9

CCR on Fock space

This chapter is devoted to the study of the *Fock representation of the canonical commutation relations*. This representation is used as the basic tool in quantum many-body theory and quantum field theory. Unlike the Schrödinger CCR representation, it allows us to consider phase spaces of infinite dimension.

Throughout this chapter, $\mathcal{Z}$ is a Hilbert space. This space will be called the *one-particle space*. The Fock CCR representation will act in the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

As in Sect. 1.3, we introduce the space

$$\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) := \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\},$$

which will serve as the *dual phase space* of our system. It will be equipped with the structure of a Kähler space consisting of the anti-involution j, the Euclidean scalar product $\cdot$ and the symplectic form ω:

$$\mathrm{j}(z, \overline{z}) := (\mathrm{i}z, \overline{\mathrm{i}z}), \tag{9.1}$$

$$(z, \overline{z}) \cdot (w, \overline{w}) := 2\mathrm{Re}(z|w), \tag{9.2}$$

$$(z, \overline{z}) \cdot \omega (w, \overline{w}) := 2\mathrm{Im}(z|w) = -(z, \overline{z}) \cdot \mathrm{j}(w, \overline{w}). \tag{9.3}$$

In principle, we can identify $\mathcal{Z}$ with $\mathcal{Y}$ by

$$\mathcal{Z} \ni z \mapsto \frac{1}{\sqrt{2}}(z + \overline{z}) \in \mathcal{Y}, \tag{9.4}$$

but we choose not to do so.

$\mathbb{C}\mathcal{Y}$ is identified with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ by the map

$$\mathbb{C}\mathcal{Y} \ni (z_1, \overline{z}_1) + \mathrm{i}(z_2, \overline{z}_2) \mapsto (z_1 + \mathrm{i}z_2, \overline{z_1 - \mathrm{i}z_2}) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

The complexifications of (9.1), (9.2) and (9.3) are

$$\mathrm{j}_{\mathbb{C}}(z_1, \overline{z}_2) = (\mathrm{i}z_1, -\mathrm{i}\overline{z}_2),$$

$$(z_1, \overline{z}_2) \cdot_{\mathbb{C}} (w_1, \overline{w}_2) = (z_1|w_1) + (w_2|z_2), \tag{9.5}$$

$$(z_1, \overline{z}_2) \cdot \omega_{\mathbb{C}} (w_1, \overline{w}_2) = \frac{1}{\mathrm{i}}\big((z_1|w_1) - (w_2|z_2)\big). \tag{9.6}$$

$\mathcal{Y}^{\#}$, the space dual to $\mathcal{Y}$, is canonically identified with $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ by using the scalar product (9.2), and $\mathbb{C}\mathcal{Y}^{\#}$ is identified with $\overline{\mathcal{Z}} \oplus \mathcal{Z}$.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

9.1 Fock CCR representation

9.1.1 Field operators on Fock spaces

Consider the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. Recall that, for $z \in \mathcal{Z}$, $a^*(z)$, resp. $a(z)$ denote the corresponding creation, resp. annihilation operators.

Definition 9.1 *For $w = (z_1, \overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ we define the unbounded operator*

$$\phi(w) := a^*(z_1) + a(z_2) \quad \textit{with domain } \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}).$$

Proposition 9.2 (1) *For $w \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$, $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ is an invariant subspace of entire analytic vectors for $\phi(w)$.*

(2) *The operators $\phi(y)$ for $y \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ are essentially self-adjoint. We will still denote by $\phi(y)$ their closures.*

(3) *The operators $\phi(w)$ for $w \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ are closable. We will still denote by $\phi(w)$ their closures.*

(4) *The map $\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni w \mapsto \phi(w)$ is $\mathbb{C}$-linear on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.*

(5) *For $w_1, w_2 \in \mathbb{C}\mathcal{Y}$, we have*

$$[\phi(w_1), \phi(w_2)] = \mathrm{i} w_1 \cdot \omega_{\mathbb{C}} w_2 \mathbb{1} \textit{ on } \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z}). \tag{9.7}$$

(6) *If $w = y_1 + \mathrm{i} y_2$ with $y_1, y_2 \in \mathcal{Y}$, then $\mathrm{Dom}\,\phi(w) = \mathrm{Dom}\,\phi(y_1) \cap \mathrm{Dom}\,\phi(y_2)$.*

Proof Let $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. From Thm. 3.51 we obtain

$$\|\phi(w)\Psi\| \leq \|w\| \|(N + \mathbb{1})^{\frac{1}{2}} \Psi\|.$$

By induction on n we obtain then that

$$\|\phi(w)^n \Psi\| \leq \|w\|^n \|(\tfrac{(N+n)!}{N!})^{\frac{1}{2}} \Psi\|. \tag{9.8}$$

This proves (1).

Now (2) follows from Nelson's commutator theorem; see Thm. 2.74 (1).

To prove (3) note that $\phi(\overline{w}) \subset \phi(w)^*$. So $\phi(w)$ is closable.

(4) and (5) follow by direct computation. (6) follows from (5) by repeating the argument of the proof of Prop. 8.31. □

Corollary 9.3 *Let $z \in \mathcal{Z}$. Then $a(z)$, $a^*(z)$ are closable. Denoting their closures with the same symbols, for $y = (z, \overline{z})$, we have*

$$a^*(z) = \frac{1}{2}\big(\phi(y) - \mathrm{i}\phi(\mathrm{j}y)\big), \quad a(z) = \frac{1}{2}\big(\phi(y) + \mathrm{i}\phi(\mathrm{j}y)\big),$$
$$\mathrm{Dom}\, a^*(z) = \mathrm{Dom}\, a(z) = \mathrm{Dom}\,\phi(y) \cap \mathrm{Dom}\,\phi(\mathrm{j}y).$$

Remark 9.4 *We have seen in Subsect. 1.3.9 that the map (9.4) is unitary. Using this identification, one can parametrize field operators by vectors of $\mathcal{Z}$ instead of vectors of $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. This leads to the definition*

$$\phi(z) := \frac{1}{\sqrt{2}}\big(a^*(z) + a(z)\big), \quad z \in \mathcal{Z},$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

which is commonly found in the literature. In most of our work we will try to avoid this definition.

