THE REPRESENTATION OF D. G. NEAR-RINGS

Dedicated to the memory of Hanna Neumann

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In this paper we study the problem of the representation of d.g. near-rings, and in particular the problem of a faithful representation, which is equivalent to the adjoining of an identity. This problem has been considered by Malone [5] and Malone and Heatherly [6] and [7]. They have shown that a finite near-ring with two sided zero can be embedded in the d.g. near-ring generated by the inner automorphisms of a suitable group, and that an identity can always be adjoined to a near-ring with two sided zero. They have also given some special conditions under which a faithful representation of a d.g. near-ring exists.

From another point of view, Fröhlich has studied groups over a d.g. near-ring in [3] and [4]. If (R, S) is a d.g. near-ring, where S is the distributive semigroup generating R, then he showed that free (R, S) groups exist. We use free (R,S) groups to show that not every d.g. near-ring (R, S) can have a faithful representation on a group, if we insist that S should be a semigroup of distributive elements, i.e. endomorphisms on the group. This is true even in the finite case.

We start by setting the work of Fröhlich on free (R, S) groups in the context of varieties, using methods differing substantially from his. Using these ideas, we construct in each non-abelian variety a d.g. near-ring without a faithful representation. This opens up the problem of determining those d.g. near-rings which do have a faithful representation. It also leaves open the question of whether it is possible to embed a d.g. near-ring (R, S) in a d.g. near-ring with identity, if we do not insist that the elements of S should be distributive in the larger near-ring.

We finish by establishing that for every d.g. near-ring (R, S), there exist 'nearest' d.g. near-rings $(\bar{R}, S)$, $(\underline{R}, S)$ which have faithful representations and such that (R, S) is a homomorphic image of $(\bar{R}, S)$ and $(\underline{R}, S)$ is a homomorphic image of (R, S). For those d.g. near-rings to which an identity can be adjoined, there is a natural way of doing it. If the near-ring is a ring, then it is interesting to note that this method of adjoining the identity is the standard one.

1. Definitions and preliminary results

A near-ring R is a set on which are defined two operations: addition $+$ and multiplication $\cdot$, a symbol which will generally be omitted, such that $(R, +)$ is a group (not necessarily commutative), $(R, \cdot)$ is a semigroup and the left distributive law

$$x(y + z) = xy + xz \text{ for all } x, y, z \in R$$

is satisfied. The additive identity will be denoted by 0, and the multiplicative identity, if it exists, by 1. An element $r \in R$ is called distributive if we have

$$(x + y)r = xr + yr \text{ for all } x, y \in R.$$

A distributively generated (d.g.) near-ring is a near-ring R such that $(R, +) = Gp\{S; S$ a (multiplicative) semigroup of distributive elements$\}$, i.e. R is generated as an additive group by the set S. S need not be the semigroup of all distributive elements. As the set S which is chosen can be important, we denote a d.g. near-ring by (R, S).

A common example of a near-ring is the set of all mappings of a group G (all groups will be written additively) into itself, with point-wise addition, and with multiplication being composition of maps. Distributive elements in this case are the endomorphisms, and these generate a d.g. near-ring, which we will denote by $E(G)$. A pair G, θ consisting of a group G and a near-ring homomorphism θ is called an (R, S) group, where (R, S) is a d.g. near-ring, if θ is a homomorphism from R into $E(G)$ such that $S\theta \subseteq \text{End}(G)$, the semigroup of endomorphisms of G. Such a map θ is called a representation of (R, S) on the group G. The map will often be omitted and we will write gr for $g(r\theta)$, where $g \in G$, $r \in R$, and speak of an (R, S) group G. A representation θ is faithful if Ker θ, the kernel of θ, is trivial. If R has an identity 1, the representation θ will be called unitary, and G will be called a unitary (R, S) group, if θ maps 1 to the identity map of G onto itself. Every d.g. near-ring (R, S) has an obvious representation, namely the right regular representation on the group $(R, +)$, which is faithful if R has a left identity, i.e. an element e such that $er = r$ for all $r \in R$.

Let S be a multiplicative semigroup. Then a group G is called an S group if there exists a homomorphism θ from S into End(G). We then write gs for $g(s\theta)$, where $g \in G, s \in S$. A homomorphism ϕ from an (R, S) group (S group) G to another (R, S) group (S group) H is an (R, S) homomorphism (S homomorphism) if

$$(gr)\phi = (g\phi)r \text{ for all } r \in R,$$

$((gs)\phi = (g\phi)s$ for all $s \in S)$. Fröhlich has proved ([2], 2.1.1) that ϕ is an (R, S) homomorphism if and only if it is an S homomorphism. Near-ring homomorphisms are not sufficient for our purposes. So we define a d.g. near-ring homomorphism as follows. Let (R, S) and (T, U) be two d.g. near-rings. Then a d.g. near-ring homomorphism θ is a near-ring homomorphism from R to T such that $S\theta \subseteq U$.

We will now define the varieties with which we will work. A variety of groups is the class of all groups satisfying a given set of laws or words, e.g. the variety of abelian groups is the class of all groups satisfying the law $[x, y] = -x-y+x+y=0$. See H. Neumann [8]. Varieties of d.g. near-rings can be defined in the same way, but by using laws involving both addition and multiplication. These have been considered by Fröhlich in [4], where the definition used is somewhat different although equivalent to that used here. The only varieties of d.g. near-rings we will use will be those satisfying additive laws.

