INFINITE NON-LINEAR PROGRAMMING

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1. Introduction

In recent years there has been extensive development in the theory and techniques of mathematical programming in finite spaces. It would be very useful in practice to extend this development to infinite spaces, in order to treat more realistically the problems that arise for example in economic situations involving infinitely divisible processes, and in particular problems involving time as a continuous variable. A more mathematical reason for seeking such generalisation is possibly that of obtaining a unification of mathematical programming with other branches of mathematics concerned with extrema, such as the calculus of variations.

Some early results in infinite programming were obtained by Duffin [1] who investigated infinite linear programmes. In the present paper the duality theorems of convex programming are formulated in Banach spaces, based on Hurwicz' generalisation [2] of the Kuhn-Tucker theorem. Duality theorems of varying generality in finite Euclidean spaces have been developed by Kuhn and Tucker [3], Karlin [4], Dennis [5], Dorn [6], [7], Wolfe [8], and Hanson [9].

2. Notation and definitions

Let x denote an element of the set X in a Banach space $\mathscr{X}$. The conjugate, or dual, space of $\mathscr{X}$ will be denoted by $\mathscr{X}^*$ whose typical element is x^*. Positive elements of $\mathscr{X}$ are defined to be the elements of some specified convex cone $P_x \subset \mathscr{X}$. It will be assumed that P_x contains the origin, which will be denoted by 0 for all spaces.

Three ordering relationships are subsequently defined:

$$x_1 \geqq x_2 \quad \text{means that} \quad x_1 - x_2 \in P_x$$

$$x_1 \geq x_2 \quad \text{means that} \quad x_1 \geqq x_2 \quad \text{but} \quad x_2 \ngeqq x_1$$

and

$$x_1 > x_2 \quad \text{means that} \quad x_1 - x_2 \in \text{Int. } P_x.$$

In the conjugate space the expression $x_0^* \geqq 0$ means

$$x_0^*(x) \geqq 0 \quad \text{for all} \quad x \geqq 0.$$

A programme in Banach space is represented by the couple $\langle f, g \rangle$ where f and g are defined on $X \subset \mathscr{X}$ into the Banach spaces $\mathscr{Y}$ and $\mathscr{Z}$ respectively; and a maximal value of the programme is defined to be an element y_0 of the set $Y = f(P_x \cap g^{-1}(P_z))$ such that for $y \in Y$, $y \geqq y_0$ implies $y \leqq y_0$. Such an element y_0 is said to be maximal over Y, and if $y_0 = f(x_0)$ then x_0 is said to maximise $f(x)$ over $f(P_x \cap g^{-1}(P_z))$. (It should be mentioned that the simultaneous inequalities $x \geqq y$ and $y \geqq x$ do not necessarily imply that $x = y$.) In the terminology of economics the spaces $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ are called the activity, objective, and constraint spaces respectively. The value x_0 is called the optimal activity, and y_0 the optimal value of the programme.

The symbol $\delta f(x_0; x^1)$ will denote the Fréchet differential of $f(x)$ at x_0 with increment x^1.

A concave function $f : X \to \mathscr{Y}$ is such that

$$f(\lambda x_1 + (1-\lambda)x_2) \geqq \lambda f(x_1) + (1-\lambda) f(x_2) \text{ for all } x_1, x_2 \text{ in } X$$

and $0 \leqq \lambda \leqq 1$. It follows that if f is differentiable then

$$f(x_1) \leqq f(x_2) + \delta f(x_2; x_1 - x_2).$$

A convex function has the relevant above inequalities reversed.