9.1.2 Weyl operators on Fock spaces

Theorem 9.5 (1) *If $w_1, w_2 \in \mathbb{C}\mathcal{Y}$ and $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$, then the relationship*

$$\mathrm{e}^{\mathrm{i}\phi(w_1)}\mathrm{e}^{\mathrm{i}\phi(w_2)}\Psi = \mathrm{e}^{-\frac{\mathrm{i}}{2}w_1\cdot\omega_{\mathbb{C}}w_2}\mathrm{e}^{\mathrm{i}\phi(w_1+w_2)}\Psi \tag{9.9}$$

holds, where the exponentials are defined in terms of the power series and all the series involved in (9.9) are absolutely convergent.

(2) *Set*

$$W(y) := \mathrm{e}^{\mathrm{i}\phi(y)}, \quad y \in \mathcal{Y}.$$

Then the map

$$\mathcal{Y} \ni y \mapsto W(y) \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big) \tag{9.10}$$

is a regular irreducible CCR representation, if we equip $\mathcal{Y}$ with the symplectic form ω defined in (9.3).

(3) *If $p \in U(\mathcal{Z})$, $(z, \overline{z}) \in \mathcal{Y}$, we have*

$$\Gamma(p)W(z, \overline{z}) = W(pz, \overline{pz})\Gamma(p).$$

(4) *The map (9.10) is strongly continuous if we equip $\mathcal{Y}$ with the norm topology.*

Definition 9.6 *(9.10) is called the* Fock CCR representation on $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

Proof To prove (1), we use the *Baker–Campbell formula*, which says the following: if A, B are operators such that $[A, B]$ commutes with A and B, then

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{\frac{1}{2}[A,B]}\mathrm{e}^{A+B} \tag{9.11}$$

as an identity between formal power series. We apply this formula to $A = \mathrm{i}\phi(w_1)$, $B = \mathrm{i}\phi(w_2)$, using (9.7). We use (9.8) to prove the norm convergence of the series appearing in (9.11).

Let us now prove (2). For $y_1, y_2 \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, both sides of (9.9) extend to unitary operators, so (9.9) is valid on the whole space $\Gamma_{\mathrm{s}}(\mathcal{Z})$. Therefore, (9.10) is a CCR representation. Since $W(y) = \mathrm{e}^{\mathrm{i}\phi(y)}$, this representation is regular.

Let us prove that it is irreducible. Let P be an orthogonal projection acting on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ such that $[P, W(y)] = 0$ for all $y \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. Then $[P, \phi(y)] = 0$ on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ for all $y \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, and hence $[P, a^*(z)] = [P, a(z)] = 0$ for all $z \in \mathcal{Z}$. It follows that $a(z)P\Omega = 0$. Hence, by (3.25), $P\Omega = 0$ or $P\Omega = \Omega$. By (3.26) and the fact that $[P, a^*(z)] = 0$, we obtain that $P = 0$ or $P = \mathbb{1}$.

To prove (4), we first see using the CCR that it suffices to prove the continuity of (9.10) at $y = 0$. Now, for $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ we have

$$\|(W(y) - \mathbb{1})\Psi\| \leq \|\phi(y)\Psi\| \leq \|y\|\|(N + \mathbb{1})^{\frac{1}{2}}\Psi\|. \qquad \square$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Recall that we defined the parity operator as $I := (-1)^N$ in (3.10). If $\mathcal{Y}$ is finite-dimensional, we defined the parity operator as $I := \mathrm{Op}(\pi^d \delta_0)$ in (8.46).

Proposition 9.7 *In the finite-dimensional case, the definitions of the parity operator of (3.10) and of (8.46) coincide.*

9.1.3 Exponentials of creation and annihilation operators

Theorem 9.8 *Let $z \in \mathcal{Z}$.*

(1) *The operators $\mathrm{e}^{\phi(z,\overline{z})}$ are essentially self-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.*
(2) $\mathrm{e}^{a^*(z)}$ *and* $\mathrm{e}^{a(z)}$ *are closable operators on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ and their closures have the domains*

$$\mathrm{Dom}\,\mathrm{e}^{a^*(z)} = \mathrm{Dom}\,\mathrm{e}^{a(z)} = \mathrm{Dom}\,\mathrm{e}^{\frac{1}{2}\phi(z,\overline{z})}.$$

(3) *In the sense of quadratic forms, we can write*

$$W(-\mathrm{i}z, \mathrm{i}\overline{z}) = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\mathrm{e}^{a^*(z)}\mathrm{e}^{-a(z)}. \tag{9.12}$$

(4)

$$(\Omega|W(z,\overline{z})\Omega) = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}. \tag{9.13}$$

Proof (1) Using the exponential law in Prop. 3.56, it suffices to consider the case when $\dim\mathcal{Z} = 1$. For $z \in \mathcal{Z}$, we consider the unique conjugation τ such that $\tau z = z$ and introduce the associated real-wave representation defined in Thm. 9.20. This allows us to identify $\Gamma_{\mathrm{s}}(\mathcal{Z})$ with $L^2(\mathbb{R}, (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$, $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ with the space of polynomials, and $\phi(z,\overline{z})$ with the operator of multiplication by αx for some $\alpha \in \mathbb{R}$. Then (1) is equivalent to the fact that the space of polynomials is dense in $L^2(\mathbb{R}, \mathrm{d}\mu)$ for $\mathrm{d}\mu = (2\pi)^{-\frac{1}{2}}(1 + \mathrm{e}^{\alpha x})^2\mathrm{e}^{-x^2/2}\mathrm{d}x$, which is well known.