We will be using presentations of groups in $\mathfrak{B}$, a variety of groups. Free groups possessing the universal property exist in varieties, ([8], §4). So we can speak of $Fr(X)$, the free $\mathfrak{B}$ group on the set X. A presentation of a group $G \in \mathfrak{B}$ is a definition of G given by

$$G = Gp\{X;\ w_i(x_1^{(i)}, \cdots, x_{n(i)}^{(i)}) = 0,\ i \in I\}$$

which means that $G \cong Fr(X)/K$ where K is the normal closure of the set of words $\{w_i(x_1^{(i)}, \cdots, x_{n(i)}^{(i)};\ i \in I\}$ in $Fr(X)$. This corresponds to ordinary presentations, except that we omit from the relations $\{w_i(x_1^{(i)}, \cdots, x_{n(i)}^{(i)});\ i \in I\}$ all the laws of $\mathfrak{B}$. We will also use the fact that free products of groups exist in $\mathfrak{B}$, the $\mathfrak{B}$ free product, generally called the verbal product ([8], definition 18.31, theorem 18.42).

We now prove a result which will enable us to link a variety of groups with a corresponding variety of d.g. near-rings. First we need

Lemma 1.1. *Let $w(x_1, \cdots, x_n)$ be a word in n variables. Then*

$$gw(r_1, \cdots, r_n) = w(gr_1, \cdots, gr_n)$$

whenever $r_1, \cdots, r_n$ lie in the d.g. near-ring (R, S), $g \in G$ and G is an (R, S) group.

Proof. We prove this lemma by induction on the length $l(w(x_1, \cdots, x_n))$ of the word $w(x_1, \cdots, x_n)$. If $l(w(x_1, \cdots, x_n)) = 1$, then the result comes from the definition. So we assume that the lemma holds for all words with length at most m. Suppose $l(w(x_1, \cdots, x_n)) = m + 1$. Then

$$w(x_1, \cdots, x_n) = w'(x_1, \cdots, x_n) + y_j$$

where $y_j = \pm x_j$ and $l(w'(x_1, \cdots, x_n)) = m$. Then

$$\begin{aligned} gw(r_1, \cdots, r_n) &= g(w'(r_1, \cdots, r_n) \pm r_j) \\ &= w'(gr_1, \cdots, gr_n) \pm gr_j \\ &= w(gr_1, \cdots, gr_n) \end{aligned}$$

using the induction hypothesis. This finishes the induction argument and the proof.

Let $\mathfrak{V}$ be a variety of groups, and let (R, S) be a d.g. near-ring. Then we define the variety $\mathfrak{V}$ of d.g. near-rings by $(R, S) \in \mathfrak{V}$ if $(R, +) \in \mathfrak{V}$. There will be no confusion in using the same symbol for a variety of groups and a variety of d.g. near-rings.

THEOREM 1.2. *Let (R, S) be a d.g. near-ring with a faithful representation on the (R, S) group G. Let $G \in \mathfrak{V}$, a variety of groups. Then $(R, S) \in \mathfrak{V}$.*

PROOF. Let $\mathfrak{V}$ be defined by the set of words $\{w_i(x_1, \cdots, x_{n(i)});\ i \in I\}$. If $r_1, \cdots, r_{n(i)} \in R$, then, by lemma 1.1,

$gw_i(r_1, \cdots, r_{n(i)}) = w_i(gr_1, \cdots, gr_{n(i)}) = 0$ for all $g \in G$. As the representation of (R, S) on G is faithful, this shows that $w_i(r_1, \cdots, r_{n(i)}) = 0$ for all choices of $r_1, \cdots, r_{n(i)}$ in R, all $i \in I$. Hence the set of words $\{w_i(x_1, \cdots, x_{n(i)});\ i \in I\}$ is satisfied in $(R, +)$ for all $i \in I$ and so $(R, S) \in \mathfrak{V}$.

Theorem 1.2 provides the reason for considering groups in $\mathfrak{V}$ together with d.g. near-rings in $\mathfrak{V}$.

Before we start the work of constructing free (R, S) groups, we will prove a result which reduces the amount of work needed to show that a given mapping between d.g. near-rings is a d.g. near-ring homomorphism.

THEOREM 1.3. *Let (R, S) and (T, U) be two d.g. near-rings. If θ is a group homomorphism from $(T, +)$ into $(R, +)$ which is also a semigroup homomorphism from U into S, then it is a d.g. near-ring homomorphism from (T, U) into (R, S).*

PROOF. Since θ is a group homomorphism from $(T, +)$ into $(R, +)$, we only need to show that θ is a semigroup homomorphism from T into R. We first note that, if $u \in U$, then $(-u)\theta = -(u\theta)$. Now let $\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n$ and $\eta_1 v_1 + \cdots + \eta_m v_m$ be two elements of T, where $u_i, v_j \in T$ and $\varepsilon_i = \pm 1 = \eta_j$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. Then

$$\begin{aligned}(\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)&(\eta_1 v_1 + \cdots + \eta_m v_m)\\ &= (\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)\eta_1 v_1 + \cdots + (\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)\eta_m v_m\\ &= \eta_1(\varepsilon_1 u_1 v_1 + \cdots + \varepsilon_n u_n v_n) + \cdots + \eta_m(\varepsilon_1 u_1 v_m + \cdots + \varepsilon_n u_n v_m).\end{aligned}$$