3. The dual problem

Let f and g be Fréchet differentiable functions on the Banach space $\mathscr{X}$ into the Banach spaces $\mathscr{Y}$ and $\mathscr{Z}$ respectively, and let x_0 maximise the programme $\langle f, g \rangle$. In the problems to be considered it will be assumed that the constraints of the programme satisfy Hurwicz' generalisation of the Kuhn-Tucker constraint conditions, namely that g is regular ([2], p. 95) and there is regular convexity ([2], p. 61) of the set

$$\{w^* : w^* = T^*(v^*), v^* \geqq 0, v \in \mathscr{Z} \times \mathscr{W}\}$$

where

$$\mathscr{W} = \{w : w = (\rho, x), \rho \text{ real}, \quad x \in \mathscr{X}\}$$

and

$$T(\rho, x) = [\delta g(x_0; x) - \rho(\delta g(x_0; x_0) - g(x_0)), (\rho, x)].$$

Further the space $\mathscr{Y}$ will be restricted to be such that to each closed convex cone in $\mathscr{Y}$ there exists a continuous linear functional $y^* \in \mathscr{Y}^*$ which is strictly positive on this cone.

Define the Lagrangian expression

$$\Phi(x, z^*) = y_0^*(f(x)) + z^*(g(x)).$$

Hurwicz has shown ([2] Theorem V.3.3.4.) that for each $y_0^* > 0$ there exists

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$z_0^* \geqq 0$ such that $\Phi(x, z^*)$ has a non-negative quasi-saddle-point at $(x_0, z_0^*; y_0^*)$; that is, the following relations hold:

$$\delta_x \Phi((x_0, z_0^*); x) \leqq 0 \quad \text{for all} \quad x \geqq 0 \tag{1}$$

$$\delta_x \Phi((x_0, z_0^*); x_0) = 0 \tag{2}$$

$$\delta_z^* \Phi((x_0, z_0^*); z^*) \geqq 0 \quad \text{for all} \quad z^* \geqq 0 \tag{3}$$

$$\delta_z^* \Phi((x_0, z_0^*); z_0^*) = 0. \tag{4}$$

These results will be used to establish a dual programme to the programme $\langle f, g \rangle$ in the case where f and g are differentiable concave functions.

The activity, objective, and constraint spaces of the dual programme will be denoted by $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{U}$ respectively, defined by their elements:

$$\begin{aligned} s &\in \mathscr{S} = \mathscr{X} \times \mathscr{Z}^* \\ t(s) &= \Phi(x, z^*) - \delta_x \Phi((x, z^*); x) \\ u(s) &= -\delta_x \Phi((x, z^*); x^1) \end{aligned}$$

where $u(s)$ is defined for elements $x^1 \in P_x$.

It will be shown that the dual problem is to find $s_0 \in \mathscr{S}$ which minimises $t(s)$ over $t(P_s \cap u^{-1}(P_u))$ where $P_s = P_x \times P_z^*$; and that a solution of either primal or dual problem, if such exists, implies the existence, under conditions to be stated, of a solution of the other problem, the optimal values of primal and dual objectives are the same, and the optimal activity x_0 of the primal is a component of the optimal activity $s_0 = (x_0, z_0^*)$ of the dual. The evaluation of s_0 may also of course be considered as a maximisation problem, namely, the programme $\langle -t, u \rangle$.

THEOREM 1. If f and g are Fréchet differentiable concave functions and if for some $y_0^* > 0$ there exists x_0 which maximises $y_0^*(f(x))$ over $y_0^*(f(P_x \cap g^{-1}(P_z)))$ then there exists z_0^* such that $s_0 = (x_0, z_0^*)$ minimises $t(s)$ over $t(P_s \cap u^{-1}(P_u))$, and $y_0^*(f(x_0)) = t(s_0)$.

PROOF. Let z_0^* be the functional introduced in the expressions (1) to (4). Then $s_0 = (x_0, z_0^*)$ satisfies the dual constraints. Let $s = (x, z^*)$ be any other element of $\mathscr{S}$ satisfying the constraints. Then

$$\begin{aligned} t(s_0) - t(s) &= \Phi(x_0, z_0^*) - \delta_x \Phi((x_0, z_0^*); x_0) - \Phi(x, z^*) + \delta_x \Phi((x, z^*); x) \\ &= \Phi(x_0, z_0^*) - \Phi(x, z^*) - \delta_x \Phi((x, z^*); x) \\ &\quad \text{using (2),} \\ &= \Phi(x_0, z_0^*) - \Phi(x_0, z^*) + \Phi(x_0, z^*) - \Phi(x, z^*) + \delta_x \Phi((x, z^*); x) \\ &\leqq z_0^*(g(x_0)) - z^*(g(x_0)) + \delta_x \Phi((x, z^*); x_0 - x) + \delta_x \Phi((x, z^*); x) \quad (5) \\ &\quad \text{since } \Phi \text{ is concave,} \\ &= \delta_x \Phi((x, z^*); x_0) \\ &\quad \text{using (3) and (4),} \\ &\leqq 0 \end{aligned}$$

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since (x, z^*) satisfies the dual constraints. Hence s_0 minimises the dual problem.