(2) We have

$$\mathrm{e}^{a(z)} \subset (\mathrm{e}^{a^*(z)})^*, \qquad \mathrm{e}^{a^*(z)} \subset (\mathrm{e}^{a(z)})^*.$$

Hence $\mathrm{e}^{a^*(z)}$ and $\mathrm{e}^{a(z)}$ are closable on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. Next we use the Baker–Campbell formula (9.11) on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ to get

$$\mathrm{e}^{a(z)}\mathrm{e}^{a^*(z)} = \mathrm{e}^{\frac{1}{2}\overline{z}\cdot z}\mathrm{e}^{\phi(z,\overline{z})}, \qquad \mathrm{e}^{a^*(z)}\mathrm{e}^{a(z)} = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\mathrm{e}^{\phi(z,\overline{z})}.$$

Thus, for $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$,

$$\|\mathrm{e}^{a^*(z)}\Psi\|^2 = \mathrm{e}^{\frac{1}{2}\overline{z}\cdot z}\|\mathrm{e}^{\frac{1}{2}\phi(z,\overline{z})}\Psi\|^2, \qquad \|\mathrm{e}^{a(z)}\Psi\|^2 = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\|\mathrm{e}^{\frac{1}{2}\phi(z,\overline{z})}\Psi\|^2.$$

Then we apply (1).

(3) follows from (9.11) and implies (4). □

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

9.1.4 Gaussian coherent vectors on Fock spaces

Let $z \in \mathcal{Z}$.

Definition 9.9 *We define*

$$\Omega_z := W(-\mathrm{i}z, \mathrm{i}\overline{z})\Omega = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\mathrm{e}^{a^*(z)}\Omega = \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\sum_{n=0}^{\infty}\frac{z^{\otimes n}}{\sqrt{n!}}. \tag{9.14}$$

The vectors Ω_z will be called Glauber's *or* Gaussian coherent vectors. *Let P_z be the orthogonal projection onto Ω_z, so that*

$$P_z = W(-\mathrm{i}z, \mathrm{i}\overline{z})|\Omega)(\Omega|W(\mathrm{i}z, -\mathrm{i}\overline{z}).$$

Note that $(-\mathrm{i}z, \mathrm{i}\overline{z}) = -\omega^{-1}(\overline{z}, z)$. Hence, in the notation of Sect. 8.5, Ω_z equals $\Psi_{\overline{z},z}$ for $\Psi_{0,0} = \Omega$. Gaussian coherent vectors are eigenvectors of annihilation operators. Besides, one can say that Ω_z is localized in phase space around $(\overline{z}, z)$. This is expressed in the following proposition:

Proposition 9.10 *Let $w, z \in \mathcal{Z}$. Then $a(w)\Omega_z = (w|z)\Omega_z$. Therefore,*

$$\begin{aligned}(\Omega_z|a^*(w)\Omega_z) &= (z|w),\\ (\Omega_z|a(w)\Omega_z) &= (w|z),\\ \big(\Omega_z|\phi(w, \overline{w})\Omega_z\big) &= 2\mathrm{Re}(z|w) = (z, \overline{z})\cdot(w, \overline{w}).\end{aligned}$$

9.2 CCR on anti-holomorphic Gaussian $\mathbf{L}^2$ spaces

Let $\mathcal{Z}$ be a separable Hilbert space. We will use z as the generic variable in $\mathcal{Z}$.

Recall that if $\dim\mathcal{Z} < \infty$, then $(2\mathrm{i})^{-d}\mathrm{d}\overline{z}\mathrm{d}z$ is the volume form on $\mathcal{Z}_{\mathbb{R}}$ and $(2\pi\mathrm{i})^{-d}\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z$ defines the Gaussian measure for the covariance $\mathbb{1}$, which is a probability measure on $\mathcal{Z}_{\mathbb{R}}$. We can also define the corresponding Hilbert space of anti-holomorphic functions, denoted $L^2_{\mathbb{C}}(\overline{\mathcal{Z}}, (2\pi\mathrm{i})^{-d}\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. Thus if $F, G \in L^2_{\mathbb{C}}(\overline{\mathcal{Z}}, (2\pi\mathrm{i})^{-d}\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$, then their scalar product is given by

$$(F|G) := (2\pi\mathrm{i})^{-d}\int\overline{F(\overline{z})}G(\overline{z})\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z.$$

Recall from Subsect. 5.5.4 that this Hilbert space has a natural generalization to the case of an arbitrary dimension, denoted $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ and called the anti-holomorphic Gaussian $\mathbf{L}^2$ space over the space $\mathcal{Z}$.

The bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is naturally isomorphic to $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. This makes it possible to interpret Fock CCR representations in terms of operators acting on anti-holomorphic Gaussian $\mathbf{L}^2$ spaces.

This section can be viewed as a continuation of Sect. 5.5 on Gaussian measures on complex Hilbert spaces.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

9.2.1 Bosonic complex-wave representation

Theorem 9.11 (1) *The map* $T^{\mathrm{cw}} : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to \mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *given by*

$$\begin{aligned} T^{\mathrm{cw}}\Psi(\overline{z}) &:= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}(z^{\otimes n}|\Psi), \\ &= \mathrm{e}^{\frac{1}{2}\overline{z}\cdot z}(\Omega_z|\Psi), \quad \Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z}), \end{aligned}$$

is unitary. (In the second line we use Gaussian coherent vectors Ω_z*.)*

(2) *For* $w \in \mathcal{Z}$ *we have*

$$\begin{aligned} T^{\mathrm{cw}}\Omega &= 1, \\ T^{\mathrm{cw}}a^*(w) &= w\cdot\overline{z}\,T^{\mathrm{cw}}, \\ T^{\mathrm{cw}}a(w) &= \overline{w}\cdot\nabla_{\overline{z}}T^{\mathrm{cw}}, \\ (T^{\mathrm{cw}}\Gamma(p)\Psi)(\overline{z}) &= T^{\mathrm{cw}}\Psi(p^{\#}\overline{z}), \quad p \in B(\mathcal{Z}), \quad \Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z}). \end{aligned}$$

(3) *We have a regular irreducible CCR representation*

$$\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}}) \ni (w,\overline{w}) \mapsto \mathrm{e}^{\mathrm{i}(w\cdot\overline{z}+\overline{w}\cdot\nabla_{\overline{z}})} \in U(\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)). \tag{9.15}$$

(4) *The CCR representation (9.15) is equivalent to the Fock representation:*

$$T^{\mathrm{cw}}\mathrm{e}^{\mathrm{i}\phi(w,\overline{w})} = \mathrm{e}^{\mathrm{i}(w\cdot\overline{z}+\overline{w}\cdot\nabla_{\overline{z}})}T^{\mathrm{cw}}, \quad w \in \mathcal{Z}.$$

