \downarrow \\ \text{Fun}_{d+1}(\mathcal{C}_1, \mathcal{C}_2) \times \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_1) & \xrightarrow{\circ} & \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_2) \end{array}$$

gives a strong morphism of d -cocartesian fibrations. To see this, let $f, g \in \text{Map}(p_0, p_1)$ and $f', g' \in \text{Map}(p_1, p_2)$. We will abuse notation and write $f, g: \mathcal{D}_0 \rightarrow \mathcal{D}_1$ and $f', g': \mathcal{D}_1 \rightarrow \mathcal{D}_2$ for the underlying functors, which all preserve lateral k -morphisms. Let $\alpha: f \rightarrow g$ and $\beta: f' \rightarrow g'$ be lateral k -morphisms (which we depict as 1-morphisms instead of higher cells, for simplicity). Then the image of (α, β) under \circ is given by the horizontal composition

$$f' f \xrightarrow{f' \alpha} f' g \xrightarrow{\beta g} g' g.$$

Then $f' \alpha$ is lateral since f' preserves lateral k -morphisms and βg is lateral since lateral natural transformations are detected pointwise. Consequently, their horizontal composite is a lateral k -morphism as well.

It remains to check that $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ is a pre-cartesian fibration. Given a map $f: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and a d -cocartesian fibration $p_2: \mathcal{D}_2 \rightarrow \mathcal{C}_2$, let $p_1: \mathcal{D}_1 = \mathcal{D}_2 \times_{\mathcal{C}_2} \mathcal{C}_1 \rightarrow \mathcal{C}_1$ be the base change. We have to show that the canonical (strong) map of cocartesian fibrations $\tilde{f}: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ defines a cartesian lift of f , i.e. that for any $p_0: \mathcal{D}_0 \rightarrow \mathcal{C}_0$, the square

$$\begin{array}{ccc} \text{Map}_{\mathbf{Cocart}_d}(p_0, p_1) & \xrightarrow{\tilde{f}_*} & \text{Map}_{\mathbf{Cocart}_d}(p_0, p_2) \\ \downarrow & & \downarrow \\ \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_1) & \xrightarrow{f_*} & \text{Fun}_{d+1}(\mathcal{C}_0, \mathcal{C}_2) \end{array}$$

is cartesian. Using (5.27), one sees that this comes down to verifying that

$$\begin{array}{ccc} \text{Fun}_{d+1}^{\natural}(\mathcal{D}_0, \mathcal{D}_2 \times_{\mathcal{C}_2} \mathcal{C}_1) & \longrightarrow & \text{Fun}_{d+1}^{\natural}(\mathcal{D}_0, \mathcal{D}_2) \\ \downarrow & & \downarrow \\ \text{Fun}_{d+1}^{\natural}(\mathcal{D}_0, \mathcal{C}_1) & \longrightarrow & \text{Fun}_{d+1}(\mathcal{D}_0, \mathcal{C}_2) \end{array}$$

is cartesian. This follows immediately from the fact that a functor $\mathcal{D}_0 \rightarrow \mathcal{D}_2 \times_{\mathcal{C}_2} \mathcal{C}_1$ preserves lateral k -morphisms if and only if the corresponding map $\mathcal{D}_0 \rightarrow \mathcal{D}_2$ does. \square

6. Straightingening and unstraightening

Finally, we turn to straightening and unstraightening: for a small $(d + 1)$ -category \mathcal{C} , we will establish an equivalence between d -cocartesian fibrations over \mathcal{C} and copresheaves $\mathcal{C} \rightarrow \mathbf{Cat}_d$. We will start by describing this as an equivalence of 1-categories (Theorem 6.1). We then use this to establish a more structured version of straightening–unstraightening that provides an equivalence of $(d + 1)$ -categories and is furthermore $(d + 2)$ -functorial in \mathcal{C} (Theorem 6.20).

6.1 Straightening and unstraightening: 1-categorical version

Our first goal will be to prove the following.

THEOREM 6.1. *There is an equivalence of cartesian fibrations between 1-categories as follows.*

$$\begin{array}{ccc} \text{coPSh}_d^{\text{Seg}} & \xrightarrow{\sim} & \text{Cocart}_d \\ & \searrow & \swarrow \\ & \text{Cat}_{d+1} & \end{array}$$

In particular, for each $(d + 1)$ -category \mathcal{C} this induces equivalences of 1-categories

$$\mathbf{c}_1\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \xrightarrow{\sim} \text{coPSh}_d^{\text{Seg}}(\mathcal{C}) \xrightarrow{\sim} \text{Cocart}_d(\mathcal{C})$$

between (Segal) copresheaves of d -categories on \mathcal{C} and d -cocartesian fibrations over \mathcal{C} .

Remark 6.2. Similarly, for every $(d + 1)$ -category \mathcal{C} , there is an equivalence of 1-categories between cartesian fibrations and presheaves:

$$\mathbf{c}_1\text{Fun}_{d+1}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_d) \simeq \mathbf{c}_1\text{Fun}_{d+1}(\mathcal{C}^{(1, \dots, d+1)\text{-op}}, \mathbf{Cat}_d) \simeq \text{Cocart}_d(\mathcal{C}^{(1, \dots, d+1)\text{-op}}) \simeq \text{Cart}_d(\mathcal{C}).$$

Here the first equivalence sends a functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}_d$ to the functor $c \mapsto F(c)^{(1, \dots, d)\text{-op}}$ (which changes the variance with respect to higher morphisms). The second equivalence is Theorem 6.1 and the last equivalence takes $(1, \dots, d + 1)$ -opposites to obtain a cartesian fibration (Remark 5.14), whose fibers are equivalent to the values of F .

We will prove this theorem by a repeated application of Theorem 3.1 in different simplicial directions. Concretely, the above equivalence will arise as a composition of d equivalences, whose intermediate categories consist of certain fibrations of $(d + 1)$ -fold categories.

DEFINITION 6.3. Let \mathcal{C} be a $(d + 1)$ -category and let $p: \mathbb{D} \rightarrow \mathcal{C}$ be a map of $(d + 1)$ -fold categories. By induction, we will say that p is a fibration of the following type.

- A *0th stage fibration* if $\mathbb{D}(1) \rightarrow \mathcal{C}(1) \times_{\mathcal{C}(\{0\})} \mathbb{D}(\{0\})$ is an equivalence of d -fold categories. A *strong map* of 0th stage fibrations is simply a commuting square.
- A *k th stage fibration*, for $1 \leq k \leq d$, if it satisfies the following two conditions.
 - (a1) Each $\mathbb{D}(n)^{(1, \dots, d)\text{-op}} \rightarrow \mathcal{C}(n)^{(1, \dots, d)\text{-op}}$ is a $(k - 1)$ th stage fibration and each $[n] \rightarrow [n']$ induces a strong map between $(k - 1)$ th stage fibrations.
 - (a2) The map of categories $\mathbb{D}(-, \vec{0}_d) \rightarrow \mathcal{C}(-, \vec{0}_d)$ is a cocartesian fibration and each map $\mathbb{D}(-, \vec{0}_d) \rightarrow \mathbb{D}(-, \vec{n}_d) \times_{\mathcal{C}(-, \vec{n}_d)} \mathcal{C}(-, \vec{0}_d)$ preserves cocartesian arrows (over $\mathcal{C}(-, \vec{0}_d)$). A strong map between k th stage fibrations is a commuting square such that:
 - (b1) evaluating at $[n] \in \mathbf{\Delta}$ in the first coordinate yields a strong map of $(k - 1)$ th stage fibrations;
 - (b2) the map $\mathbb{D}(-, \vec{0}_d) \rightarrow \mathbb{D}'(-, \vec{0}_d)$ preserves cocartesian arrows.

We will write $\text{Fib}_d^{(k)} \subseteq \text{Fun}([1], \text{Cat}^{\otimes d+1})$ for the subcategory whose objects are k th stage fibrations (in particular, the codomain is a $(d + 1)$ -category) and whose maps are commuting squares satisfying conditions (b1) and (b2). The codomain projection defines a cartesian fibration

$$\text{codom}: \text{Fib}_d^{(k)} \longrightarrow \text{Cat}_{d+1}.$$

PROPOSITION 6.4. *For each $1 \leq k \leq d$, there is an equivalence of categories as follows.*

$$\begin{array}{ccc} \text{Fib}_d^{(k-1)} & \xrightarrow{\sim} & \text{Fib}_d^{(k)} \\ \text{codom} \searrow & & \swarrow \text{codom} \\ & \text{Cat}_{d+1} & \end{array}$$

Proof. Let us fix $k \geq 1$ and start with the following observation: if $p: \mathbb{D} \rightarrow \mathcal{C}$ is a k th stage fibration, then each

$$\mathbb{D}(\vec{n}_{k-1}, -, -, \vec{n}_{d-k}) \longrightarrow \mathcal{C}(\vec{n}_{k-1}, -, -, \vec{n}_{d-k}) \tag{6.5}$$

is a (cart, left)-fibration if k is even, and a (cocart, right)-fibration if k is odd. Furthermore, each $\vec{n}_{k-1} \rightarrow \vec{n}'_{k-1}$ and $\vec{n}_{d-k} \rightarrow \vec{n}'_{d-k}$ induces a map preserving (co)cartesian morphisms. When $k = 1$ this follows from condition (a2) and the fact that $\mathcal{C}(-, \vec{0}_d) \simeq \mathcal{C}(-, 0, \vec{n}_{d-1})$; for higher k it follows inductively, by repeatedly applying condition (a1).

From this point on, let us assume that k is odd; when k is even, the same argument applies up to taking opposites (in all dimensions). Note that for a $(k - 1)$ th stage fibration $p: \mathbb{D} \rightarrow \mathcal{C}$, the map (6.5) is a (left, cart)-fibration and each $\vec{n}_{k-1} \rightarrow \vec{n}'_{k-1}$ and $\vec{n}_{d-k} \rightarrow \vec{n}'_{d-k}$ induces a strong map of (left, cart)-fibrations. Indeed, the previous paragraph shows that p is a left fibration in the k th variable. Furthermore, setting the k th entry equal to 0 yields a map $\mathbb{D}(\vec{n}_{k-1}, 0, -, \vec{n}_{d-k}) \rightarrow \mathcal{C}(\vec{n}_{k-1}, 0, -, \vec{n}_{d-k})$ from a 1-category to a space. In particular, each such map is a cartesian fibration and since its cartesian arrows are the equivalences, each $\vec{n}_{k-1} \rightarrow \vec{n}'_{k-1}$ and $\vec{n}_{d-k} \rightarrow \vec{n}'_{d-k}$ induces a map preserving cartesian arrows.