Also

$$\begin{aligned}(\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)&\theta(\eta_1 v_1 + \cdots + \eta_m v_m)\theta\\ &= (\varepsilon_1(u_1\theta) + \cdots + \varepsilon_n(u_n\theta))(\eta_1(v_1\theta) + \cdots + \eta_m(v_m\theta))\end{aligned}$$

as θ is a group homomorphism,

$$= (\varepsilon_1(u_1\theta) + \cdots + \varepsilon_n(u_n\theta))\eta_1(v_1\theta) + \cdots + (\varepsilon_1(u_1\theta) + \cdots + \varepsilon_n(u_n\theta))\eta_m(v_m\theta)$$

$$= \eta_1(\varepsilon_1(u_1\theta)(v_1\theta) + \cdots + \varepsilon_n(u_n\theta)(v_1\theta)) + \cdots + \eta_m(\varepsilon_1(u_1\theta)(v_m\theta) + \cdots + \varepsilon_n(u_n\theta)(v_m\theta))$$

$$= \eta_1(\varepsilon_1(u_1 v_1)\theta + \cdots + \varepsilon_n(u_n v_1)\theta) + \cdots + \eta_m(\varepsilon_1(u_1 v_m)\theta + \cdots + \varepsilon_n(u_n v_m)\theta)$$

as θ is a semigroup homomorphism from U into S,

$$= \eta_1(\varepsilon_1(u_1v_1) + \cdots + \varepsilon_n(u_nv_1))\theta + \cdots + \eta_m(\varepsilon_1(u_1v_m) + \cdots + \varepsilon_n(u_nv_m))\theta$$

as θ is a group homomorphism,

$$= ((\varepsilon_1u_1 + \cdots + \varepsilon_nu_n)\eta_1v_1)\theta + \cdots + ((\varepsilon_1u_1 + \cdots + \varepsilon_nu_n)\eta_mv_m)\theta$$

$$= ((\varepsilon_1u_1 + \cdots + \varepsilon_nu_n)(\eta_1v_1 + \cdots + \eta_mv_m))\theta.$$

We have used the fact that if $s \in S$, then $r(-s) = -(rs)$, and have shown that θ is a semigroup homomorphism from T into R, hence completing the proof of the theorem.

2. Free (R, S) groups

Throughout we will work within a given variety $\mathfrak{B}$, which we will consider both as a variety of groups and of d.g. near-rings as defined above.

Let S be a multiplicative semigroup. We will first define the free $\mathfrak{B}$ d.g. near-ring on the semigroup S. This has already been done for the variety of all groups by Fröhlich in [3]. We generalize his results to arbitrary varieties, and use a different method of proof. Let X be a set. Define $Fr(X, S)$ to be the free $\mathfrak{B}$ group on the set of symbols $\{x, s_x;\ x \in X,\ s \in S\}$. For $t \in S$, we define $\bar{t}$ as an endomorphism of $Fr(X, S)$ by

$$x \to \cdot_x,\ s_x \to (st)_x \tag{2.1}$$

for all $x \in X$, $s \in S$. As the symbols $\{x, s_x;\ x \in X,\ s \in S\}$ are a free generating set in $\mathfrak{B}$ for $Fr(X, S)$, we can extend the map defined in (2.1) uniquely to be an endomorphism of $Fr(X, S)$.

Let $\bar{S} = \{\bar{s};\ s \in S\}$. We will show that $\bar{S}$ is a semigroup of endomorphisms of $Fr(X, S)$, isomorphic as a multiplicative semigroup to S. To do this it is enough to show that $\overline{t_1t_2} = \bar{t_1}\,\bar{t_2}$ for all $t_i \in S$, $i = 1, 2$, and also that if $t_1 \neq t_2$ then $\bar{t_1} \neq \bar{t_2}$. But

$$\begin{aligned} x\overline{t_1t_2} &= (t_1t_2)_x = t_{1x}\,\bar{t_2} = x\bar{t_1}\,\bar{t_2}, \\ s_x\overline{t_1t_2} &= (st_1t_2)_x = (st_1)_x\,\bar{t_2} = s_x\,\bar{t_1}\,\bar{t_2} \end{aligned} \tag{2.2}$$

for all $x \in X$, $s \in S$. As this is a generating set for $Fr(X, S)$, (2.2) shows that $\overline{t_1t_2} = \bar{t_1}\,\bar{t_2}$. If $t_1 \neq t_2$, then

$$x\,\bar{t_1} = t_{1x} \neq t_{2x} = x\,\bar{t_2}.$$

Hence $\bar{t_1} \neq \bar{t_2}$ and we have shown that $\bar{S}$ is a semigroup of endomorphisms of $Fr(X, S)$ isomorphic to S. Because of this we will omit $^-$ and write s for the endomorphism of $Fr(X, S)$ denoted by $\bar{s}$ above. We now write $(Fr(S), S)$ for the d.g. near-ring generated by the semigroup S of endomorphisms of $Fr(X, S)$.

The following theorem generalizes Theorem 2.1 of [3].