(5) *(9.15) acts on* $F \in \mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *as follows:*

$$\mathrm{e}^{\mathrm{i}(w\cdot\overline{z}+\overline{w}\cdot\nabla_{\overline{z}})}F(\overline{z}) = \mathrm{e}^{\mathrm{i}w\cdot\overline{z}-\frac{1}{2}\overline{w}\cdot w}F(\overline{z}+\mathrm{i}\overline{w}), \qquad w \in \mathcal{Z}.$$

Proof (1) follows from Thm. 5.88. (2)–(4) follow immediately from Thm. 5.88 and Subsect. 3.5.2. To prove (5) we use the Baker–Campbell–Hausdorff formula. □

Definition 9.12 *Following Segal, we will call* $T^{\mathrm{cw}}\Psi$ *the* complex-wave transform of Ψ. *(It is also sometimes called the* Bargmann *or* Bargmann–Segal transform of $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$. *Berezin calls it the* generating functional of Ψ.*)*

(9.15) will be called the complex-wave CCR representation. *(It is also called the* Bargmann *or* Bargmann–Segal representation.*)*

9.2.2 Coherent vectors in the complex-wave representation

Let $w \in \mathcal{Z}$. The complex-wave transform of the Gaussian coherent vector Ω_w is

$$T^{\mathrm{cw}}\Omega_w(\overline{z}) = \mathrm{e}^{-\frac{1}{2}\overline{w}w}\mathrm{e}^{\overline{z}\cdot w}.$$

As an exercise in the complex-wave representation let us calculate the scalar product of two such vectors:

$$\begin{aligned} (\Omega_{w_1}|\Omega_{w_2}) &= (2\pi\mathrm{i})^{-d}\int \mathrm{e}^{-\frac{1}{2}|w_1|^2-\frac{1}{2}|w_2|^2+z\cdot\overline{w}_1+\overline{z}\cdot w_2-|z|^2}\mathrm{d}\overline{z}\mathrm{d}z \\ &= \mathrm{e}^{-\frac{1}{2}|w_1|^2-\frac{1}{2}|w_2|^2+\overline{w}_1\cdot w_2}. \end{aligned}$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Definition 9.13 *Let* $\dim_{\mathbb{C}} \mathcal{Z} = d$ *be finite. The* Gaussian FBI transform *is the map* $T^{\mathrm{FBI}} : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to L^2(\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ *defined by*

$$\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z}) \ni (\overline{z}, z) \mapsto T^{\mathrm{FBI}}\Psi(\overline{z}, z) := (2\pi)^{-\frac{d}{2}} (\Omega_z|\Psi). \tag{9.16}$$

Clearly, the Gaussian FBI transform is a special case of the FBI transform defined in Subsect. 8.5.1, where we put $\Psi_0 = \Omega$.

By (9.16), in the finite-dimensional case we have the following simple relationship between the Gaussian FBI transformation and the complex-wave transformation:

$$T^{\mathrm{FBI}}\Psi(z, \overline{z}) = (2\pi)^{\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z} T^{\mathrm{cw}}\Psi(\overline{z}). \tag{9.17}$$

This gives the following alternative proof of the unitarity of T^{cw}:

$$\begin{aligned}(\Psi_1|\Psi_2) &= \mathrm{i}^{-d} \int \overline{T^{\mathrm{FBI}}\Psi_1(\overline{z}, z)} T^{\mathrm{FBI}}\Psi_2(\overline{z}, z) \mathrm{d}\overline{z}\mathrm{d}z & (9.18)\\ &= (2\pi\mathrm{i})^{-d} \int \mathrm{e}^{-\frac{1}{2}|z|^2} \overline{T^{\mathrm{cw}}\Psi_1(\overline{z})} \mathrm{e}^{-\frac{1}{2}|z|^2} T^{\mathrm{cw}}\Psi_2(\overline{z}) \mathrm{d}\overline{z}\mathrm{d}z & (9.19)\\ &= (T^{\mathrm{cw}}\Psi_1|T^{\mathrm{cw}}\Psi_2)_{\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)}.\end{aligned}$$

In (9.18) we used that $\mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z$ is the canonical measure on the symplectic space $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ and that T^{FBI} is isometric; see (8.53).

9.3 CCR on real Gaussian $\mathbf{L}^2$ spaces

If the complex dimension of $\mathcal{Z}$ is finite and equals the real dimension of $\mathcal{X}$, then the Fock representation on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is unitarily equivalent to the Schrödinger representation on $L^2(\mathcal{X})$. In order to describe this equivalence, one needs to fix a conjugation on the Kähler space $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, which allows us to separate field operators into "momentum" and "position" operators. In addition, one needs to fix a Euclidean structure on $\mathcal{X}$, which allows us to distinguish the Gaussian vector that is mapped to the Fock vacuum.

In the case of an infinite dimension we do not have a Schrödinger representation, since there is no Lebesgue measure on infinite-dimensional vector spaces. However, in this case we have the so-called *real-wave representations*, which can serve as a substitute for Schrödinger representations. Real-wave representations will be the main topic of this section. They are CCR representations acting on real Gaussian $\mathbf{L}^2$ spaces. They are unitarily equivalent to Fock representations.

Throughout this section, $\mathcal{X}$ is a real Hilbert space and $c \in B_{\mathrm{s}}(\mathcal{X})$ is invertible and positive. x will be used as the generic variable in $\mathcal{X}$.

Recall that if $\dim \mathcal{X} < \infty$, then $(2\pi)^{-\frac{d}{2}}(\det c)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x$ is a probability measure on $\mathcal{X}$. Thus we can define the corresponding Hilbert space

$$L^2(\mathcal{X}, (2\pi)^{-\frac{d}{2}}(\det c)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x).$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

As described in Def. 5.72, this can be generalized to the case of an arbitrary dimension, and then it is called the Gaussian $\mathbf{L}^2$ space for the covariance c and denoted

$$\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x). \tag{9.20}$$

In this section we describe the real-wave representation acting on (9.20).

This section can be viewed as a continuation of Sect. 5.4 on Gaussian measures on real Hilbert spaces.