Denoting $\mathcal{J} = (\Delta^{\times k-1} \times \Delta^{\times d-k})^{\text{op}}$, we therefore obtain the following diagram.

$$\begin{array}{ccccc} \text{Fib}_d^{(k-1)} & \hookrightarrow & \text{Fun}(\mathcal{J}, \text{Fib}^{\text{left, cart}}) & \xrightarrow{\Psi_{(k,k+1)}^\perp} & \text{Fun}(\mathcal{J}, \text{Fib}^{\text{cocart, right}}) \\ & \searrow \text{dotted} & & \sim & \\ & & \text{Fib}_d^{(k)} & \hookrightarrow & \text{Fun}(\mathcal{J}, \text{Fib}^{\text{cocart, right}}) \\ \text{codom} \searrow & & \downarrow \text{codom} & & \downarrow \text{codom} \\ & & \text{Cat}_d & \hookrightarrow & \text{Cat}^{\otimes d} \end{array}$$

Here the functors from left to right are subcategory inclusions and the top right equivalence is the reflection functor from Theorem 3.1, applied to the k th and $(k + 1)$ th variable. We will prove by induction on k that the equivalence $\Psi_{(k,k+1)}^\perp$ identifies the subcategories of $(k - 1)$ th stage fibrations and k th stage fibrations.

When $k = 1$, note that the bottom and front square are pullbacks: a map $\mathbb{D} \rightarrow \mathcal{C}$ is a 0th stage fibration if and only if it is a natural (left, cart)-fibration in the first two variables, and a 1st stage fibration if and only if it is a natural (cocart, right)-fibration in the first two variables. In particular, $\Psi_{(1,2)}^\perp$ identifies 0th stage fibrations and 1st stage fibrations.

For $k > 1$, the inductive hypothesis implies that $p: \mathbb{D} \rightarrow \mathcal{C}$ satisfies condition (a1) for $(k - 1)$ th stage fibrations if and only if $\Psi_{(k,k+1)}^\perp(p): \Psi^\perp(\mathbb{D}) \rightarrow \mathcal{C}$ satisfies condition (a1) for k th stage fibrations. For condition (a2), note that $\mathbb{D} \times_{\mathcal{C}} \mathbf{c}_1(\mathcal{C}) \rightarrow \mathbf{c}_1(\mathcal{C})$ is given in the k th and $(k + 1)$ th variables by a (left, cart)-fibration over a space. For such (left, cart)-fibrations over spaces, the functor Ψ^\perp is equivalent to the functor that exchanges the k th and $(k + 1)$ th variable (Proposition 3.15). Using that Ψ^\perp is compatible with base change, we therefore find

$$\begin{aligned} \Psi_{(k,k+1)}^\perp(\mathbb{D})(-,\vec{n}_{k-2},m,n,\vec{n}_{d-k}) \times_{\mathcal{C}(-,\vec{n}_{k-2},m,n,\vec{n}_{d-k})} \mathcal{C}(-,\vec{0}_d) \\ \simeq \mathbb{D}(-,\vec{n}_{k-2},n,m,\vec{n}_{d-k}) \times_{\mathcal{C}(-,\vec{n}_{k-2},n,m,\vec{n}_{d-k})} \mathcal{C}(-,\vec{0}_d). \end{aligned}$$

It follows that p satisfies condition (a2) if and only if $\Psi_{(k,k+1)}^\perp(p)$ does. A similar argument shows that $\Psi_{(k,k+1)}^\perp$ identifies strong morphisms between $(k - 1)$ th stage fibrations and k th stage fibrations, so that $\Psi_{(k,k+1)}^\perp$ indeed restricts to an equivalence $\text{Fib}_d^{(k-1)} \xrightarrow{\sim} \text{Fib}_d^{(k)}$. \square

COROLLARY 6.6. Consider the following strong morphism of k th stage fibrations.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & \mathbb{D}' \\ p \downarrow & & \downarrow p' \\ \mathcal{C} & \longrightarrow & \mathcal{C}' \end{array}$$

This square is cartesian if and only if the map f induces equivalences on fibers $\mathbb{D}_c \rightarrow \mathbb{D}'_c$ for each object $c \in \mathcal{C}$.

Proof. This is evident for $k = 0$. For higher k , it follows from Proposition 6.4. \square

LEMMA 6.7. Let $p: \mathbb{D} \rightarrow \mathcal{C}$ be a d th stage fibration from a $(d + 1)$ -fold category to a $(d + 1)$ -category. Then the following two conditions are equivalent.

- (1) For any object $c \in \mathcal{C}$, the fiber \mathbb{D}_c is a d -category in the first d -variables, and constant in the last variable.
- (2) The $(d + 1)$ -fold category \mathbb{D} is a $(d + 1)$ -category.

Proof. We proceed by induction on d , the case $d = 0$ being obvious. Note that a d th stage fibration is either a left or a right fibration in the last variable, so that $\mathbb{D}(\vec{n}_d, p) \simeq \mathbb{D}(\vec{n}_d, 0) \times_{\mathcal{C}(\vec{n}_d, 0)} \mathcal{C}(\vec{n}_d, p)$. This immediately shows that condition (2) implies condition (1). For the converse, note that condition (1) implies that the fibers of $\mathbb{D}(0) \rightarrow \mathcal{C}(0)$ are spaces, so that $\mathbb{D}(0)$ is a space. It then remains to verify that $\mathbb{D}(1)$ is a d -category. Since $\mathbb{D}(1) \rightarrow \mathcal{C}(1)$ is the opposite of a $(d - 1)$ th stage fibration, the inductive hypothesis says that it suffices to verify that its fibers are $(d - 1)$ -categories. In particular, we can reduce to the case where $\mathcal{C} \simeq \mathbf{c}_1(\mathcal{C})$ is a 1-category.

In this case, take a morphism $\gamma: c_0 \rightarrow c_1$ in \mathcal{C} and consider the map between fibers $\mathbb{D}(1)_\gamma \rightarrow \mathbb{D}(\{0\})_{c_0}$. Since the target is a space, it suffices to verify that the fiber over each point $d_0 \in \mathbb{D}_{c_0}$ is a $(d - 1)$ -category. Using condition (a2), we can take a cocartesian lift $\alpha: d_0 \rightarrow d_1$ of γ such that precomposition with α determines the following equivalence of $(d - 1)$ -fold categories:

$$\alpha^*: \mathbb{D}(1)_{\text{id}_{c_1}} \times_{\mathbb{D}(\{0\})} \{d_1\} \xrightarrow{\sim} \mathbb{D}(1)_\gamma \times_{\mathbb{D}(\{0\})} \{d_0\}.$$

The domain is a $(d - 1)$ -category since the fiber \mathbb{D}_{c_1} is a d -category by assumption. \square

PROPOSITION 6.8. There is a fully faithful inclusion $\text{Cocart}_d \hookrightarrow \text{Fib}_d^{(d)}$, whose essential image consists of the d th stage fibrations satisfying the equivalent conditions of Lemma 6.7.

Proof. Both Cocart_d and the full subcategory of $\text{Fib}_d^{(d)}$ defined by Lemma 6.7 are subcategories of $\text{Fun}([1], \text{Cat}^{\otimes d+1})$. It therefore suffices to show that these two subcategories have the same objects and morphisms. We will do this by induction on d , the case $d = 0$ being evident.

Objects. Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a map between $(d + 1)$ -categories. By inductive hypothesis and Lemma 5.21, p satisfies condition (a1) if and only if p is a homwise $(d - 1)$ -cartesian fibration. Assuming that p satisfies these equivalent conditions, it remains to show that p satisfies condition (a2) if and only if it is a pre-cocartesian fibration. To see this, consider the base change

$$p': \mathcal{D}' = \mathcal{D} \times_{\mathcal{C}} \mathbf{c}_1(\mathcal{C}) \longrightarrow \mathbf{c}_1(\mathcal{C}).$$

For any 1-morphism $\alpha: d_0 \rightarrow d_1$ and object d_2 in \mathcal{D}' (or, equivalently, in \mathcal{D}), we have the following commutative diagram.

$$\begin{array}{ccccc}
 \text{Map}_{\mathcal{D}'}(d_1, d_2) & \xrightarrow{\alpha^*} & \text{Map}_{\mathcal{D}'}(d_0, d_2) & & \\
 \downarrow p' & \searrow & \downarrow & \searrow & \\
 & \text{Map}_{\mathcal{D}}(d_1, d_2) & \xrightarrow{\quad} & \text{Map}_{\mathcal{D}}(d_0, d_2) & \\
 & \downarrow & & \downarrow p & \\
 \text{Map}_{\mathbf{c}_1(\mathcal{C})}(pd_1, pd_2) & \xrightarrow{\quad} & \text{Map}_{\mathbf{c}_1(\mathcal{C})}(pd_0, pd_2) & & \\
 & \searrow & \downarrow & \searrow & \\
 & \text{Map}_{\mathcal{C}}(pd_1, pd_2) & \xrightarrow{p(\alpha)^*} & \text{Map}_{\mathcal{C}}(pd_0, pd_2) &
 \end{array}$$

The left and right face of the cube are cartesian. By our assumption on p , the vertical arrows are $(d - 1)$ -cartesian fibrations and the front and back face determine strong maps of $(d - 1)$ -cartesian fibrations. By inductive hypothesis, this means equivalently that the vertical arrows are $(d - 1)$ th stage fibrations between d -categories and that the front and back face determine strong morphisms between $(d - 1)$ th stage fibrations. Since the back and front faces are equivalent at the level of objects, Corollary 6.6 then implies that the front face is cartesian if and only if the back face is cartesian. By Corollary 5.7, this means that α is p' -cocartesian if and only if it is p -cocartesian. It follows that p is a pre-cocartesian fibration if and only if p' is a pre-cocartesian fibration, which, in turn, is equivalent to condition (a2), by Remark 5.8.

Morphisms. A commuting square determines a strong morphism between d -cocartesian fibrations if and only if the induced square on d -categories of morphisms induces a strong morphism between $(d - 1)$ -cartesian fibrations and the map $\mathbf{c}_1(\mathcal{D}) \rightarrow \mathbf{c}_1(\mathcal{D}')$ preserves cocartesian 1-morphisms. Comparing this with conditions (b1) and (b2) shows inductively that such strong morphisms coincide with strong morphisms of d th stage fibrations. \square

Proof (of Theorem 6.1). Composing the equivalences from Proposition 6.4 yields

$$\text{Fib}^{(0)} \xrightarrow{\sim} \text{Fib}^{(d)} \tag{6.9}$$

that commutes with the codomain projection. Note that on both sides, the fiber over $* \in \text{Cat}_{d+1}$ can be identified with the category $\text{Cat}^{\otimes d}$ of d -fold categories: in $\text{Fib}^{(0)}$, a d -fold category is viewed as a $(d + 1)$ -fold category which is constant in the first variable, and in $\text{Fib}^{(d)}$ it is constant in the last variable. Since the above equivalence is given by repeatedly applying the functor Ψ^\perp , Proposition 3.15 shows that the induced equivalence on fibers over $* \in \text{Cat}_{d+1}$ simply sends a d -fold category to itself.