THEOREM 2.1. (i) *$Fr(S)$ is a d.g. near-ring in $\mathfrak{V}$ generated by the distributive semigroup S;*

(ii) *$(Fr(S), +)$ is the free $\mathfrak{V}$ group on the set S;*
(iii) *every S group $H \in \mathfrak{V}$ is a $(Fr(S), S)$ group;*
(iv) *every semigroup homomorphism θ of S into T, where (R, T) is a d.g. near-ring in $\mathfrak{V}$, can be extended to a d.g. near-ring homomorphism from $(Fr(S), S)$ to (R, T);*
(v) *$(Fr(S), S)$ is uniquely determined to within d.g. near-ring isomorphism by* (i) *and either* (ii) *or* (iii) *or* (iv).

PROOF. (i) follows from the definition, and theorem 1.2.
(ii) Let $\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n$ be a word in the elements of S, where $\varepsilon_i = \pm 1$, $1 \leqq i \leqq n$. Then

$$x(\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n) = \varepsilon_1 s_{1x} + \cdots + \varepsilon_n s_{nx}. \tag{2.3}$$

As $S_x = \{s_x;\, s \in S\}$ is part of a free $\mathfrak{V}$ basis of $Fr(X, S)$, the right hand side of (2.3) is equal to 0 only if $\varepsilon_1 y_1 + \cdots + \varepsilon_n y_n$ is a law in $\mathfrak{V}$ ([8], Corollary 13.25). This suffices to prove (ii) since we already know by theorem 1.2 that $(Fr(S), +) \in \mathfrak{V}$. In particular (2.3) shows that $Gp\{S_x\} \cong (Fr(S), +)$ for each $x \in X$, under the obvious isomorphism.

(iii) Let H be an S group. Then there is a semigroup homomorphism θ from S into $\mathrm{End}(H)$. As $H \in \mathfrak{V}$, we have $(E(H), +) \in \mathfrak{V}$. As S is a free $\mathfrak{V}$ basis of $(Fr(S), +)$, we can extend the mapping $\theta : S \to \mathrm{End}(H)$ to be a group homomorphism from $(Fr(S), +)$ into $(E(H), +)$. Then by theorem 1.3, θ is a d.g. near-ring homomorphism from $(Fr(S), S)$ to $(E(H), \mathrm{End}(H))$, i.e. H is a $(Fr(S), S)$ group.

(iv) As (R, T) is in $\mathfrak{V}$, we have $(R, +) \in \mathfrak{V}$. Then the same argument as that used in (iii) gives the result.

(v) Let (R, S) be any d.g. near-ring in $\mathfrak{V}$. By (iv) there is a d.g. near-ring epimorphism θ from $(Fr(S), S)$ onto (R, S) which extends the identity map on S. If (R, S) satisfies (ii), then $(Fr(S), +) \cong (R, +)$ under θ and so $(R, S) \cong (Fr(S), S)$ as a d.g. near-ring. If (R, S) satisfies (iii), then $Fr(X, S)$ is an (R, S) group and, by the argument used in (ii), $(R, +)$ is the free $\mathfrak{V}$ group on the set S. So (R, S) satisfies (ii) and is d.g. near-ring isomorphic to $(Fr(S), S)$. If (R, S) satisfies (iv), it is immediate that θ has a two sided inverse which is a d.g. near-ring homomorphism, arising from the identity map from $S \subseteq (R, S)$ to $S \subseteq (Fr(S), S)$. Hence θ is a d.g. near-ring isomorphism. This finishes the proof of the theorem.

We now consider an arbitrary d.g. near-ring (R, S), generated by a distributive semigroup S. By theorem 2.1 (iv), the identity map on S extends to a d.g. near-ring epimorphism θ from $(Fr(S), S)$ to (R, S). Let Ker θ, the kernel of this epimorphism be the ideal I of $(Fr(S), S)$. Then $\overline{Fr(X,S)I}$, the normal closure of $Fr(X,S)I = Gp\{gr;\ g \in Fr(X,S),\ r \in I\}$ in $Fr(X,S)$ is easily seen to be a normal S subgroup, and hence an (R,S) subgroup ([2], 1.2.3). This enables us to define

$$Fr(X,S)/\overline{Fr(X,S)I} = Fr(X,R,S)$$

as an (R,S) group, where the action of S on $Fr(X,R,S)$ is defined in the natural way from the action of S on $Fr(X,S)$, namely

$$(g + \overline{Fr(X,S)I})s = gs + \overline{Fr(X,S)I}.$$

To check that this defines $Fr(X,R,S)$ as an (R,S) group is a routine matter, and was essentially done by Fröhlich in [3], result (2.2). In the same paper, theorems 3.4 and 5.1, and in [4], theorem 2.4, Fröhlich establishes the existence of the free (R,S) sum of (R,S) groups and of free (R,S) groups in a category. We will state and prove the result in the setting of varieties, as it is a fairly short result and for completeness.