9.3.1 Real-wave CCR representation

Let $\eta, q \in \mathcal{X}$. We set

$$\begin{aligned} \eta\cdot x_{\mathrm{rw}} &:= \eta\cdot x, \\ q\cdot D_{\mathrm{rw}} &:= q\cdot(\frac{1}{\mathrm{i}}\nabla_x + \frac{\mathrm{i}}{2}c^{-1}x), \quad \text{as operators on } \mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x). \end{aligned}$$

Theorem 9.14 (1) *The operator $\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}}$ is essentially self-adjoint on $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$.*

(2) *The map*

$$\mathcal{X}\oplus\mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{\mathrm{i}(\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}})} \in U(\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)) \tag{9.21}$$

is an irreducible regular CCR representation.

(3) *For $F \in \mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)$ one has*

$$\mathrm{e}^{\mathrm{i}(\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}})}F(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}q\cdot(\eta + \frac{\mathrm{i}}{2}c^{-1}q)}\mathrm{e}^{\mathrm{i}x\cdot(\eta + \frac{\mathrm{i}}{2}c^{-1}q)}F(x+q).$$

Proof We consider the one-parameter group

$$U_tF(x) := \mathrm{e}^{\frac{\mathrm{i}}{2}t^2q\cdot(\eta + \frac{\mathrm{i}}{2}c^{-1}q)}\mathrm{e}^{\mathrm{i}tx\cdot(\eta + \frac{\mathrm{i}}{2}c^{-1}q)}F(x+tq), \quad t \in \mathbb{R}.$$

Let $\mathcal{D} := \mathrm{Span}\{\mathrm{e}^{w\cdot x}, \quad w \in \mathbb{C}\mathcal{X}\}$. From Subsect. 5.2.5, we know that $\mathcal{D}$ is dense in $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)$. Clearly, $\mathcal{D}$ is invariant under U_t, and U_t is a strongly continuous group of isometries of $\mathcal{D}$, hence it extends to a strongly continuous unitary group. $\mathcal{D}$ is included in the domain of its generator, which equals $\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}}$ on $\mathcal{D}$. By Nelson's invariant domain theorem, Thm. 2.74 (2), we obtain that $\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}}$ is essentially self-adjoint on $\mathcal{D}$.

To show the essential self-adjointness on $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$, we note that $\mathcal{D}$ is in the closure of $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$ for the graph norm: in fact, for $w \in \mathbb{C}\mathcal{X}$, the series

$$\sum_{n=0}^{+\infty}\frac{(w\cdot x)^n}{n!}$$

converges to $\mathrm{e}^{w\cdot x}$ for the graph norm of $\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}}$. This proves (1) and (3). (2) follows immediately from (3). □

Definition 9.15 *The CCR representation (9.21) is called the* real-wave representation of covariance c.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Note that the operators x_{rw}, D_{rw} are examples of abstract position and momentum operators considered in Subsect. 8.2.6.

We equip $\mathcal{X}\oplus\mathcal{X}$ with the complex structure

$$\mathrm{j}=\begin{bmatrix}0 & -(2c)^{-1}\\ 2c & 0\end{bmatrix}, \tag{9.22}$$

which is Kähler. Thus $\mathcal{X}\oplus\mathcal{X}$ becomes a Kähler space with a conjugation. Therefore, as in Subsect. 8.2.7, for $w\in\mathbb{C}\mathcal{X}$ we can introduce the associated Schrödinger-type creation and annihilation operators:

$$a_{\mathrm{rw}}(w)=\overline{w}\cdot c\nabla_x,\quad a_{\mathrm{rw}}^*(w)=w\cdot x-w\cdot c\nabla_x.$$

Proposition 9.16 *Let $w,w_1,w_2\in\mathbb{C}\mathcal{X}$.*

(1) *The operators $a_{\mathrm{rw}}(w)$ and $a_{\mathrm{rw}}^*(w)$ are closable on $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$.*
(2) *We have*

$$[a_{\mathrm{rw}}(w_1),a_{\mathrm{rw}}^*(w_2)]=(w_1|cw_2)\mathbb{1},$$

$$[a_{\mathrm{rw}}(w_1),a_{\mathrm{rw}}(w_2)]=[a_{\mathrm{rw}}^*(w_1),a_{\mathrm{rw}}^*(w_2)]=0.$$

(3) $F\in\mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)$ *satisfies*

$$a_{\mathrm{rw}}(w)F=0,\quad w\in\mathbb{C}\mathcal{X},$$

iff F is proportional to 1.

Proof (1) follows from Prop. 8.31 and (2) is a special case of (8.30).

Let F be such that $a_{\mathrm{rw}}(w)F=0$ for $w\in\mathbb{C}\mathcal{X}$ and $(F|1)=0$. In particular, for each $G\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$,

$$(a_{\mathrm{rw}}^*(w)G|F)=0.$$

Clearly, the span of vectors of the form $\prod\limits_{i=1}^{n}a_{\mathrm{rw}}^*(w_i)1$ equals the space of polynomials in $\mathbb{C}\mathrm{Pol}(\mathcal{X})$ of degree greater than 1. So F is orthogonal to $\mathbb{C}\mathrm{Pol}(\mathcal{X})$, and hence $F=0$, which proves (3). □

The usual choice is $c=\mathbb{1}$, which leads to the complex structure

$$\mathrm{j}=\begin{bmatrix}0 & -\frac{1}{2}\mathbb{1}\\ 2\mathbb{1} & 0\end{bmatrix}.$$

Remark 9.17 *The advantage of the real-wave representation is the fact that we can make an identification*

$$\mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)\simeq L^2(Q,\mu)$$

for an L^2 space over some true measure space $(Q,\mathfrak{S},\mu)$. There is no unique choice of the measure space $(Q,\mathfrak{S},\mu)$, especially in the case of an infinite-dimensional $\mathcal{X}$, but it essentially does not matter which one we take. A class

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

of possible choices is described in Subsect. 5.4.2: we can set $Q = B^{\frac{1}{2}}\mathcal{X}$, where $B > 0$ is an operator on $\mathcal{X}$ with B^{-1} trace-class, but there are many others; see the discussion in Simon (1974). Therefore, the real-wave representation is sometimes called the Q-space representation *of the bosonic Fock space.*