Now observe that there is a fully faithful inclusion $\text{coPSh}_d^{\text{Seg}} \hookrightarrow \text{Fib}^{(0)}$, whose essential image consists of all those 0th stage fibrations $p: \mathbb{D} \rightarrow \mathcal{C}$ with the property that for any object $c \in \mathcal{C}$, the fiber \mathbb{D}_c is a d -category (rather than a d -fold category). The essential image of $\text{coPSh}_d^{\text{Seg}}$ under (6.9) then consists of d th stage fibrations whose fibers are d -categories. By Proposition 6.8, this means that (6.9) restricts to an equivalence $\text{coPSh}_d^{\text{Seg}} \xrightarrow{\sim} \text{Cocart}_d$, as desired. The second part of Theorem 6.1 now follows directly from Corollary 4.33. \square

6.2 Straightening and unstraightening: uniqueness and further properties

The equivalence of Theorem 6.1 arises from an iteration of explicit combinatorial maneuvers. The purpose of this intermezzo is to argue that at the level of objects, the precise construction of this equivalence is essentially irrelevant.

PROPOSITION 6.10. *There is a contractible space of equivalences of right fibrations as follows.*

$$\begin{array}{ccc} \text{coPSh}_d^{\text{Seg, cart}} & \xrightarrow{\sim} & \text{Cocart}_d^{\text{cart}} \\ & \searrow & \swarrow \\ & \text{Cat}_{d+1} & \end{array}$$

Combining this with Proposition 4.40 (in the large setting) and rephrasing everything in terms of presheaves rather than right fibrations (using straightening), we obtain the following.

COROLLARY 6.11. *For any small $(d + 1)$ -category \mathcal{C} , there is a unique 1-natural equivalence*

$$\text{Un}: \mathbf{c}_0\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d) \xrightleftharpoons{\sim} \mathbf{c}_0\text{Cocart}_d(\mathcal{C}): \text{St}$$

between the spaces of copresheaves $\mathcal{C} \rightarrow \mathbf{Cat}_d$ and of d -cocartesian fibrations $\mathcal{D} \rightarrow \mathcal{C}$. We will refer to these functors as *unstraightening* and *straightening*.

Remark 6.2 also gives a unique 1-natural equivalence $\mathbf{c}_0\text{Fun}_{d+1}(\mathcal{C}^{\text{op}}, \mathbf{Cat}_d) \simeq \mathbf{c}_0\text{Cart}_d(\mathcal{C})$.

Remark 6.12. In particular, the straightening of a d -cocartesian fibration $p: \mathcal{D} \rightarrow \mathcal{C}$ between $(d + 1)$ -categories is naturally equivalent to the straightening of p , viewed as a special kind of $(d + 1)$ -cocartesian fibration between $(d + 2)$ -categories.

We will deduce Proposition 6.10 from the following (simpler) version of a result of Barwick and Schommer-Pries [BS21].

PROPOSITION 6.13. *Let \mathbf{CAT}_{d+1} be the (very large) $(d + 2)$ -category of large $(d + 1)$ -categories and let $\text{Aut}_{\mathbf{CAT}_{d+1}}(\mathbf{Cat}_d) \subseteq \text{Fun}_{d+1}(\mathbf{Cat}_d, \mathbf{Cat}_d)$ be the full sub- $(d + 1)$ -category on those $(d + 1)$ -functors that are equivalences. Then $\text{Aut}_{\mathbf{CAT}_{d+1}}(\mathbf{Cat}_d) \simeq *$.*

Proof. Let $\phi: \mathbf{Cat}_d \rightarrow \mathbf{Cat}_d$ be an equivalence of $(d + 1)$ -categories. Since ϕ preserves the terminal object, we can pick a (unique) element $u \in \phi(*)$. By the enriched Yoneda lemma [Hin20, 6.2.7] (or Example 4.48), u induces a natural transformation of $(d + 1)$ -functors $u: \text{Map}_{\mathbf{Cat}_d}(*, -) \rightarrow \phi(-)$. Note that the domain can be identified with the identity on \mathbf{Cat}_d , so that u is, in particular, a natural transformation between two automorphisms of the 1-category Cat_d . By [BS21, Theorem 10.1], u is a natural equivalence.

We conclude that $\text{Aut}_{\mathbf{CAT}_{d+1}}(\mathbf{Cat}_d)$ has an essentially unique object, given by $\text{id}_{\mathbf{Cat}_d} \simeq \text{Map}_{\mathbf{Cat}_d}(*, -)$. Using the enriched Yoneda lemma once more, the endomorphism d -category of this object can be identified with $\text{Map}_{\mathbf{Cat}_d}(*, *) \simeq *$, which concludes the proof. \square

Proof of Proposition 6.10. By Theorem 6.1, the space of such equivalences is a nonempty torsor over the automorphism group G of the right fibration $\text{coPSh}_d^{\text{Seg, cart}} \rightarrow \text{Cat}_{d+1}$. Note that this right fibration fits into a pullback square

$$\begin{array}{ccc} \text{coPSh}_d^{\text{Seg, cart}} & \hookrightarrow & \text{coPSh}_d^{\text{Seg, cart}} \\ \pi \downarrow & & \downarrow \pi' \\ \text{Cat}_{d+1} & \hookrightarrow & \text{Cat}_{d+1} \end{array}$$

where $\text{coPSh}_d^{\text{Seg, cart}}$ is the category of Segal copresheaves $X \rightarrow \mathcal{C}$ over large $(d + 1)$ -categories, whose fibers are essentially small.

Applying Proposition 4.40 in the large setting (with $\kappa = \kappa_{\text{sm}}$ the supremum of all small cardinals), the right fibration π' corresponds under (some 1-natural version of) unstraightening to the presheaf $F: \text{CAT}_{d+1}^{\text{op}} \rightarrow \mathcal{S}^{\text{big}}$ representable by the $(d + 1)$ -category \mathbf{Cat}_d . We then obtain equivalences of spaces of automorphisms as follows:

$$\text{Aut}_{\text{CAT}_{d+1}}(\mathbf{Cat}_d) \xrightarrow{\sim} \text{Aut}(F) \xrightarrow{\sim} \text{Aut}(F|_{\text{Cat}_{d+1}}) \simeq G.$$

Here the first equivalence is the Yoneda embedding (4.45) and the last equivalence is unstraightening. The middle equivalence uses that CAT_{d+1} is a (large) ind-completion of Cat_{d+1} ; in particular, restriction along $\text{Cat}_{d+1} \rightarrow \text{CAT}_{d+1}$ is fully faithful on presheaves sending large κ_{sm} -filtered colimits in CAT_{d+1} to large limits in \mathcal{S}^{big} (e.g. on representable presheaves). Finally, Proposition 6.13 implies that the left space is contractible. \square

Theorems 4.32 and 6.1 imply that the (un)straightening equivalence of Corollary 6.11 admits some extension to an equivalence of 1-categories, rather than spaces. Using this, let us record some properties of the (un)straightening at the level of objects.

Example 6.14. The proof of Theorem 6.1 shows that over $\mathcal{C} = *$, the (un)straightening equivalence of Corollary 6.11 is homotopic to the identity on Cat_d . Since (un)straightening intertwines restriction of copresheaves with base change of d -cocartesian fibrations, it follows that the straightening of a cocartesian fibration $p: \mathcal{D} \rightarrow \mathcal{C}$ is given pointwise by $\text{St}(p)_c \simeq \mathcal{D}_c$.

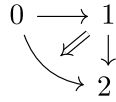
Example 6.15. Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a d -cocartesian fibration and let $\gamma: c_0 \rightarrow c_1$ be a 1-morphism in \mathcal{C} . The straightening of p determines a functor $\gamma_*: \mathcal{D}_{c_0} \rightarrow \mathcal{D}_{c_1}$ that can be understood explicitly as follows. For any diagram $\sigma_0: \mathcal{K} \rightarrow \mathcal{D}_{c_0}$, there exists a unique extension of σ_0 to a map $\sigma: [1] \times \mathcal{K} \rightarrow \mathcal{D}$ that covers $\gamma: [1] \rightarrow \mathcal{C}$ and preserves cocartesian arrows. The restriction $\sigma_1: \{1\} \times \mathcal{K} \rightarrow \mathcal{D}_1$ to the fiber over 1 then presents the composite diagram $\gamma_* \circ \sigma_0$. In particular, applying this when $\sigma_0 = \text{id}_{\mathcal{D}_{c_0}}$ gives a description of γ_* .

To see this, it suffices to treat the case where $\mathcal{C} = [1]$. By naturality with respect to the projection $[1] \rightarrow *$, the unstraightening of a constant copresheaf $\Delta_{\mathcal{K}}: [1] \rightarrow \mathbf{Cat}_d$ with value \mathcal{K} is given by the projection $\pi_1: [1] \times \mathcal{K} \rightarrow [1]$. Theorem 6.1 then shows that we have the following commuting diagram of mapping spaces.

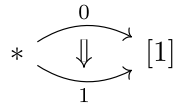
$$\begin{array}{ccc} & \text{Map}_{\text{Fun}([1], \text{Cat}_d)}(\Delta_{\mathcal{K}}, \text{St}(p)) & \\ \text{ev}_0 \swarrow \sim & \downarrow \sim & \searrow \text{ev}_1 \\ \text{Map}_{\text{Cat}_d}(\mathcal{K}, \mathcal{D}_0) & & \text{Map}_{\text{Cat}_d}(\mathcal{K}, \mathcal{D}_1) \\ \swarrow 0^* \sim & & \searrow 1^* \\ & \text{Map}_{\text{Cocart}_d([1])}(\pi_1, p) & \end{array}$$

Now note that the top zigzag is homotopic to postcomposition with γ_* , while the bottom zigzag corresponds precisely to the procedure described above.

Example 6.16. Let C_2 be the 2-cell and let $[2]^{\text{lax}}$ denote the following 2-category.



The projection $p: [2]^{\text{lax}} \rightarrow C_2$ sending $0 \mapsto 0$ and $1, 2 \mapsto 1$ is a 1-cocartesian fibration. By Examples 6.14 and 6.15, the straightening $\text{St}(p): C_2 \rightarrow \mathbf{Cat}_1$ is given by the diagram



where the natural transformation is the only possible one. In other words, $\text{St}(p) = \text{Map}_{C_2}(0, -)$ is representable by 0. Likewise, let \mathcal{K} be a d -category and consider the diagram $F_{\mathcal{K}}: C_2 \rightarrow \mathbf{Cat}_d$ encoding the evident natural transformation from $(0, \text{id})$ to $(1, \text{id}): \mathcal{K} \rightarrow [1] \times \mathcal{K}$. Then $\text{Un}(F_{\mathcal{K}}) \simeq [2]^{\text{lax}} \times \mathcal{K}$.

Now let $q: \mathcal{D} \rightarrow C_2$ be a general d -cocartesian fibration, so that $\text{St}(q)$ describes a natural transformation $\mu: f_0 \rightarrow f_1$ between two functors $f_0, f_1: \mathcal{D}_0 \rightarrow \mathcal{D}_1$. As in Example 6.15, one can obtain μ as follows: there is a unique map $[2]^{\text{lax}} \times \mathcal{D}_0 \rightarrow \mathcal{D}$ over C_2 restricting to the identity on $\{0\} \times \mathcal{D}_0$, and μ is given by its restriction $\{1 \leq 2\} \times \mathcal{D}_0 \rightarrow \mathcal{D}_1$.