THEOREM 2.2. *$Fr(X,R,S)$ is the free (R,S) group on the set X, in the variety $\mathfrak{B}$.*

PROOF. Let H be an (R,S) group, and let $\theta : X \to H$ map X into H. Define $\phi : Fr(X,S) \to H$ by

$$x\phi = x\theta,\ s_x\phi = (x\theta)s \tag{2.4}$$

and extend this map to a homomorphism from $Fr(X,S)$ to H, possible since $Fr(X,S)$ is freely generated by $\{x, s_x;\ x \in X,\ s \in S\}$. Then H is an $(Fr(S),S)$ group by theorem 2.1 (iii), and from (2.4), ϕ commutes with the action of S on a generating set of $Fr(X,S)$, and hence on $Fr(X,S)$. Hence ϕ is an S homomorphism, and so a $(Fr(S),S)$ homomorphism ([2], 2.1.1). Since $HI = 0$,

$$(gr)\phi = (g\phi)r = 0$$

for all $g \in Fr(X,S)$, $r \in I$. Hence $Fr(X,S)I \subseteq \operatorname{Ker} \phi$ and so $\overline{Fr(X,S)I} \subseteq \operatorname{Ker} \phi$. Thus ϕ induces an S homomorphism $\mu : Fr(X,R,S) \to H$ which will also be defined by (2.4). Again μ is an (R,S) homomorphism by [2], 2.1.1. Since μ must agree with θ on X, and is an (R,S) homomorphism the definition (2.4) is forced and μ is uniquely defined. Hence the result is true.

If S and hence R has an identity, and we impose the condition that all representations of (R,S) are to be unitary, then using theorems 5.7 and 5.8 of [3], or directly, it is easy to see that the following results hold.

THEOREM 2.3. *Let S be a semigroup with identity. Then $Fr_1(X,S)$, the free $\mathfrak{B}$ group on the symbols $\{s_x;\ x \in X, s \in S\}$ is the free unitary $(Fr(S),S)$ group on X.*

THEOREM 2.4. *Let S be a semigroup with identity. Then $Fr_1(X,R,S) = Fr_1(X,S)/\overline{Fr_1(X,S)I}$ is the free unitary (R,S) group on X, where*

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$(R, S) \cong (Fr(S), S)/I$ and $\overline{Fr_1(X, S)I}$ is the normal closure of $Fr_1(X, S)I = Gp\{gr;$ $g \in Fr_1(X, S), r \in I\}$ in $Fr_1(X, S)$.

3. The counterexample

Before we proceed to construct our counterexample, we will prove two elementary results which will be needed at a later stage.

LEMMA 3.1. *Let $(R, S) \in \mathfrak{V}$ have a faithful representation. Then the representation of (R, S) on the free (R, S) group on one generator is faithful.*

PROOF. Let $G \in \mathfrak{V}$ give rise to the faithful representation for (R, S) and let $H = Fr(x, R, S)$ be the free (R, S) group on one generator, x. Given $r \neq 0$, $r \in R$, we can choose $g \in G$ such that $gr \neq 0$. Map $x \to g$ and extend this mapping to an (R, S) homomorphism θ from H to G by theorem 2.2. Then $xr = 0$ would imply

$$0 = (xr)\theta = (x\theta)r = gr \neq 0,$$

a contradiction. Hence $xr \neq 0$ and H gives rise to a faithful representation for (R, S).

LEMMA 3.2. *Let (R, S) be a d.g. near-ring, and let $X = \{x_\lambda; \lambda \in \Lambda\}$ be a set of elements in R. Then the ideal I of (R, S) generated by X is the normal subgroup of $(R, +)$ generated by*

$$RXS = \{rx_\lambda s, rx_\lambda, x_\lambda s, x_\lambda; \lambda \in \Lambda, r \in R, s \in S\}.$$

PROOF. If I is the normal subgroup generated by RXS, it is easy to check that $RI \subseteq I$ and $IS \subseteq I$, and so, by Fröhlich [1], result 1.3.2, I is an ideal. Any ideal containing X must contain RXS and hence I. This finishes the proof.

We now come to the counterexample. Let $\mathfrak{V}$ be any non-abelian variety. Let S be a four element semigroup $S = \{a, b, c, 0\}$ with all products equal to 0. Let $(Fr(S), S)$ be the free d.g. near-ring in $\mathfrak{V}$ on the semigroup S, with 0 taken as the additive 0 of $(Fr(S), +)$. We define (R, S) as the homomorphic image of $(Fr(S), S)$ given by the ideal I generated by the element $a + b + c$ of $Fr(S)$. In this case $(Fr(S), S)$ is the zero d.g. near-ring. By lemma 3.2, I is the normal closure of the element $a + b + c$ in $(Fr(S), +)$ since RXS reduces to $a + b + c$. Hence $(R, +)$ is the free $\mathfrak{V}$ group on two generators, which we can take to be a and b.

Now let $G = Fr(x, S)$ be the free $(Fr(S), S)$ group on one generator, x, and let $H = Fr(x, R, S)$ be the free (R, S) group on the element x. Then $H = G/\overline{GI}$, where $\overline{GI}$ is the normal closure of GI in G. Then the free (R, S) generator of H is $\bar{x} = x + \overline{GI}$. From lemma 3.1, we know that if (R, S) has a faithful representation, then $\bar{x}r \neq 0$ for $0 \neq r \in R$. And then $\bar{x}R$ is a subgroup of H isomorphic to $(R, +)$ under the mapping $r \to \bar{x}r$.