9.3.2 Real-wave CCR representation in finite dimension

If the dimension of $\mathcal{X}$ is finite, then the real-wave representation is a special case of a weighted Schrödinger representation with

$$m(x) = (2\pi)^{-\frac{d}{4}}(\det c)^{-\frac{1}{4}}\mathrm{e}^{-\frac{1}{4}x\cdot c^{-1}x}. \tag{9.23}$$

(9.23) is the pointwise positive ground state of

$$H = -\Delta + \frac{1}{4}x\cdot c^{-2}x - \frac{1}{2}\mathrm{Tr}\, c^{-1}.$$

The Dirichlet form for (9.23) in the Hilbert space $L^2\left(\mathcal{X}, (2\pi)^{-\frac{d}{2}}(\det c)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x\right)$ equals

$$-\Delta + x\cdot c^{-1}\nabla_x.$$

The unitary operator

$$L^2\left(\mathcal{X}, (2\pi)^{-\frac{d}{4}}(\det c)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x\right) \ni F \mapsto T^{\mathrm{sch}}F := m(x)F \in L^2(\mathcal{X})$$

intertwines the Schrödinger and the real-wave representations:

$$\mathrm{e}^{\mathrm{i}(\eta\cdot x + q\cdot D)}T^{\mathrm{sch}} = T^{\mathrm{sch}}\mathrm{e}^{\mathrm{i}(\eta\cdot x_{\mathrm{rw}} + q\cdot D_{\mathrm{rw}})}.$$

9.3.3 Wick transformation

The real-wave representation on $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)$ is unitarily equivalent to the Fock representation on $\Gamma_{\mathrm{s}}(c^{-\frac{1}{2}}\mathbb{C}\mathcal{X})$. This follows by a general argument from Prop. 9.16 and the fact that polynomials are dense; see Subsect. 5.2.6.

In this subsection we will construct an explicit unitary transformation that intertwines the real-wave representation and the Fock representation.

Definition 9.18 *For $F \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$, we define*

$$:F: = a^*_{\mathrm{rw}}(F)1 \in \mathbb{C}\mathrm{Pol}_s(\mathcal{X}).$$

The map $F \mapsto {:F:}$ is called the Wick transformation w.r.t. the covariance c.

The following proposition shows how one can compute $:G:$.

Proposition 9.19 (1) *For $G \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$, one has*

$$:G(x): = \mathrm{e}^{-\frac{1}{2}\nabla_x\cdot c\nabla_x}G(x) = \mathrm{e}^{\frac{1}{2}x\cdot c^{-1}x}G(-c\nabla_x)\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}. \tag{9.24}$$

(2) *For $G(x) \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$, one has*

$$G(x) = \mathrm{e}^{\frac{1}{2}\nabla_x\cdot c\nabla_x}:G(x): = \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}:G:(c\nabla_x)\mathrm{e}^{\frac{1}{2}x\cdot c^{-1}x}.$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Proof Let $w \in \mathbb{C}\mathcal{X}$. The following operator identities are valid on $\mathbb{C}\mathrm{Pol}_s(\mathcal{X})$:

$$\begin{aligned} a^*_{\mathrm{rw}}(w) &= w \cdot x - w \cdot c\nabla_x \\ &= \mathrm{e}^{-\frac{1}{2}\nabla_x \cdot c\nabla_x}(w \cdot x)\mathrm{e}^{\frac{1}{2}\nabla_x \cdot c\nabla_x} = \mathrm{e}^{\frac{1}{2}x\cdot c^{-1}x}(-w \cdot c\nabla_x)\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}. \end{aligned}$$

This yields, for $G \in \mathbb{C}\mathrm{Pol}_\mathrm{s}(\mathcal{X})$, the operator identity

$$a^*_{\mathrm{rw}}(G) = \mathrm{e}^{-\frac{1}{2}\nabla_x \cdot c\nabla_x}G(x)\mathrm{e}^{\frac{1}{2}\nabla_x \cdot c\nabla_x} = \mathrm{e}^{\frac{1}{2}x\cdot c^{-1}x}G(-c\nabla_x)\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}.$$

By applying it to the polynomial 1, we obtain

$$:G: = a^*_{\mathrm{rw}}(G)1 = \mathrm{e}^{-\frac{1}{2}\nabla_x \cdot c\nabla_x}G = \mathrm{e}^{\frac{1}{2}x\cdot c^{-1}x}G(-c\nabla_x)\mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x},$$

which proves (1). Clearly, (2) follows from (1). □

Note that the space $\mathbb{C}\mathrm{Pol}_\mathrm{s}(\mathcal{X})$ can be identified with $\mathrm{Pol}_\mathrm{s}(\overline{\mathbb{C}\mathcal{X}})$ (by analytic continuation/restriction; see Subsect. 3.5.6). Let z denote the generic variable in $\mathbb{C}\mathcal{X}$. The following theorem is immediate:

Theorem 9.20 (1) *The map*

$$\mathrm{Pol}_\mathrm{s}(\overline{\mathbb{C}\mathcal{X}}) \ni F \mapsto :F: \in \mathbb{C}\mathrm{Pol}_\mathrm{s}(\mathcal{X})$$

extends to a unitary map

$$\mathbf{L}^2_{\mathbb{C}}(\overline{\mathbb{C}\mathcal{X}}, \mathrm{e}^{-\overline{z}\cdot c^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z) \ni F \mapsto :F: \in \mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x). \tag{9.25}$$

(2) *(9.25) intertwines the complex-wave and real-wave CCR representations:*

$$:\mathrm{e}^{\mathrm{i}(w\cdot\overline{z}+\overline{w}\cdot\nabla_{\overline{z}})}F: = \mathrm{e}^{\mathrm{i}(a^*_{\mathrm{rw}}(w)+a_{\mathrm{rw}}(w))}:F:,\ \ F \in \mathbf{L}^2_{\mathbb{C}}(\overline{\mathbb{C}\mathcal{X}}, \mathrm{e}^{-\overline{z}\cdot c^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z),\ \ w \in \mathbb{C}\mathcal{X}.$$

(3) *For $w \in \mathbb{C}\mathcal{X}$, we have*

$$:\mathrm{e}^{w\cdot x}: = \mathrm{e}^{w\cdot x}\mathrm{e}^{-\frac{1}{2}w\cdot cw}. \tag{9.26}$$

Remark 9.21 *(9.26) is often used as the definition of the Wick transformation.*

Using Subsect. 9.2.1, we can unitarily identify the real-wave representation on $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x)$ and the Fock representation on $\Gamma_\mathrm{s}(c^{-\frac{1}{2}}\mathbb{C}\mathcal{X})$. This is described in the next theorem.