Example 6.17. Given a 2-category \mathcal{C} , let us denote by $\text{Tw}_{\text{lax}}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ the 1-cocartesian fibration arising as the unstraightening of $\text{Map}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Cat}_1$. We will refer to $\text{Tw}_{\text{lax}}(\mathcal{C})$ as the *lax twisted arrow 2-category* of \mathcal{C} .

Explicitly, $\text{Tw}_{\text{lax}}(\mathcal{C})$ is the image under the functor Ψ^\perp of the Segal copresheaf $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$ (Example 4.36). Unraveling the definitions, one arrives at the following explicit description of $\text{Tw}_{\text{lax}}(\mathcal{C})$ as a bisimplicial space. For each $[m], [n] \in \mathbf{\Delta}$, consider the bisimplicial space (in fact, set)

$$J[m, n] = \text{colim}_{\substack{\alpha \star \beta: [q] \star [p] \rightarrow [m] \\ \alpha': [q] \rightarrow [n]}} ([p]^{\text{op}} \star [p]) \boxtimes [q].$$

This comes with natural inclusions $[m]^{\text{op}} \boxtimes [n] \rightarrow J[m, n] \leftarrow [m] \boxtimes [n]$, arising as parts of the bisimplex associated to $(\alpha, \alpha', \beta) = (\text{cst}(0), \text{id}_{[m]}, \text{id}_{[n]})$. One then has that

$$\text{Tw}_{\text{lax}}(\mathcal{C})(m, n) \subseteq \text{Map}_{\text{Fun}(\mathbf{\Delta} \times 2, \text{op}, \mathcal{S})}(J[m, n], \mathcal{C})$$

is the subspace of maps $J[m, n] \rightarrow \mathcal{C}$ whose restriction to any bisimplex of the form $([0]^{\text{op}} \star [0]) \boxtimes [q] \rightarrow J[m, n]$ (i.e. a summand for which $p = 0$) is degenerate in the q -direction.

Note that in the definition of $J[m, n]$, it suffices to take the colimit over the cofinal subcategory of (α, α', β) such that $(\alpha, \alpha'): [q] \rightarrow [m] \times [n]$ is injective and $\beta: [p] \rightarrow \{\alpha(q) \leq \dots \leq m\}$ is the inclusion of an upwards closed subset. Using this for low values of m and n , one sees that the 2-category $\text{Tw}_{\text{lax}}(\mathcal{C})$ has objects given by morphisms in \mathcal{C} , and 1- and 2-morphisms can be depicted by diagrams in \mathcal{C} of the following form.



Here the left diagram defines an arrow from $c_0 \rightarrow c_1$ to $d_0 \rightarrow d_1$ and the right (cylindrical) diagram defines a 2-morphism from λ to μ . The projection to $\mathcal{C}^{\text{op}} \times \mathcal{C}$ restricts to the left and right column.

By restricting to $\mathcal{C}^{\text{op}} \times \{c\}$ or $\{c\} \times \mathcal{C}$, we obtain an explicit description of the unstraightening of the representable presheaf $\text{Map}_{\mathcal{C}}(-, c)$ and copresheaf $\text{Map}_{\mathcal{C}}(c, -)$ in terms of lax versions of the over- and under-category of c .

LEMMA 6.18. *Let $p: \mathcal{D} \rightarrow \mathcal{C}$ be a d -cocartesian fibration between $(d + 1)$ -categories. Then the following assertions hold.*

- (1) *The induced map $p': \mathbf{c}_1\mathcal{D} \rightarrow \mathbf{c}_1\mathcal{C}$ is a 1-cocartesian fibration, whose straightening $\text{St}(p')$ can be identified with the 1-functor*

$$\text{St}(p'): \mathbf{c}_1\mathcal{C} \xrightarrow{\text{St}(p)} \text{Cat}_d \xrightarrow{\text{core}_1} \text{Cat}_1.$$

- (2) *The restriction to cocartesian arrows $p'': (\mathbf{c}_1\mathcal{D})^{\text{cocart}} \rightarrow \mathbf{c}_1\mathcal{C}$ is a 0-cocartesian fibration, whose straightening $\text{St}(p'')$ can be identified with the 1-functor*

$$\text{St}(p''): \mathbf{c}_1\mathcal{C} \xrightarrow{\text{St}(p)} \text{Cat}_d \xrightarrow{\text{core}_0} \mathcal{S}.$$

Proof. By replacing p by its base change $\mathcal{D} \times_{\mathcal{C}} \mathbf{c}_1\mathcal{C} \rightarrow \mathbf{c}_1\mathcal{C}$, we can reduce to the case where \mathcal{C} is a 1-category. In this case, Lemma 5.17 implies that a d -cocartesian fibration $q: \mathcal{E} \rightarrow \mathcal{C}$ is a 1-cocartesian fibration if and only if \mathcal{E} is a 1-category. Consequently, p' is the terminal 1-cocartesian fibration over \mathcal{C} equipped with a natural map to p . Under straightening, this corresponds to the terminal Cat_1 -valued copresheaf on \mathcal{C} equipped with a natural transformation to $\text{St}(p)$. This is precisely the composite functor given in assertion (1).

Likewise, Lemma 5.17 implies p'' is the terminal 0-cocartesian fibration with a map to p' . Under straightening, this corresponds to the terminal space-valued presheaf with a natural map to $\text{St}(p')$, which is precisely the functor given in assertion (2). □

6.3 Straightening and unstraightening: higher functoriality

Theorem 6.1 shows that for every $(d + 1)$ -category \mathcal{C} , there is an equivalence between the 1-category of d -cocartesian fibrations over \mathcal{C} and the 1-category underlying $\text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d)$. We will now refine this equivalence to a natural equivalence of $(d + 2)$ -functors with values in $(d + 1)$ -categories, using Theorem 6.1 one categorical dimension higher.

DEFINITION 6.19. Consider the $(d + 1)$ -cartesian fibration $\pi: \mathbf{Cocart}_d \rightarrow \mathbf{Cat}_{d+1}$ from Proposition 5.25. We will write

$$\mathbf{Cocart}_d(-): \mathbf{Cat}_{d+1}^{\text{op}} \longrightarrow \mathbf{CAT}_{d+1}$$

for the $(d + 2)$ -functor corresponding to π under the (unique) 1-natural straightening equivalence of Corollary 6.11 (in the large setting).

We now come to our main result.

THEOREM 6.20 (Straightening and unstraightening). *The $(d + 1)$ -category of natural equivalences of $(d + 2)$ -functors $\mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$,*

$$\text{St}: \mathbf{Cocart}_d(-) \xrightarrow[\sim]{\sim} \text{Fun}_{d+1}(-, \mathbf{Cat}_d): \mathbf{Un},$$

is equivalent to a single point.

Since \mathbf{Cat}_d is itself a large $(d + 1)$ -category, it will be convenient to first study a version of the above result where we bound the cardinality.

Notation 6.21. Let κ be a regular uncountable cardinal and consider the $(d + 2)$ -functor $\pi: \mathbf{Cocart}_d(\kappa) \rightarrow \mathbf{Cat}_{d+1}$ whose domain is the full subcategory of \mathbf{Cocart}_d on d -cocartesian fibrations with κ -small fibers. This is a $(d + 1)$ -cartesian fibration, and we will denote its image under the (natural) straightening of Corollary 6.11 by $\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{Cat}_{d+1}$. Note that the values of this functor are indeed small categories: Corollary 6.11 identifies their spaces of objects with the spaces of enriched functors $\mathcal{C} \rightarrow \mathbf{Cat}_d(\kappa)$.

Let us start by recording some properties of the $(d + 2)$ -functor \mathbf{Cocart}_d^κ .

LEMMA 6.22. *Let $f: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor of $(d + 1)$ -categories. Then the $(d + 1)$ -functor*

$$f^* = \mathbf{Cocart}_d^\kappa(f): \mathbf{Cocart}_d^\kappa(\mathcal{C}') \rightarrow \mathbf{Cocart}_d^\kappa(\mathcal{C})$$

is tensored over $\mathbf{Cat}_d(\kappa)$ (in particular, so are its domain and codomain).

Heuristically, this simply comes down to the fact that the tensoring of $\mathbf{Cocart}_d^\kappa(\mathcal{C})$ is given by $K \otimes (\mathcal{D} \rightarrow \mathcal{C}) = (K \times \mathcal{D} \rightarrow \mathcal{C})$ and that such products are stable under base change. However, since we defined the $(d + 1)$ -functor f^* somewhat abstractly, in terms of the straightening from Corollary 6.11, we will provide a more detailed argument using point-set models.

Proof. The $(d + 2)$ -functor $\pi: \mathbf{Cocart}_d(\kappa) \rightarrow \mathbf{Cat}_{d+1}$ can be modeled by a strict functor of \mathbf{Cat}_{d+1} -enriched categories $\underline{\pi}: \mathbf{Cocart}_d(\kappa) \rightarrow \mathbf{Cat}_{d+1}$. Here $\mathbf{Cocart}_d(\kappa)$ is the enriched subcategory of $\text{Fun}([1], \mathbf{Cat}_d)$, whose objects are fibrations in \mathbf{Cat}_{d+1} that present d -cocartesian fibrations with κ -small fibers and with

$$\text{Map}_{\mathbf{Cocart}_d(\kappa)}(p, q) \subseteq \text{Map}_{\text{Fun}([1], \mathbf{Cat}_d)}(p, q)$$

the maximal subobject in \mathbf{Cat}_{d+1} that is a model for the full sub- d -category of strong morphisms. Example 6.14 identifies the values of $\mathbf{Cocart}_d^\kappa(-)$ with the fibers of π . Since $\underline{\pi}$ is a fibration in the model category of \mathbf{Cat}_{d+1} -enriched categories, the fibers of π can be modeled by the strict fibers of $\underline{\pi}$. Furthermore, as we already know that the fibers $\mathbf{Cocart}_d^\kappa(\mathcal{C})$ are $(d + 1)$ -categories (instead of $(d + 2)$ -categories), we can model each $\mathbf{Cocart}_d^\kappa(-)$ by the \mathbf{Cat}_d -enriched category underlying $\mathbf{Cocart}_d^\kappa(\mathcal{C}) = \underline{\pi}^{-1}(\mathcal{C})$.

As a \mathbf{Cat}_d -enriched category (rather than a \mathbf{Cat}_{d+1} -enriched category), this fiber is strictly tensored over $\mathbf{Cat}_d(\kappa)$: the tensoring of $p: \mathcal{D} \rightarrow \mathcal{C}$ with $K \in \mathbf{Cat}_d(\kappa)$ is simply given by $\mathcal{D} \times K \rightarrow \mathcal{C}$. Note that for any map $f: \mathcal{C} \rightarrow \mathcal{C}'$ in \mathbf{Cat}_{d+1} , taking (strict) base change along f defines an enriched functor $f^*: \mathbf{Cocart}_d^\kappa(\mathcal{C}') \rightarrow \mathbf{Cocart}_d^\kappa(\mathcal{C})$ which strictly preserves the tensoring over $\mathbf{Cat}_d(\kappa)$. It therefore remains to verify that this f^* indeed presents the value of $\mathbf{Cocart}_d^\kappa(-)$ on f .