But GI contains $x(a+b+c)$ and $(2x)(a+b+c) = 2xa + 2xb + 2xc$. Hence in H we have

$$-\bar{x}c = \bar{x}a + \bar{x}b$$

and

$$-(2\bar{x}c) = 2\bar{x}a + 2\bar{x}b.$$

Hence $\bar{x}a$ and $\bar{x}b$ commute. As $\bar{x}R$ is generated by $\bar{x}a$ and $\bar{x}b$, it follows that $\bar{x}R$ is abelian. But $(R, +)$ is the free $\mathfrak{V}$ group on two generators, and $\mathfrak{V}$ is a non-abelian variety. Hence $(R, +)$ is not abelian, since any relation which holds between free generators of a free $\mathfrak{V}$ group is a law in $\mathfrak{V}$ ([8], 13.25). Hence $\bar{x}R$ is not isomorphic to $(R, +)$ and we have a contradiction if we assume that (R, S) has a faithful representation.

THEOREM 3.3. *In every non-abelian variety $\mathfrak{V}$, there exists a d.g. near-ring (R, S) which does not have a faithful representation.*

The only restriction on $\mathfrak{V}$ is that it is not an abelian variety. So if $\mathfrak{V}$ is a locally finite variety, i.e. a variety whose finitely generated groups are finite, then we have a finite d.g. near-ring (R, S) which does not have a faithful representation. Compare this with the results in Malone [5] and Malone and Heatherly [7], that any finite near-ring with a two sided zero can be embedded in a near-ring generated by the inner automorphisms of a suitable group. This gives a faithful representation of any finite d.g. near-ring (R, S) in the wider sense of an embedding of R as a near-ring in a near-ring generated by the endomorphisms of some group G. But the elements of S no longer remain distributive in the larger near-ring. The question as to whether a d.g. near-ring can be embedded in a d.g. near-ring with identity by means of a near-ring monomorphism is still open.

To return to d.g. near-ring embeddings, the question now arises as to what conditions on a d.g. near-ring (R, S) are necessary or sufficient for (R, S) to have a faithful representation. Two obvious sufficient conditions are

THEOREM 3.4. (i) *If S has a left identity, then any d.g. near-ring (R, S) on S has a faithful representation.*

(ii) *If $(R, +)$ has S as a set of free $\mathfrak{V}$ generators, then (R, S) has a faithful representation.*

PROOF. (i) The left identity of S is a left identity for R, hence $(R, +)$ gives rise to a faithful representation for (R, S)

(ii) This follows from theorem 2.1.

These two results give examples of two kinds of sufficient conditions we may have: a condition on the multiplicative structure of S or a condition on the presentation of $(R, +)$ in terms of S as a set of generators. There would seem to be a lot of work to do in this direction.

4. The upper and lower faithful d.g. near-rings for (R, S)

Throughout this section, we will work within a given variety $\mathfrak{B}$.

Although a d.g. near-ring (R, S) may not have a faithful representation, it is the homomorphic image of a d.g. near-ring with a faithful representation, namely $(Fr(S), S)$, and has as homomorphic image a d.g. near-ring with a faithful representation, in fact the representation on any (R, S) group, including the free (R, S) group. This motivates the following work.

We first define a faithful d.g. near-ring (R, S) to be a d.g. near-ring (R, S) with a faithful representation. We can now define the upper and lower faithful d.g. near-rings for a d.g. near-ring (R, S).

The upper faithful d.g. near-ring for (R, S) is a faithful d.g. near-ring $(\bar{R}, S)$ such that $\bar{\theta} : (\bar{R}, S) \to (R, S)$ is a d.g. near-ring epimorphism with $\bar{\theta}|_S =$ identity, and if θ is any d.g. near-ring epimorphism from a faithful d.g. near-ring (T, S) to (R, S) with $\theta|_S =$ identity, then $\theta = \phi\bar{\theta}$ for a uniquely defined d.g. near-ring epimorphism $\phi : (T, S) \to (\bar{R}, S)$.

The lower faithful d.g. near-ring for (R, S) is a faithful d.g. near-ring $(\underline{R}, S)$ such that $\underline{\theta} : (R, S) \to (\underline{R}, S)$ is a d.g. near-ring epimorphism with $\underline{\theta}|_S =$ identity, and if θ is any d.g. near-ring epimorphism from (R, S) to a faithful d.g. near-ring (T, S) with $\theta|_S =$ identity, then $\theta = \underline{\theta}\phi$ for a uniquely defined d.g. near-ring epimorphism $\phi : (\underline{R}, S) \to (T, S)$.

We will show that for any given d.g. near-ring (R, S), upper and lower faithful d.g. near-rings exist, and we will determine them in terms of (R, S). We start with the lower faithful d.g. near-ring. First we need some preliminary results.

LEMMA 4.1. *Let θ be a d.g. near-ring homomorphism from (R, S) into (T, U). Let G be a (T, U) group with representation ϕ. Then G can be defined as an (R,S) group, and the kernel of the representation μ of (R, S) on G is the inverse image under θ of* Ker ϕ.

PROOF. We define $r\mu$ by $g(r\mu) = g(r\theta\phi)$ for all $r \in R$, i.e. $\mu = \theta\phi$. The rest follows easily.

LEMMA 4.2. *Let (R, S) be a d.g. near-ring, and let $G = Fr(x, R, S)$ be the free (R, S) group on one element, x. Write $A = A(R, S) = \{r; Gr = 0\}$. If θ is a representation of (R, S), then* Ker $\theta \supseteq A$.