Theorem 9.22 *Set*

$$\Gamma_\mathrm{s}(c^{-\frac{1}{2}}\mathbb{C}\mathcal{X}) \ni \Phi \mapsto T^{\mathrm{rw}}\Phi := :T^{\mathrm{cw}}\Phi: \in \mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot c^{-1}x}\mathrm{d}x). \tag{9.27}$$

Then

(1) T^{rw} *is unitary.*
(2) T^{rw} *is the unique bounded linear map such that*

$$T^{\mathrm{rw}}\Omega = 1,\ \ \textit{and}\ T^{\mathrm{rw}}\mathrm{e}^{\mathrm{i}a^*(\eta)+a(\eta)} = \mathrm{e}^{\mathrm{i}\eta\cdot x_{\mathrm{rw}}}T^{\mathrm{rw}},\quad \eta \in c^{-\frac{1}{2}}\mathcal{X}.$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

(3) T^{rw} *is the unique bounded linear map such that*

$$T^{\mathrm{rw}}\Omega = 1, \text{ and } T^{\mathrm{rw}} \prod_{i=1}^{n} a^*(w_i) = \prod_{i=1}^{n} a^*_{\mathrm{rw}}(w_i) T^{\mathrm{rw}}, \quad w_i \in c^{-\frac{1}{2}}\mathbb{C}\mathcal{X}.$$

Remark 9.23 *In the case of a single variable, that is, $\mathcal{X} = \mathbb{R}$, and $c = \mathbb{1}$, the Wick transformation for monomials is the same as the Gram–Schmidt orthogonalization procedure with the weight $\mathrm{e}^{-\frac{1}{2}x^2}$. The polynomials $:x^n:$ are rescaled Hermite polynomials. More precisely, if one adopts the following definition of Hermite polynomials:*

$$\mathrm{e}^{2xt-t^2} =: \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x),$$

then

$$:x^n: = \sqrt{2}^n H_n\left(\tfrac{x}{\sqrt{2}}\right).$$

9.3.4 Integrals of polynomials with a Gaussian weight

In this subsection, for simplicity, we assume that $c = \mathbb{1}$.

In physics one often computes integrals of a polynomial times the Gaussian weight. The Wick transformation helps to perform such an integral, as is seen from (9.29):

Theorem 9.24 *Let $F \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$. Then*

$$\int_{\mathcal{X}} F(x)\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x = \left(\mathrm{e}^{\frac{1}{2}\nabla_x^2}F\right)(0), \tag{9.28}$$

$$\int_{\mathcal{X}} :F(x):\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x = F(0). \tag{9.29}$$

Proof We can assume that $\mathcal{X}$ is of finite dimension. Recall the identity (4.14):

$$\mathrm{e}^{\frac{1}{2}\nabla_x^2}F(y) = (2\pi)^{-\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2}(y-x)^2}F(x)\mathrm{d}x. \tag{9.30}$$

In (9.30) we set $y = 0$, which proves (9.28).

To prove (9.29) we use (9.28) and Prop. 9.19. □

Note that the r.h.s. of (9.28) can be expanded in a finite sum and leads to the well-known sum over all possible “pairings”. This is the simplest version of what is usually called the *Wick theorem.*

A more complicated version of the Wick theorem is given below. It has a well-known graphical interpretation in terms of diagrams, which we will discuss in Chap. 20.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Theorem 9.25 *Let $F_1,\dots,F_n \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$. Then*

$$
\begin{aligned}
&{:}F_1(x){:}\cdots{:}F_n(x){:} \\
&= {:}\exp\Big(\sum_{i<j}\nabla_{x_i}\nabla_{x_j}\Big)F_1(x_1)\cdots F_n(x_n)\big|_{x=x_1=\cdots=x_n}{:}\,,
\end{aligned}
\tag{9.31}
$$

$$
\begin{aligned}
&(2\pi)^{-\frac{d}{2}}\int {:}F_1(x){:}\cdots{:}F_n(x){:}\mathrm{e}^{-\frac12 x^2}\mathrm{d}x \\
&= \exp\Big(\sum_{i<j}\nabla_{x_i}\nabla_{x_j}\Big)F_1(x_1)\cdots F_n(x_n)\big|_{0=x_1=\cdots=x_n}.
\end{aligned}
\tag{9.32}
$$

Proof To prove (9.31), we write

$$
\begin{aligned}
&{:}F_1(x){:}\cdots{:}F_n(x){:} \\
&= \mathrm{e}^{-\frac12\nabla_{x_1}^2}F_1(x_1)\cdots\mathrm{e}^{-\frac12\nabla_{x_n}^2}F_n(x_n)\big|_{x=x_1=\cdots=x_n} \\
&= {:}\mathrm{e}^{\frac12\nabla_x^2}\Big(\mathrm{e}^{-\frac12\nabla_{x_1}^2}F_1(x_1)\cdots\mathrm{e}^{-\frac12\nabla_{x_n}^2}F_n(x_n)\big|_{x=x_1=\cdots=x_n}\Big){:} \\
&= {:}\mathrm{e}^{\frac12(\nabla_{x_1}+\cdots+\nabla_{x_n})^2-\frac12\nabla_{x_1}^2-\cdots-\frac12\nabla_{x_n}^2}F_1(x_1)\cdots F_n(x_n)\big|_{x=x_1=\cdots=x_n}{:}.
\end{aligned}
$$

In the last step we used that

$$\nabla_x f(x,\dots,x) = (\nabla_{x_1}+\cdots+\nabla_{x_n})f(x_1,\cdots,x_n)\big|_{x=x_1=\cdots=x_n}.$$

(9.32) follows from (9.31) and (9.29). □

9.3.5 Operators in the real-wave representation

Definition 9.26 *For an operator a on $\mathcal{X}$, we will write*

$$\Gamma_{\mathrm{rw}}(a) := T^{\mathrm{rw}}\Gamma(a_{\mathbb{C}})T^{\mathrm{rw}*},$$

where we recall that $a_{\mathbb{C}}$ denotes the extension of a to $\mathbb{C}\mathcal{X}$.