To see this, note that associated to f is the following strict diagram of \mathbf{Cat}_{d+1} -enriched categories.

$$\begin{array}{ccc} \mathbf{Cocart}_d^\kappa(\mathcal{C}') \times [1] & \xrightarrow{\phi} & \mathbf{Cocart}_d(\kappa) \\ \downarrow & & \downarrow \\ [1] & \xrightarrow{f} & \mathbf{Cat}_{d+1} \end{array}$$

Here ϕ is given on $\mathbf{Cocart}_d^\kappa(\mathcal{C}') \times \{1\}$ by the fiber inclusion, and on $\mathbf{Cocart}_d^\kappa(\mathcal{C}') \times \{0\}$ by the strict base change functor f^* (which comes with an evident natural transformation to the fiber

inclusion). One easily sees that this enriched functor ϕ models a map of $(d + 1)$ -cartesian fibrations (i.e. it preserves cartesian 1-morphisms). Example 6.15 then shows that the strict functor f^* indeed presents (up to homotopy) the value of the straightening $\mathbf{Cocart}_d^\kappa(-)$ on the arrow f , so that the latter is tensored as well. \square

LEMMA 6.23. *The underlying functor between 1-categories $\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{Cat}_{d+1}$ preserves all small limits.*

Proof. By Lemma 6.22, the functor of 1-categories $\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{Cat}_{d+1}$ takes values in $(d + 1)$ -categories and functors which are tensored over $\mathbf{Cat}_d(\kappa)$. Using Lemma 4.57, Remark 4.59 and the fact that taking cores preserves limits, it then suffices to verify that the composite

$$F: \mathbf{Cat}_{d+1}^{\text{op}} \xrightarrow{\mathbf{Cocart}_d^\kappa} \mathbf{Cat}_{d+1} \xrightarrow{c_1} \mathbf{Cat}_1$$

preserves limits. To see this it suffices to verify that for $i = 0, 1$,

$$\text{Map}_{\mathbf{Cat}_1}([i], F(-)): \mathbf{Cat}_{d+1}^{\text{op}} \longrightarrow \mathcal{S} \tag{6.24}$$

preserves limits. By Lemma 6.18, the functor F arises as the unstraightening of the cartesian fibration of 1-categories $\pi: \mathbf{Cocart}(\kappa)_d \rightarrow \mathbf{CAT}_{d+1}$. Theorem 6.1 implies that this can be identified with $\text{coPSH}_d^{\text{Seg}}(\kappa) \rightarrow \mathbf{Cat}_{d+1}$, where the domain consists of Segal copresheaves whose fibers are κ -small.

The two functors (6.24) therefore arise as the straightening of the right fibrations $\text{coPSH}_d^{\text{Seg}}(\kappa)^{\text{cart}} \rightarrow \mathbf{Cat}_{d+1}$ and $\mathcal{X}^{\text{cart}} \rightarrow \mathbf{Cat}_{d+1}$ from Proposition 4.40 and Variant 4.44, respectively. It was shown there that the straightenings of these two right fibrations are representable, by $\mathbf{Cat}_d(\kappa)$ and $\text{Fun}_{d+1}([1], \mathbf{Cat}_d(\kappa))$, respectively. In particular, they preserve all limits, as required. \square

THEOREM 6.25. *The $(d + 2)$ -functor $\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{Cat}_{d+1}$ from Notation 6.21 is representable by $\mathbf{Cat}_d(\kappa)$.*

Proof. By Proposition 4.58, it suffices to verify that the functor of $(d + 2)$ -categories $\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{Cat}_{d+1}$ is cotensored over \mathbf{Cat}_{d+1} and that the functor of 1-categories $c_0\mathbf{Cocart}_d^\kappa: \mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathcal{S}$ is representable. We have already seen in the proof of Lemma 6.23 that this latter functor is representable by the $(d + 1)$ -category $\mathbf{Cat}_d(\kappa)$.

It remains to verify that for any $K \in \mathbf{Cat}_{d+1}$ and $\mathcal{C} \in \mathbf{Cat}_{d+1}$, the comparison map

$$\mu: \mathbf{Cocart}_d^\kappa(K \times \mathcal{C}) \longrightarrow \text{Fun}_{d+1}(K, \mathbf{Cocart}_d^\kappa(\mathcal{C}))$$

from Definition 4.54 is an equivalence of $(d + 1)$ -categories. This will follow immediately from Lemma 4.57 once we show the following two claims:

- (a) μ induces an equivalence on the underlying 1-categories;
- (b) both the domain and codomain of μ are tensored over $\mathbf{Cat}_d(\kappa)$ and μ is tensored over $\mathbf{Cat}_d(\kappa)$.

Let us start with assertion (a). As a first reduction, note that μ depends, in particular, 1-functorially on K and \mathcal{C} . Since both its domain and codomain send colimits in K and \mathcal{C} to limits (Lemma 6.23 and Corollary 4.9), it suffices to verify that μ is an equivalence on underlying 1-categories for generating objects K and \mathcal{C} . We can therefore reduce to the case where K and \mathcal{C} are contained in Θ_{d+1} . In particular, we may assume that K is κ -small and that for any object $k \in K$, $K(k, k) \simeq *$, and likewise for \mathcal{C} .

By Theorem 4.32, Theorem 6.1 and Lemma 6.18, the 1-cartesian fibration between 1-categories $\pi: \mathbf{c}_1\mathbf{Cocart}_d(\kappa) \rightarrow \mathbf{c}_1\mathbf{Cat}_{d+1}$ arises as the localization of the homotopy cartesian fibration $\mathbf{EnrFun}_d(\kappa) \rightarrow \mathbf{Cat}(\mathbf{Cat}_d)$, where $\mathbf{EnrFun}_d(\kappa)$ consists of enriched categories \mathbf{C} together with an enriched functor $\mathbf{C} \rightarrow \mathbf{Cat}_d$ whose values are homotopically κ -small, as in the proof of Proposition 4.40. In particular, this implies that there are equivalences

$$\begin{aligned} \mathbf{c}_1\mathbf{Cocart}_d^\kappa(K \times \mathcal{C}) &\simeq \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)) \\ \mathbf{c}_1\mathbf{Fun}_{d+1}(K, \mathbf{Cocart}_d^\kappa(\mathcal{C})) &\simeq \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)) \end{aligned}$$

such that for any two maps $K \rightarrow K'$ and $\mathcal{C} \rightarrow \mathcal{C}'$, the functoriality on the left-hand side is homotopic to restriction of functors on the right-hand side (we will not need any information about composition or homotopy coherence).

Using these identifications, the map μ therefore determines an endofunctor that fits into the following commuting diagram of 1-categories.

$$\begin{array}{ccc} \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)) & \xrightarrow{\mu} & \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)) \\ r \downarrow & & \downarrow r \\ \prod_{k \in K, c \in \mathcal{C}} \mathbf{Cat}_d(\kappa) & \xrightarrow{\Pi\mu} & \prod_{k \in K, c \in \mathcal{C}} \mathbf{Cat}_d(\kappa) \end{array}$$

Here r denotes the functor restricting to the sets of (isomorphism classes of) objects of K and \mathcal{C} . Note that μ is (by construction) homotopic to the identity when applied to monoidal unit $K = *$. Up to homotopy, we may therefore assume that the bottom map in the square is the identity.

The restriction functor r preserves κ -small limits and colimits and detects equivalences. By our assumption that K and \mathcal{C} are κ -small, it has a left adjoint $r_!$ sending each (k, c, \mathcal{A}) to the representable functor $K(k, -) \times \mathcal{C}(c, -) \times \mathcal{A}$; this follows directly from the point-set presentation of r in terms of enriched functor categories. The above square then shows that $\mu: \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)) \rightarrow \mathbf{c}_1\mathbf{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa))$ preserves κ -small limits and colimits and detects equivalences as well, and admits a left adjoint $\mu_!$. It suffices to show that for any $G: K \times \mathcal{C} \rightarrow \mathbf{Cat}_d(\kappa)$, the unit map $\eta: G \rightarrow \mu\mu_!(G)$ is an equivalence. Since μ and $\mu_!$ both preserve κ -small colimits, it suffices to verify this when $G = r_!(k, c, \mathcal{A})$ is left Kan extended from an object. By adjunction, η is then determined by a map

$$\phi_{k,c}: \mathcal{A} \rightarrow K(k, k) \times \mathcal{C}(c, c) \times \mathcal{A}$$

depending (1-)naturally on \mathcal{A} . Since K and \mathcal{C} were assumed to have trivial endomorphisms, we therefore obtain an endomorphism $\phi_{k,c}: \text{id} \rightarrow \text{id}$ of the identity functor on $\mathbf{Cat}_d(\kappa)$. There is a contractible space of such natural endomorphisms by [BS21, Theorem 10.1].

The map $\phi_{k,c}$ is therefore naturally (in \mathcal{A}) homotopic to $\{\text{id}_k\} \times \{\text{id}_c\} \times \text{id}_{\mathcal{A}}$. This means precisely that η is homotopic to the identity, so in particular an equivalence. We conclude that $\mu_!$ and μ form an adjoint equivalence, so that μ does indeed induce an equivalence on the underlying 1-categories, as asserted.

We now turn to assertion (b). Note that the construction of μ (Definition 4.54) implies that for any object $k \in K$, the composite with evaluation at k

$$\mathbf{Cocart}_d^\kappa(K \times \mathcal{C}) \xrightarrow{\mu} \mathbf{Fun}_{d+1}(K, \mathbf{Cocart}_d^\kappa(\mathcal{C})) \xrightarrow{\text{ev}_k} \mathbf{Cocart}_d^\kappa(\mathcal{C})$$

is equivalent to the value of $\mathbf{Cocart}_d^\kappa(-)$ on the inclusion $\{k\} \times \mathcal{C} \hookrightarrow K \times \mathcal{C}$. By Lemma 6.22, this implies that the domain and codomain of each $\text{ev}_k \circ \mu$ are tensored over $\mathbf{Cat}_d(\kappa)$ and that $\text{ev}_k \circ \mu$ preserves the tensoring. Claim (b) will therefore follow once we know that

$\text{Fun}_{d+1}(K, \mathbf{Cocart}_d^\kappa(\mathcal{C}))$ is tensored over $\mathbf{Cat}_d(\kappa)$ and that evaluation at each $k \in K$ preserves the tensoring.