PROOF. Assume that Ker $\theta \not\supseteq A$. Let $r \in A -$Ker θ. Then we have a group H such that $hr \neq 0$ for some $h \in H$. Map x to h and extend this to ϕ, an (R, S) homomorphism from G to H. Then

$$0 = (xr)\phi = (x\phi)r = hr \neq 0.$$

This is a contradiction, and so we deduce that Ker $\theta \supseteq A$.

THEOREM 4.3. *Let (R,S) be a d.g. near-ring. Then its lower faithful d.g. near-ring is $(R,S)/A$ where $A = A(R,S) = \{r;\ Gr = 0\}$ and G is the free (R,S) group on one generator.*

PROOF. Certainly $(R,S)/A$ is a faithful d.g. near-ring. Denote $(R,S)/A$ by $(\underline{R},S)$ and let $\underline{\theta}$ be the natural d.g. near-ring epimorphism from (R,S) to $(\underline{R},S)$ with $\underline{\theta}|_S$ = identity. Let θ be a d.g. near-ring epimorphism from (R,S) to a faithful d.g. near-ring (T,S) with $\theta|_S$ = identity. Let $H = Fr(x,T,S)$ be the free (T,S) group on one element x. Then as (T,S) is a d.g. homomorphic image of (R,S), we deduce by lemma 4.1 that H is an (R,S) group with the action of $s \in S$ defined in the same way for both (R,S) and (T,S). This gives rise to a representation of (R,S) whose kernel is Ker θ, again by lemma 4.1. But by lemma 4.2, this means that $A \subseteq$ Ker θ. Hence $\theta = \underline{\theta}\phi$ for a d.g. near-ring epimorphism $\phi : (\underline{R},S) \to (T,S)$. As $\theta = \underline{\theta}\phi$, we must have $\phi|_S$ = identity and this will define ϕ uniquely since S generates $(\underline{R},+)$.

We now turn to the upper faithful d.g. near-ring. Again, we will first prove some preliminary results.

LEMMA 4.4. *Let (T,S) be a faithful d.g. near-ring, and let $G = Fr(x,T,S)$. Then $G = G_1 * G_2$ is the free $\mathfrak{V}$ product of G_1, the free $\mathfrak{V}$ group on one generator and G_2, a group isomorphic to $(T,+)$.*

PROOF. Let $(F,S) = (Fr(S),S)$ be the free $\mathfrak{V}$ d.g. near-ring on the semigroup S, and let H be the free (F,S) group on one element x. By theorem 2.1, H is the free $\mathfrak{V}$ group on the set $\{x, s_x;\ s \in S\}$. Also $(T,S) = (F,S)/I$ for some ideal I of (F,S) and $G = H/(HI)^H$, where $(HI)^H$ is the normal subgroup of H generated by $HI = Gp\{hr;\ h \in H, r \in I\}$, by theorem 2.2. We write $H = H_1 * H_2$ where $H_1 = Gp\{x\}$, $H_2 = Gp\{s_x;\ s \in S\}$ are both free $\mathfrak{V}$ groups on the elements mentioned, and $*$ indicates the free $\mathfrak{V}$ product. By the definition of the action of (F,S) on H, we know that $HF = H_2$. Hence $HI \subseteq H_2$ and by lemma 4.5, we obtain $G \cong H_1 * H_2/(HI)^{H_2}$, where $(HI)^{H_2}$ is the normal subgroup of H_2 generated by HI. Again $GT = H_2/(HI)^{H_2}$, identifying G and $H_1 * H_2/(HI)^{H_2}$. In fact $xT = H_2/(HI)^{H_2} \cong (T,+)$ from the first part of the proof of theorem 3.3. Hence $G \cong H_1 * (T,+)$ which is the result we want.

We now prove lemma 4.5. It is a fairly standard result, but there does not seem to be a reference for it.

LEMMA 4.5. *Let $G = H * K$, let N be normal in K and let $M = N^G$ be the normal closure of N in G. Then $G/M \cong H * K/N$. All groups are in $\mathfrak{V}$ and $*$ indicates the free $\mathfrak{V}$ product.*

PROOF. Let $G' = H * K/N$. Map $G \to G'$ by θ extending $h \to h$, $k \to k + N$. Then Ker $\theta \supseteq N$. Hence Ker $\theta \supseteq M$, the normal closure of N. We know that θ is a homomorphism by the universal property of free $\mathfrak{V}$ products ([8], 18.42).

Now consider G/M. As $K^G \cap H = 0$ ([8], 18.36), we have $M \cap H = 0$. So if $\phi : G \to G/M$ is the natural homomorphism, then $H\phi = HM/M \cong H/H \cap M = H$. Hence $G/M = Gp\{H, KM/M\}$ where we are substituting H for HM/M. We now define $\mu : G' \to G/M$ by $h \to h$ and $k + N \to k + M$. Then μ extends to a homomorphism from G' to G/M, by the universal property of free products ([8], 18.42). It is immediate that μ is onto G/M. Hence $\theta\mu$ is an epimorphism from G onto G/M and

$$h\theta\mu = h\mu = h,\ k\theta\mu = (k + N)\mu = k + M, \text{ for all } h \in H,\ k \in K.$$

Hence $\theta\mu = \phi$. So Ker $\phi \supseteq$ Ker θ, i.e. $M \supseteq$ Ker θ. This means that $M =$ Ker ϕ = Ker θ. As $\theta\mu = \phi$, this means that μ is $1 - 1$ and so is an isomorphism.