Further, the optimal value of the dual objective function is $\Phi(x_0, z_0^*) - \delta_x \Phi((x_0, z_0^*); x_0)$ which by (2) and (4) is equal to the optimal value of the primal objective function $y_0^*(f(x_0))$.

Note that although this theorem has been stated for the positive cone in the $\mathscr{S}$ space defined as $P_x \times P_z^*$, the proof applies equally well to the wider cone $\mathscr{X} \times P_z^*$.

THEOREM 2. If

(i) Φ is continuously differentiable with respect to $x \in \mathscr{X}$

(ii) there exists for any $z^* \in P_z^*$ and $x^1 \in \mathscr{X}$ a continuous one-to-one inversion of P_u into $u^{-1}(P_u)$,

and

(iii) there exists $s_0 = (x_0, z_0^*)$ which minimises $t(s)$ over $t(P_s \cap u^{-1} P_u)$ then x_0 maximises $y_0^*(f(x))$ over $y_0^*(f(P_x \cap g^{-1}(P_z)))$.

PROOF. It follows from the hypothesis that (x_0, z_0^*) is a feasible solution of the problem:

$$\text{minimise } \Phi(x_0, z^*) - \delta_x \Phi((x, z_0^*); x) \tag{6}$$

$$\text{subject to } -\delta_x \Phi((x, z_0^*); x) \geqq 0 \tag{7}$$

and

$$z^* \geqq 0. \tag{8}$$

It will be shown that (x_0, z_0^*) is the optimum solution of the problem. Put

$$u^0(x) = -\delta_x \Phi((x, z_0^*); x). \tag{9}$$

Suppose (z_1^*, u_1) is a feasible solution of the above problem such that

$$\Phi(x_0, z_1^*) + u_1 \leq \Phi(x_0, z_0^*) + u_0 \tag{10}$$

where u_1 is some particular value of $-\delta_x \Phi((x, z_0^*); x)$ and

$$u_0 = -\delta_x \Phi((x_0, z_0^*); x_0), \tag{11}$$

that is,

$$z_1^*(g(x_0)) - z_0^*(g(x_0)) + u_1 - u_0 \leq 0 \tag{12}$$

Let

$$z_2^* = z_0^* + k(z_1^* - z_0^*) \tag{13}$$

and

$$u_2 = u_0 + k(u_1 - u_0) \tag{14}$$

for $0 < k \leqq 1$.

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Thus (z_2^*, u_2) is a feasible solution of the problem (6), (7), (8). Therefore by hypothesis there exists $(x_2, z_2^*) \in P_s \cap u^{-1}(P_u)$ such that

$$u_2 = -\delta_x \Phi((x_2, z_2^*); x_0) \tag{15}$$

Define

$$x_1 = x_0 + k^{-1}(x_2 - x_0) \tag{16}$$

Then

$$[\Phi(x_2, z_2^*) - \delta_x \Phi((x_2, z_2^*); x_2)] - [\Phi(x_0, z_0^*) - \delta_x \Phi((x_0, z_0^*); x_0)] \tag{17}$$

$$\begin{aligned} = [\Phi(x_2, z_2^*) - \Phi(x_0, z_2^*)] + [\Phi(x_0, z_2^*) - \Phi(x_0, z_0^*)] \\ - \delta_x \Phi((x_2, z_2^*); x_2) + \delta_x \Phi((x_0, z_0^*); x_0) \end{aligned} \tag{18}$$

$$\begin{aligned} \leqq \delta_x \Phi((x_0, z_2^*); x_2 - x_0) - \delta_x \Phi((x_2, z_2^*); x_2 - x_0) \\ + z_2^*(g(x_0)) - z_0^*(g(x_0)) + u_2 - u_0 \end{aligned} \tag{19}$$

since Φ is concave,

$$\begin{aligned} = \delta_x \Phi((x_0, z_2^*); k(x_1 - x_0)) - \delta_x \Phi((x_0 + k(x_1 - x_0), z_2^*); \\ k(x_1 - x_0)) + k z_1^*(g(x_0)) - k z_0^*(g(x_0)) + k u_1 - k u_0 \end{aligned} \tag{20}$$

using (13), (14) and (16).