To see this, one can either use the formalism from [Hin20] or use the following trick. We will first prove assertion (b) in the case where $\mathcal{C} = *$, so that the target of μ reduces to $\text{Fun}_{d+1}(K, \mathbf{Cat}_d(\kappa))$. By Example 4.56, this is indeed tensored over $\mathbf{Cat}_d(\kappa)$ and restriction along $\{k\} \rightarrow K$ preserves the tensoring. Combined with assertion (a), it follows that μ yields an equivalence $\mathbf{Cocart}_d^\kappa(K) \simeq \text{Fun}_{d+1}(K, \mathbf{Cat}_d^\kappa)$. Using this equivalence (with K replaced by \mathcal{C}), the target of the map μ is then equivalent to

$$\text{Fun}_{d+1}(K, \mathbf{Cocart}_d^\kappa(\mathcal{C})) \simeq \text{Fun}_{d+1}(K, \text{Fun}_{d+1}(\mathcal{C}, \mathbf{Cat}_d(\kappa))) \simeq \text{Fun}_{d+1}(K \times \mathcal{C}, \mathbf{Cat}_d(\kappa)).$$

Applying Example 4.56 once more then shows that this $(d + 1)$ -category is indeed tensored over $\mathbf{Cat}_d(\kappa)$ and that evaluation at $k \in K$ preserves the tensoring. We conclude that the natural transformation μ satisfies both assertions (a) and (b), so that it is a natural equivalence as desired. \square

Proof of Theorem 6.20. Let us apply Theorem 6.25 in the large setting, with $\kappa = \kappa_{\text{sm}}$ the supremum of all small cardinals. It follows that the $(d + 2)$ -functor on large $(d + 1)$ -categories

$$\mathbf{Cocart}_d^{\kappa_{\text{sm}}} : \mathbf{CAT}_{d+1}^{\text{op}} \longrightarrow \mathbf{CAT}_{d+1}$$

is representable by \mathbf{Cat}_d . Note that the restriction of $\mathbf{Cocart}_d^{\kappa_{\text{sm}}}$ to $\mathbf{Cat}_{d+1} \subseteq \mathbf{CAT}_{d+1}$ is precisely the functor \mathbf{Cocart}_d from Theorem 6.20. This proves the existence of a natural equivalence $\mathbf{Cocart}_d(-) \simeq \text{Fun}_{d+1}(-, \mathbf{Cat}_d)$ between $(d + 2)$ -functors $\mathbf{Cat}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$.

For uniqueness, let us write $F : \mathbf{CAT}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}$ for the $(d + 2)$ -functor represented by \mathbf{Cat}_d . Then let us write

$$\begin{aligned} \text{Iso} &\subseteq \text{Map}_{\text{Fun}_{d+2}(\mathbf{Cat}_{d+1}^{\text{op}}, \mathbf{CAT}_{d+1})}(\mathbf{Cocart}_d, F | \mathbf{Cat}_{d+1}), \\ \text{Aut}(F | \mathbf{Cat}_{d+1}) &\subseteq \text{Map}_{\text{Fun}_{d+2}(\mathbf{Cat}_{d+1}^{\text{op}}, \mathbf{CAT}_{d+1})}(F | \mathbf{Cat}_{d+1}, F | \mathbf{Cat}_{d+1}) \end{aligned}$$

for the full sub- $(d + 1)$ -categories spanned by the natural equivalences between $(d + 2)$ -functors. We have to show that $\text{Iso} \simeq *$. We have just verified that Iso is nonempty, and precomposing with any element $\phi \in \text{Iso}$ determines an equivalence $\text{Iso} \simeq \text{Aut}(F | \mathbf{Cat}_{d+1})$. It therefore remains to verify that the $(d + 1)$ -category of automorphisms of $F | \mathbf{Cat}_{d+1}$ is equivalent to a point. As in Proposition 6.10, there are natural maps of $(d + 1)$ -categories as follows:

$$\text{Aut}_{\mathbf{CAT}_{d+1}}(\mathbf{Cat}_d) \longrightarrow \text{Aut}(F) \longrightarrow \text{Aut}(F | \mathbf{Cat}_{d+1}).$$

The first map is an equivalence by the enriched Yoneda lemma (§ 4.4). The second map arises by restricting the natural automorphisms of F along $i : \mathbf{Cat}_{d+1}^{\text{op}} \hookrightarrow \mathbf{CAT}_{d+1}^{\text{op}}$. To see that this is an equivalence, note that together with right Kan extension, restriction along i defines a coreflective localization of (very large) $(d + 2)$ -categories as follows:

$$i_* : \text{Fun}_{d+2}(\mathbf{CAT}_{d+1}^{\text{op}}, \mathbf{CAT}_{d+1}) \overset{\longrightarrow}{\longleftarrow} \text{Fun}_{d+2}(\mathbf{Cat}_{d+1}^{\text{op}}, \mathbf{CAT}_{d+1}) : i_*$$

The essential image of i_* consists of enriched functors $\mathbf{CAT}_{d+1}^{\text{op}} \rightarrow \mathbf{CAT}_{d+1}$ whose underlying 1-functor preserves large, κ_{sm} -cofiltered limits; this can be verified by presenting everything by enriched (model) categories of strictly enriched functors. In particular, the representable functor F is contained in the essential image of i_* , so that restriction along i induces an equivalence on its automorphism $(d + 1)$ -categories. The result now follows since $\text{Aut}_{\mathbf{CAT}_{d+1}}(\mathbf{Cat}_d) \simeq *$ by Proposition 6.13. \square

ACKNOWLEDGEMENTS

This text finds its origins (via Proposition 2.12) in joint work with Rune Haugseng, Fabian Hebestreit and Sil Linskens; I thank them for several helpful discussions, especially about constructions with spans. I would also like to thank Damien Calaque, Harry Gindi, Yonatan Harpaz, Corina Keller, Claudia Scheimbauer and Pelle Steffens for various useful conversations. Finally, I am grateful to the anonymous referee for their useful comments that helped improve the paper. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement number 768679).

CONFLICTS OF INTEREST

None.

Appendix A. Technicalities on categories of fibrant objects

The purpose of this appendix is to describe a version of the Grothendieck construction for categories of fibrant objects, in the following sense.

DEFINITION A.1. By a *category of fibrant objects* \mathcal{M} we will mean a $(1, 1)$ -category with two classes of maps, called *weak equivalences* and *fibrations*, both containing the isomorphisms and closed under composition, such that the following conditions hold.

- (1) The weak equivalences have the 2-out-of-6 property.
- (2) The category \mathcal{M} admits pullbacks along fibrations and the base change of an (acyclic) fibration is again an (acyclic) fibration.
- (3) The category \mathcal{M} admits a terminal object and each $x \rightarrow *$ is a fibration.
- (4) Each map in \mathcal{M} admits a factorization into a weak equivalence, followed by a fibration.

A map between categories of fibrant objects $f: \mathcal{M} \rightarrow \mathcal{N}$ is a functor preserving fibrations, weak equivalences, the terminal object and pullbacks along fibrations. Let us write FibCat for the $(2, 1)$ -category of categories of fibrant objects.

Our condition that weak equivalences have the 2-out-of-6 property is slightly nonstandard; it is equivalent to the assertion that a morphism in \mathcal{M} is a weak equivalence if and only if its image in the localization $\mathcal{M}[W^{-1}]$ is an equivalence (see [Rad06, Theorem 7.2.7] or use the description of $\mathcal{M}[W^{-1}]$ from Remark A.8).

LEMMA A.2. *Let $\mathcal{B} \in \text{FibCat}$. Consider a (pseudo-)functor $\mathcal{B}^{\text{op}} \rightarrow \text{FibCat}; b \mapsto \mathcal{E}(b)$ and denote by $p: \mathcal{E} \rightarrow \mathcal{B}$ its (classical) Grothendieck construction. Suppose that the following two assertions hold.*

- (a) *For every weak equivalence $f: b_0 \rightarrow b_1$, the functor $f^*: \mathcal{E}(b_1) \rightarrow \mathcal{E}(b_0)$ induces an equivalence on localizations.*
- (b) *Let $f: b_0 \rightarrow b_1$ be a morphism and $g: x \rightarrow f^*y$ a morphism in $\mathcal{E}(b_0)$. Then there exists a factorization $f = pi$ with $p: b'_0 \rightarrow b_1$ a fibration and $i: b_0 \rightarrow b'_0$ a weak equivalence, together with a factorization $g = i^*(q)j$ with $q: z \rightarrow p^*y$ a fibration in $\mathcal{E}(b'_0)$ and $j: x \rightarrow i^*z$ a weak equivalence in $\mathcal{E}(b_0)$.*

Define a map $x_0 \rightarrow x_1$ in \mathcal{E} to be a *weak equivalence (fibration)* if it factors (uniquely) as $x_0 \rightarrow x \rightarrow x_1$ where the second map is a p -cartesian lift of a weak equivalence (fibration) and the first map is a weak equivalence (fibration) in $\mathcal{E}(px_0)$. This makes $p: \mathcal{E} \rightarrow \mathcal{B}$ a map between categories of fibrant objects.

Proof. One easily verifies that pullbacks along fibrations exist and that the base change of an (acyclic) fibration remains an (acyclic) fibration. The terminal object in \mathcal{E} is given by $* \in \mathcal{E}(*)$ and each $x \rightarrow *$ is a fibration. Condition (a) implies that for any weak equivalence $f: b_0 \rightarrow b_1$, the functor $f^*: \mathcal{E}(b_1) \rightarrow \mathcal{E}(b_0)$ detects weak equivalences (which are detected in the localization). Using this, one readily verifies that the weak equivalences have the 2-out-of-6 property. Finally, the factorization condition is precisely (b). \square

Remark A.3. Suppose that $\mathcal{B}^{\text{op}} \rightarrow \text{FibCat}$ arises from a diagram of model categories $\mathcal{N}(b)$ and Quillen adjunctions $f_!: \mathcal{N}(b_0) \rightleftarrows \mathcal{N}(b_1): f^*$, by letting $\mathcal{E}(b) \subseteq \mathcal{N}(b)$ be the full subcategory of fibrant objects. In this case, condition (b) follows from the following assertion: for every map $f: b_0 \rightarrow b_1$ in \mathcal{B} , there exists a factorization $f = pi$ with $p: b'_0 \rightarrow b_1$ a fibration and $i: b_0 \rightarrow b'_0$ a weak equivalence such that $i_!: \mathcal{N}(b_0) \rightarrow \mathcal{N}(b'_0)$ preserves all weak equivalences.

Indeed, for a map $g: x \rightarrow f^*y$ in $\mathcal{E}(b_0)$, we can consider the adjoint map $i_!x \rightarrow p^*y$ and factor it into $i_!x \xrightarrow{\sim} z$ followed by a fibration $q: z \rightarrow p^*y$. One then obtains the desired factorization $g = i^*(q)j$, where $j: x \rightarrow i^*i_!x \rightarrow i^*z$ is a model for the derived unit map and, hence, an equivalence by condition (a).

Example A.4. Consider a model category \mathcal{M} and a diagram of model categories and left Quillen functors $\mathcal{M} \rightarrow \text{ModCat}^{\text{L}}$ sending $b \mapsto \mathcal{N}(b)$. Let $p: \mathcal{N} \rightarrow \mathcal{M}$ be the Grothendieck construction of this diagram and let us say that a map $\alpha: x \rightarrow y$ in \mathcal{N} is:

- a (trivial) fibration if $p(\alpha): px \rightarrow py$ is a (trivial) fibration and the map $x \rightarrow p(\alpha)^*y$ is a (trivial) fibration in $\mathcal{N}(px)$;
- a (trivial) cofibration if $p(\alpha): px \rightarrow py$ is a (trivial) cofibration and the map $p(\alpha)_!x \rightarrow y$ is a (trivial) fibration in $\mathcal{N}(py)$.