We now return to our given d.g. near-ring (R, S). Let $G = Gp\{x\} * (R, +) = Gp\{x\} * R_x$, where we write R_x for the subgroup of G isomorphic to $(R, +)$ and $Gp\{x\}$ is the free $\mathfrak{B}$ group on one generator. Define $s \in S$ as an endomorphism of G by extending the map

$$x \to s_x, r_x \to (rs)_x$$

to the whole of G. This is possible by the universal property of free $\mathfrak{B}$ products ([8], 18.42), since $r_x \to (rs)_x$ is an endomorphism of R_x. It is easy to check that this defines a homomorphism from S to a semigroup of endomorphisms of G which we will still denote by S. Then S generates a d.g. near-ring $(\bar{R}, S)$ which is faithful, with G as a faithful representation.

THEOREM 4.6. *The d.g. near-ring $(\bar{R}, S)$ defined above is the upper faithful d.g. near-ring for (R, S).*

PROOF. Let $\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n$ be a word in the elements of S which is 0 in $\bar{R}$, where $\varepsilon_x = \pm 1$, $1 \leqq i \leqq n$. Then

$$\begin{aligned} 0 = x(\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n) &= \varepsilon_1 s_{1x} + \cdots + \varepsilon_n s_{nx} \\ &= (\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n)_x. \end{aligned}$$

Hence $\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n = 0$ in R. So the identity map on S extends to a homomorphism θ from $(\bar{R}, +)$ to $(R, +)$ and it is a d.g. near-ring epimorphism by theorem 1.3.

Now let θ be a d.g. near-ring epimorphism from a faithful d.g. near-ring $(T, S) \to (R, S)$ with $\theta|_S =$ identity. Then $(T, S)/I \cong (R, S)$ for some ideal I of (T, S). By lemma 4.4, if $H = Fr(x, T, S)$ is the free (T, S) group on one generator, then $H \cong Gp\{x\} * T_x$ with $T_x \cong (T, +)$. So $(R, +) \cong R_x$ is a homomorphic image of T_x, namely $R_x \cong T_x/I_x$. Then there is a homomorphism from H to G extending the map $x \to x$, $t_x \to t_x\mu$, where μ is the natural map from T_x to R_x induced by θ.

By the definition of the action of S on H, we know that I_x is an S group, and hence so is I_x^H, its normal closure in H. Hence H/I_x^H is a (T, S) group and by lemma 4.5, $H/I_x^H \cong Gp\{x\} * R_x = G$. The action of $S \subseteq T$ on G is the same as the action of $S \subseteq \bar{R}$ on G. Hence $(\bar{R}, S)$ is a homomorphic image of (T, S) under a d.g. near-ring homomorphism ϕ which extends the identity map on S. Then $\theta = \phi\bar{\theta}$, and ϕ is uniquely defined as it must act as the identity map on S, a generating set for $(T, +)$. This finishes the proof.

We will close with a fairly straightforward, but interesting result about the adjoining of identities.

LEMMA 4.7. *Let (R, S) be a faithful d.g. near-ring. Let $G = Fr(x, R, S)$ be the free (R, S) group on the generator x. If (T, U) is the d.g. near-ring contained in $E(G)$, with $U = S \cup \{1\}$, where 1 is the identity map on G, then $(T, +) \cong G$.*

PROOF. As $(R, S) \subseteq (T, U)$ with $S \subseteq U$, $(T, +)$ gives rise to a faithful representation of (R, S). So the map $x \to 1$ extends to an (R, S) homomorphism θ from G to $(T, +)$. Then $(xs)\theta = 1s = s$, i.e. $s_x\theta = s$. So $G\theta \supseteq U$ and θ is an epimorphism. But (T, U) has a faithful representation on G. Consider the map $\phi : r \to xr$. This is a homomorphism from $(T, +)$ to G such that $1\phi = x1 = x$ and $s\phi = xs = s_x$. Hence $\theta\phi$ is the identity map on $\{x, s_x; s \in S\}$ a generating set for G, and $\phi\theta$ is the identity map on U, a generating set for $(T, +)$. So θ and ϕ are both isomorphisms, giving us the result we want.

We now compare this with lemma 4.4 which states that $G = G_1 * G_2$ with $G_1 = Gp\{x\}$ the free $\mathfrak{V}$ group on one generator, and $G_2 \cong (R, +)$. From lemma 4.7, $G_2 = Gp\{S_x\} \cong (R, +)$ under the isomorphism $s_x \to s$. So (T, U), the d.g. near-ring obtained in the natural way by adjoining an identity to (R, S), where (R, S) is a faithful d.g. near-ring can be characterized as follows: $(T, +)$ is the free $\mathfrak{V}$ product of $(R, +)$ with a free $\mathfrak{V}$ group on one generator, namely 1, and multiplication is determined by $U = S \cup \{1\}$ where 1 acts as multiplicative identity. If we now let $\mathfrak{V}$ be the variety of abelian groups, then the free $\mathfrak{V}$ group on one generator is a copy of Z, the integers under addition, the free $\mathfrak{V}$ product is the direct sum and we can see that we have the standard method for adjoining an identity. Of course in an abelian variety, d.g. near-rings are rings, and all rings are faithful. So this process can always be carried out for rings.

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