$$\begin{aligned} = k\{\delta_x \Phi((x_0, z_2^*); x_1 - x_0) - \delta_x \Phi((x_0 + k(x_1 - x_0); x_1 - x_0) \\ + z_1^*(g(x_0)) - z_0^*(g(x_0)) + u_1 - u_0\} \end{aligned} \tag{21}$$

since the Fréchet differential is linear in its increment. Since by hypothesis (ii) u is continuously invertible it follows that

$$u_2 \to u_0 \Rightarrow x_2 \to x_0 \Rightarrow x_1 \to x_0, \tag{22}$$

and since Φ is continuously differentiable we can choose, using (12), the value of k to be sufficiently small that

$$\begin{aligned} \delta_x \Phi((x_0, z_2^*); x_1 - x_0) - \delta_x \Phi((x_0 + k(x_1 - x_0)); x_1 - x_0) \\ + z_1^*(g(x_0)) - z_0^*(g(x_0)) + u_1 - u_0 \leq 0. \end{aligned} \tag{23}$$

Hence

$$\Phi(x_2, z_2^*) - \delta_x \Phi((x_2, z_2^*); x_2) \leq \Phi(x_0, z_0^*) - \delta_x \Phi((x_0, z_0^*); x_0) \tag{24}$$

which contradicts hypothesis (iii).

Therefore (x_0, z_0^*) is the optimum solution of problem (6), (7), (8).

An equivalent form of this problem is:

(25) maximise $-y_0^*(f(x_0)) - z^*(g(x_0)) - u^0$

(26) subject to $u^0 \geqq 0$

(27) and $z^* \geqq 0$

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and since it has the solution (x_0, z_0^*) there exists, by theorem 1, ξ_0^* such that $((x_0, z_0^*); \xi_0^*)$ is optimal in the dual problem:

(28) minimise $-y_0^*(f(x_0))$

(29) subject to $-\delta_{(z^*, u^0)}[-z^*(g(x_0))-u^0+\xi^*(u^0); (z^{*^1}, u^{0^1})] \geqq 0$

(30) and $\xi^* \geqq 0$

(31) for all $z^{*^1} \geqq 0$

(32) and all $u^{0^1} \geqq 0$.

The inequality (29) can be written

(33) $$-\delta_{z^*}[-z^*(g(x_0)); z^{*^1}]-\delta_{u^0}[-u^0+\xi^*(u^0); u^{0^1}] \geqq 0.$$

Since the two terms in (33) are independent, they must individually satisfy the inequality; it follows that

(34) $$z^{*^1}(g(x_0)) \geqq 0$$

which together with (31) implies that

(35) $$g(x_0) \geqq 0.$$

Hence

(36) $$x_0 \in P_x \cap g^{-1}(P_z).$$

Equating the objective functions (25) and (28) at their extreme values, we have, by theorem 1,

(37) $$z_0^*(g(x_0))+\delta_x\Phi((x_0, z_0^*); x_0) = 0.$$

Hence by (31) and (32) each term in (37) is zero:

(38) $$z_0^*(g(x_0)) = 0$$

and

(39) $$\delta_x\Phi((x_0, z_0^*); x_0) = 0.$$

From (31), (32), (38), and (39) it follows that Φ has a non-negative quasi-saddle-point at (x_0, z_0^*).

Hence by Theorem V. 0 of [2] (Kuhn-Tucker Theorem 3) x_0 maximises $y_0^*(f(x))$ over $y_0^*(f(P_x \cap g^{-1}(P_z)))$.

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