Harpaz and Prasma have shown that this determines a model structure on \mathcal{N} if the diagram $b \mapsto \mathcal{N}(b)$ satisfies three conditions [HP15]: (a) every weak equivalence in \mathcal{M} induces a Quillen equivalence on fibers, (b) for every trivial cofibration in \mathcal{M} , the associated left Quillen functor preserves weak equivalences and (c) for every trivial fibration in \mathcal{M} , the associated right Quillen functor preserves weak equivalences.

The diagram of model categories indexed by \mathcal{M} induces a diagram of categories of fibrant objects $\mathcal{M}^{\text{fib,op}} \rightarrow \text{FibCat}$ sending each fibrant object b in \mathcal{M} to the full subcategory of fibrant objects in $\mathcal{N}(b)$. In light of Remark A.3, conditions (a) and (b) from [HP15] imply conditions (a) and (b) from Lemma A.2. The category of fibrant objects resulting from Lemma A.2 is then simply the category of fibrant objects associated to the model category \mathcal{N} from [HP15].

DEFINITION A.5. A map $p: \mathcal{E} \rightarrow \mathcal{B}$ of categories of fibrant objects is said to be a *homotopy cartesian fibration* if it arises as in Lemma A.2. In this situation, we will say that $\alpha: x_0 \rightarrow x_1$ in \mathcal{E} is *homotopy p-cartesian* if $x_0 \rightarrow p(\alpha)^*x_1$ is a weak equivalence in the fiber, and we will write $\mathcal{E}^{\text{hocart}} \subseteq \mathcal{E}$ for the wide subcategory of homotopy cartesian arrows.

Remark A.6. If w is a weak equivalence in \mathcal{E} , then a composition wf is homotopy cartesian if and only if f is homotopy cartesian, and likewise for a composition fw . In particular, $\mathcal{E}^{\text{hocart}}$ is a relative category (with the same weak equivalences as in \mathcal{E}) whose weak equivalences are generated under the 2-out-of-3 property by the acyclic fibrations in \mathcal{E} .

PROPOSITION A.7. *Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a homotopy cartesian fibration between categories of fibrant objects. Then the following assertions hold.*

(1) *The pullback square of relative categories*

$$\begin{array}{ccc} \mathcal{E}(b) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow p \\ \{b\} & \longrightarrow & \mathcal{B} \end{array}$$

induces a cartesian square upon localization at the weak equivalences.

- (2) *The functor between localizations $\mathcal{E}^{\text{hocart}}[W^{-1}] \rightarrow \mathcal{E}[W^{-1}]$ is a subcategory inclusion, whose image is the wide subcategory of cartesian morphisms for $p: \mathcal{E}[W^{-1}] \rightarrow \mathcal{B}[W^{-1}]$.*
- (3) *The localization $p: \mathcal{E}[W^{-1}] \rightarrow \mathcal{B}[W^{-1}]$ is a cartesian fibration.*
- (4) *Suppose that $f: b_0 \rightarrow b_1$ is a morphism in \mathcal{B} such that $f^*: \mathcal{E}(b_1) \rightarrow \mathcal{E}(b_0)$ admits a left adjoint $f_!$ that preserves acyclic fibrations. For every $x_0 \in \mathcal{E}(b_0)$, the natural arrow $x_0 \rightarrow f_!x_0$ in \mathcal{E} determines a p -cocartesian arrow in $\mathcal{E}[W^{-1}]$.*

This proposition is essentially a special case of a result of Hinich [Hin16]. We have included a proof that does not rely on Lurie’s (un)straightening machinery (in contrast to [Hin16]).

Remark A.8. For $\mathcal{M} \in \text{FibCat}$, its (∞ -categorical) localization can be modeled as follows [Hor16, Nui16]. Let us write $\text{Span}_{\mathcal{M}}$ for the $(2, 2)$ -category whose:

- (0) objects are the objects of \mathcal{M} ;
- (1) mapping categories $\text{Span}_{\mathcal{M}}(x_0, x_1)$ are the full subcategories of spans $x_0 \leftarrow x_{01} \rightarrow x_1$ where the left morphism $\alpha: x_{01} \rightarrow x_0$ is an acyclic fibration and $\beta: x_{01} \rightarrow x_1$ is any morphism (and composition by composition of spans); we will abbreviate such a span by $(\alpha, x_{01}, \beta): x_0 \dashrightarrow x_1$.

Taking classifying spaces then determines a 1-categorical algebra (i.e. a Segal space) whose completion models the localization [Hor16]. In fact, this description works for any relative category (\mathcal{C}, W) where W is generated under the 2-out-of-3 property by a class of ‘acyclic fibrations’ which is stable under base change.

Proof. Let us start by considering the map of $(2, 2)$ -categories $p: \text{Span}_{\mathcal{E}} \rightarrow \text{Span}_{\mathcal{B}}$. For each $x_0, x_1 \in \mathcal{E}$, the induced functor on mapping $(1, 1)$ -categories

$$\text{Span}_{\mathcal{E}}(x_0, x_1) \longrightarrow \text{Span}_{\mathcal{B}}(px_0, px_1) \tag{A.9}$$

is a cartesian fibration, whose fiber over a span $(\alpha, b, \beta): px_0 \dashrightarrow px_1$ can be identified with $\text{Span}_{\mathcal{E}(b)}(\alpha^*x_0, \beta^*x_1)$. Using condition (a) and Remark A.8, it follows that for each morphism $\gamma: b' \xrightarrow{\sim} b$ in $\text{Span}_{\mathcal{B}}(px_0, px_1)$, the induced map on fibers

$$\gamma^*: \text{Span}_{\mathcal{E}(b)}(\alpha^*x_0, \beta^*x_1) \longrightarrow \text{Span}_{\mathcal{E}(b')}(\gamma^*\alpha^*x_0, \gamma^*\beta^*x_1)$$

induces an equivalence on classifying spaces. It follows that (A.9) satisfies the conditions of Quillen’s theorem B; consequently, the classifying space functor $|-|: \text{Cat}_{(1,1)} \rightarrow \mathcal{S}$ preserves all pullbacks along it.

In particular, this implies that the pullback square of $(2, 2)$ -categories

$$\begin{array}{ccc} \text{Span}_{\mathcal{E}(b)} & \longrightarrow & \text{Span}_{\mathcal{E}} \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & \text{Span}_{\mathcal{B}} \end{array}$$

induces a (homotopy) pullback square of Segal spaces upon taking classifying spaces of all mapping categories. To prove assertion (1), we have to show that this also induces a homotopy pullback square on completions. This follows as soon as $\text{Span}_{\mathcal{E}} \rightarrow \text{Span}_{\mathcal{B}}$ induces an isofibration on homotopy (1, 1)-categories (cf. [HNP20, Corollary 4.4]).

To see this, note that by the 2-out-of-6 property, any isomorphism in $\text{ho}(\text{Span}_{\mathcal{B}})$ arises from a span $(\alpha, b, \beta): b_0 \dashrightarrow b_1$ in \mathcal{B} with α a trivial fibration and β a weak equivalence. It therefore suffices to prove the following stronger assertion.

- (*) For any span $(\alpha, b, \beta): b_0 \dashrightarrow b_1$ in \mathcal{B} with α a trivial fibration and any $x_1 \in \mathcal{E}_{b_1}$, there exists a lift to a span $(\tilde{\alpha}, x_{12}, \tilde{\beta}): x_0 \dashrightarrow x_1$ with $\tilde{\alpha}$ a trivial fibration and $\tilde{\beta}$ homotopy cartesian.

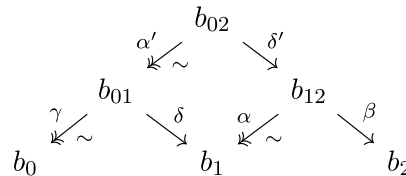
To see this, first take $\beta^*x_1 \in \mathcal{E}_b$ and recall that $\alpha^*: \mathcal{E}_{b_0} \rightarrow \mathcal{E}_b$ induces an equivalence on localizations. This implies that there exists some object $x_0 \in \mathcal{E}_{b_0}$ together with a zigzag $\alpha^*(x_0) \xleftarrow{\sim} x \xrightarrow{\sim} \beta^*x_1$. The composite $x_0 \xrightarrow{\sim} \alpha^*(x_0) \xrightarrow{\sim} x \xrightarrow{\sim} \beta^*(x_1) \rightarrow x_1$ then provides the desired lift.

This proves assertion (1). For assertions (2) and (3), note that by Remarks A.6 and A.8, the localization of $\mathcal{E}^{\text{hocart}}$ arises from the (2, 2)-category $\text{Span}_{\mathcal{E}^{\text{hocart}}}$. Since being a homotopy p -cartesian morphism is invariant under weak equivalences, $\text{Span}_{\mathcal{E}^{\text{hocart}}}(x_0, x_1) \rightarrow \text{Span}_{\mathcal{E}}(x_0, x_1)$ is an inclusion of connected components, so that $\mathcal{E}^{\text{hocart}}[W^{-1}] \rightarrow \mathcal{E}[W^{-1}]$ is indeed the inclusion of a wide subcategory. Assertions (2) and (3) then follow from (*) as soon as every arrow in $\mathcal{E}^{\text{hocart}}[W^{-1}]$ is a cartesian arrow.

Let $(\tilde{\alpha}, x_{12}, \tilde{\beta}): x_1 \dashrightarrow x_2$ be a morphism in $\text{Span}_{\mathcal{E}^{\text{hocart}}}$, so that $\tilde{\beta}$ is homotopy p -cartesian, and let $(\alpha, b_{12}, \beta): b_1 \dashrightarrow b_2$ denote its image in \mathcal{B} . To see that this map is an equivalence, we have to show that for any $x_0 \in \mathcal{E}$ with image $b_0 = px_0$ in \mathcal{B} , the square of categories

$$\begin{array}{ccc}
 \text{Span}_{\mathcal{E}}(x_0, x_1) & \xrightarrow{(\tilde{\alpha}, x_{12}, \tilde{\beta}) \circ (-)} & \text{Span}_{\mathcal{E}}(x_0, x_2) \\
 \downarrow & & \downarrow \\
 \text{Span}_{\mathcal{B}}(b_0, b_1) & \xrightarrow{(\alpha, b_{12}, \beta) \circ (-)} & \text{Span}_{\mathcal{B}}(b_0, b_2)
 \end{array} \tag{A.10}$$

induces a homotopy cartesian square of classifying spaces. Since the vertical maps are cartesian fibrations satisfying the conditions of Quillen’s Theorem B, it suffices to show that for any span $(\gamma, b_{01}, \delta): b_0 \dashrightarrow b_1$ in $\text{Span}_{\mathcal{B}}(b_0, b_1)$, the induced map between fibers induces an equivalence on classifying spaces. To see this, let us denote the composite span $(\alpha, b_{12}, \beta) \circ (\gamma, b_{01}, \delta)$ by



where $b_{02} = b_{01} \times_{b_1} b_{12}$. Recall from the beginning of the proof that the fibers of the vertical maps in (A.10) are themselves given by categories of spans. Unraveling the definition, the map between the vertical fibers over (γ, b_{01}, δ) can then be identified with the following composite.

$$\begin{array}{ccc}
 \text{Span}_{\mathcal{E}(b_{01})}(\gamma^*x_0, \delta^*x_1) & \xrightarrow{\quad\quad\quad} & \text{Span}_{\mathcal{E}(b_{02})}((\gamma\alpha')^*x_0, (\beta\delta')^*x_2) \\
 \searrow (\alpha')^* & & \nearrow f \circ (-) \\
 & \text{Span}_{\mathcal{E}(b_{02})}((\gamma\alpha')^*x_0, (\alpha\delta')^*x_2) &
 \end{array}$$

The first map is an equivalence since $(\alpha')^*$ induces an equivalence on localizations. The second map takes the postcomposition with the span $(\delta')^*\alpha^*x_1 \leftarrow (\delta')^*x_{12} \rightarrow (\delta')^*\beta^*x_2$, where

both maps are weak equivalences since $\tilde{\alpha}$ and $\tilde{\beta}$ were homotopy cartesian. This map induces an equivalence on classifying spaces as well.

Finally, for assertion (4), let us start by recalling that for any cartesian fibration $p: \mathcal{D} \rightarrow \mathcal{C}$ of 1-categories, an arrow in \mathcal{D} is p -cocartesian if and only if it is *locally* p -cocartesian, i.e. cocartesian in the base change of p to a 1-simplex [Lur09a, Corollary 5.2.2.4]. In the present situation, the observations at the beginning of the proof imply that the base change of $\mathcal{E}[W^{-1}] \rightarrow \mathcal{B}[W^{-1}]$ to the 1-simplex $f: b_0 \rightarrow b_1$ can be computed at the level of (2, 2)-categories. Unraveling the definitions as in the proofs of parts (2) and (3), we reduce to showing that for any $x_2 \in \mathcal{E}(b_1)$,

$$\text{Span}_{\mathcal{E}(b_1)}(f_!x_0, x_2) \xrightarrow{f^*} \text{Span}_{\mathcal{E}(b_1)}(f^*f_!x_0, f^*x_2) \xrightarrow{(-)\circ\eta} \text{Span}_{\mathcal{E}(b_1)}(x_0, f^*x_2) \quad (\text{A.11})$$

induces an equivalence on classifying spaces. Here the second map precomposes spans with the unit $\eta: x_0 \rightarrow f^*f_!x_0$. Explicitly, the functor (A.11) sends $f_!x_0 \leftarrow x_2 \rightarrow x_2$ to the span $x_0 \leftarrow x_0 \times_{f^*f_!x_0} f^*x_2 \rightarrow f^*x_2$. This induces an equivalence on classifying spaces since it admits a left adjoint, sending $x_0 \leftarrow x_2 \rightarrow f^*x_2$ to $f_!x_0 \leftarrow f_!x_2 \rightarrow f_!f^*x_2 \rightarrow x_2$. \square

REFERENCES

AF20 D. Ayala and J. Francis, *Fibrations of ∞ -categories*, High. Struct. **4** (2020), 168–265.
 AFR18 D. Ayala, J. Francis and N. Rozenblyum, *Factorization homology I: higher categories*, Adv. Math. **333** (2018), 1042–1177.
 AMR19 D. Ayala, A. Mazel-Gee and N. Rozenblyum, *Stratified noncommutative geometry*, Mem. Amer. Math. Soc., to appear. Preprint (2019), [arXiv:1910.14602](https://arxiv.org/abs/1910.14602).
 Bak11 I. Baković, *Fibrations of bicategories* (2011). Available at author’s website: <https://www2.irb.hr/korisnici/ibakovic/>.
 BS21 C. Barwick and C. Schommer-Pries, *On the unicity of the theory of higher categories*, J. Amer. Math. Soc. **34** (2021), 1011–1058.
 Ber07 C. Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), 230–270.
 Boa18 P. Boavida de Brito, *Segal objects and the Grothendieck construction*, in *An alpine bouquet of algebraic topology*, Contemporary Mathematics, vol. 708 (American Mathematical Society, Providence, RI, 2018), 19–44.
 Buc14 M. Buckley, *Fibred 2-categories and bicategories*, J. Pure Appl. Algebra **218** (2014), 1034–1074.
 CHS21 D. Calaque, R. Haugseng and C. Scheimbauer, *The AKSZ construction in derived algebraic geometry as an extended topological field theory*, Mem. Amer. Math. Soc., to appear. Preprint (2021), [arXiv:2108.02473](https://arxiv.org/abs/2108.02473).
 CS19 D. Calaque and C. Scheimbauer, *A note on the (∞, n) -category of cobordisms*, Algebr. Geom. Topol. **19** (2019), 533–655.
 Dug01 D. Dugger, *Universal homotopy theories*, Adv. Math. **164** (2001), 144–176.
 Ehr63 C. Ehresmann, *Catégories structurées*, Ann. Sci. Éc. Norm. Supér. (4) **80** (1963), 349–426.
 GHL20 A. Gagna, Y. Harpaz and E. Lanari, *Fibrations and lax limits of $(\infty, 2)$ -categories*, Preprint (2020), [arXiv:2012.04537](https://arxiv.org/abs/2012.04537).
 GHL21 A. Gagna, Y. Harpaz and E. Lanari, *Cartesian fibrations of $(\infty, 2)$ -categories*, Preprint (2021), [arXiv:2107.12356](https://arxiv.org/abs/2107.12356).
 GR17 D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry*, Mathematical Surveys and Monographs, vol. 221 (American Mathematical Society, Providence, RI, 2017).
 GH15 D. Gepner and R. Haugseng, *Enriched ∞ -categories via non-symmetric ∞ -operads*, Adv. Math. **279** (2015), 575–716.

- GHN17 D. Gepner, R. Haugseng and T. Nikolaus, *Lax colimits and free fibrations in ∞ -categories*, Doc. Math. **22** (2017), 1225–1266.
- HNP20 Y. Harpaz, J. Nuiten and M. Prasma, *On k -invariants for (∞, n) -categories*, Preprint (2020), [arXiv:2011.12723](https://arxiv.org/abs/2011.12723).
- HP15 Y. Harpaz and M. Prasma, *The Grothendieck construction for model categories*, Adv. Math. **281** (2015), 1306–1363.
- Hau15 R. Haugseng, *Rectification of enriched ∞ -categories*, Algebr. Geom. Topol. **15** (2015), 1931–1982.
- Hau17 R. Haugseng, *The higher Morita category of \mathbb{E}_n -algebras*, Geom. Topol. **21** (2017), 1631–1730.
- HHLN23a R. Haugseng, F. Hebestreit, S. Linskens and J. Nuiten, *Two-variable fibrations, factorisation systems and ∞ -categories of spans*, Forum Math. Sigma **11** (2023), e111.
- HHLN23b R. Haugseng, F. Hebestreit, S. Linskens and J. Nuiten, *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*, Proc. Lond. Math. Soc. (3) **127** (2023), 889–1245.
- Her99 C. Hermida, *Some properties of **Fib** as a fibred 2-category*, J. Pure Appl. Algebra **134** (1999), 83–109.
- HM15 G. Heuts and I. Moerdijk, *Left fibrations and homotopy colimits*, Math. Z. **279** (2015), 723–744.
- Hin16 V. Hinich, *Dwyer-Kan localization revisited*, Homology Homotopy Appl. **18** (2016), 27–48.
- Hin20 V. Hinich, *Yoneda lemma for enriched ∞ -categories*, Adv. Math. **367** (2020), 107129.
- Hor16 G. Horel, *Brown categories and bicategories*, Homology Homotopy Appl. **18** (2016), 217–232.
- JS17 T. Johnson-Freyd and C. Scheimbauer, *(Op)lax natural transformations, twisted quantum field theories, and “even higher” Morita categories*, Adv. Math. **307** (2017), 147–223.
- KV14 D. Kazhdan and Y. Varshavskii, *The Yoneda lemma for complete Segal spaces*, Funktsional. Anal. i Prilozhen. **48** (2014), 3–38.
- LNS17 M. Land, T. Nikolaus and K. Szumilo, *Localization of cofibration categories and groupoid C^* -algebras*, Algebr. Geom. Topol. **17** (2017), 3007–3020.
- Lur09a J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170 (Princeton University Press, Princeton, NJ, 2009).
- Lur09b J. Lurie, *($\infty, 2$)-categories and the Goodwillie calculus* (2009). Available at author’s website: <https://www.math.ias.edu/~lurie/>.
- Lur10 J. Lurie, *On the classification of topological field theories* (2010). Available at author’s website: <https://www.math.ias.edu/~lurie/>.
- Mac22 A. W. Macpherson, *A bivariant Yoneda lemma and $(\infty, 2)$ -categories of correspondences*, Algebr. Geom. Topol. **22** (2022), 2689–2774.
- Nui16 J. Nuiten, *Localizing ∞ -categories with hypercovers*, Preprint (2016), [arXiv:1612.03800](https://arxiv.org/abs/1612.03800).
- Rad06 A. Radulescu-Banu, *Cofibrations in homotopy theory*, Preprint (2006), [arXiv:math/0610009](https://arxiv.org/abs/math/0610009).
- Ras21 N. Rasekh, *Yoneda lemma for \mathcal{D} -simplicial spaces*, Preprint (2021), [arXiv:2108.06168](https://arxiv.org/abs/2108.06168).
- Rez01 C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), 973–1007.
- Rez10 C. Rezk, *A Cartesian presentation of weak n -categories*, Geom. Topol. **14** (2010), 521–571.
- RV16 E. Riehl and D. Verity, *Homotopy coherent adjunctions and the formal theory of monads*, Adv. Math. **286** (2016), 802–888.
- RV18 E. Riehl and D. Verity, *The comprehension construction*, High. Struct. **2** (2018), 116–190.
- RV22 E. Riehl and D. Verity, *Elements of ∞ -category theory*, Cambridge Studies in Advanced Mathematics, vol. 194 (Cambridge University Press, Cambridge, 2022).
- Sch14 C. I. Scheimbauer, *Factorization homology as a fully extended topological field theory*, PhD thesis, ETH Zürich (2014). Available at author’s website: <http://www.scheimbauer.at/>.

ON STRAIGHTENING FOR SEGAL SPACES

- SGA1 A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Mathematics, vol. 224 (Springer, Berlin, Heidelberg, 1971).
- Sim12 C. Simpson, *Homotopy theory of higher categories*, New Mathematical Monographs, vol. 19 (Cambridge University Press, Cambridge, 2012).
- Str80 R. Street, *Fibrations in bicategories*, Cah. Topol. Géom. Différ. Catég. **21** (1980), 111–160.

Joost Nuiten joost.nuiten@math.univ-toulouse.fr

IMT, Université Toulouse 3 Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse Cedex 9, France

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.