Suppose that $c>0$ is an operator on $\mathcal{X}$. Clearly,

$$\Gamma_{\mathrm{rw}}(c^{-\frac12}) : \mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac12 x^2}\mathrm{d}x) \to \mathbf{L}^2(c^{-\frac12}\mathcal{X},\mathrm{e}^{-\frac12 x\cdot c^{-1}x}\mathrm{d}x)$$

is a unitary operator. Therefore, in what follows we will stick to the covariance $\mathbb{1}$.

Recall from Remark 9.17 that $\mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac12 x^2}\mathrm{d}x)$ can be interpreted as $L^2(Q,\mu)$ for some measure space (Q,μ). Let F be a bounded Borel function on Q. Then one can define $F(x_{\mathrm{rw}})$, which is a bounded operator on $\mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac12 x^2}\mathrm{d}x)$. It can be also interpreted as an element of $\mathbf{L}^2(\mathcal{X},\mathrm{e}^{-\frac12 x^2}\mathrm{d}x)$, and then it will simply be written F. Clearly, $F(x_{\mathrm{rw}})1 = F$.

Proposition 9.27 *Let u be an orthogonal operator on $\mathcal{X}$. Then*

$$\Gamma_{\mathrm{rw}}(u)F(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)^{-1} = (\Gamma_{\mathrm{rw}}(u)F)(x_{\mathrm{rw}}). \tag{9.33}$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Proof A dense set of vectors in $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ is given by $G(x_{\mathrm{rw}})1 = G$ for G bounded Borel functions on $B^{\frac{1}{2}}\mathcal{X}$. We have the commutation property

$$\Gamma_{\mathrm{rw}}(u)F(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)^{-1}G(x_{\mathrm{rw}}) = G(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)F(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)^{-1}. \tag{9.34}$$

Hence, applying (9.34) to the vacuum 1 we obtain

$$\Gamma_{\mathrm{rw}}(u)F(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)^{-1}G = G(x_{\mathrm{rw}})\Gamma_{\mathrm{rw}}(u)F = (\Gamma_{\mathrm{rw}}(u)F)(x_{\mathrm{rw}})G.$$

□

Proposition 9.28 *Let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}$. Let e_1 be the orthogonal projection on $\mathcal{X}_1$. Let $\mathfrak{B}_1$ be the sub-σ-algebra of functions based in $\mathcal{X}_1$, and $E_{\mathfrak{B}_1}$ the corresponding conditional expectation. Then*

$$E_{\mathfrak{B}_1} = \Gamma_{\mathrm{rw}}(e_1).$$

Proposition 9.29 *Let $a \in B(\mathcal{X})$. Then*

(1) *If $\|a\| \leq 1$, $\Gamma_{\mathrm{rw}}(a)$ is doubly Markovian, hence it is a contraction on $L^p(Q, \mathrm{d}\mu)$ for all $1 \leq p \leq \infty$.*
(2) *If $\|a\| < 1$, then $\Gamma_{\mathrm{rw}}(a)$ is positivity improving.*

Proof We drop rw from Γ_{rw} and x_{rw}.

We first prove (1). We write a as j^*uj, where

$$\mathcal{X} \ni x \mapsto j(x) := x \oplus 0 \in \mathcal{X} \oplus \mathcal{X}$$

is isometric and

$$u = \begin{bmatrix} a & (\mathbb{1} - aa^*)^{\frac{1}{2}} \\ (\mathbb{1} - a^*a)^{\frac{1}{2}} & a^* \end{bmatrix}$$

is orthogonal. Using Subsect. 5.4.3, we see that if we take $(Q \times Q, \mu \otimes \mu)$ as the Q-space for $\mathcal{X} \oplus \mathcal{X}$, then the map $\Gamma(j)$ is

$$L^2(Q, \mathrm{d}\mu) \ni f \mapsto f \otimes 1 \in L^2(Q, \mathrm{d}\mu) \otimes L^2(Q, \mathrm{d}\mu) \simeq L^2(Q \times Q, \mathrm{d}\mu \otimes \mathrm{d}\mu),$$

which is positivity preserving.

The map $\Gamma(u)$ is clearly positivity preserving. In fact, recall that $F(x)$ is the operator of multiplication by a measurable function F on $L^2(Q, \mu)$. By (9.33) and the unitarity of u, $(\Gamma(u)F)(x) = \Gamma(u)F(x)\Gamma(u)^{-1}$. Since $F \geq 0$ a.e. iff $F(x) \geq 0$, we see that $\Gamma(u)$ is positivity preserving. Finally $\Gamma(j^*) = \Gamma(j)^*$ is also positivity preserving by the remark after Def. 5.21. Hence $\Gamma(a)$ is positivity preserving. Since $\Gamma(a)$ and $\Gamma(a)^*$ preserve 1, $\Gamma(a)$ is doubly Markovian.

Let us now prove (2). We write $\Gamma(a) = \Gamma(\|a\|)\Gamma(b)$, where $a =: \|a\|b$. Then $\|b\| \leq 1$, and thus $\Gamma(b)$ is positivity preserving by (1). If $f \geq 0$ and $f \neq 0$, then $\int_Q \Gamma(b)f\mathrm{d}\mu = \int_Q f\mathrm{d}\mu > 0$, so $\Gamma(b)$ preserves the set of non-zero positive functions. So it suffices to prove that $\Gamma(\|a\|)$ is positivity improving.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Let $f, g \geq 0$ with $f, g \neq 0$. The function $F(t) = (f|\Gamma(\mathrm{e}^{-t})g)$ is positive on $\mathbb{R}^+$ by (1). It tends to $(1|f)(1|g)$ at $+\infty$, since $\Gamma(\mathrm{e}^{-t}) = \mathrm{e}^{-tN}$, where N is the number operator. Since F extends holomorphically to $\{z : \mathrm{Re}\, z > 0\}$, it has isolated zeroes in $\mathbb{R}^+$. Let $t > 0$ and $0 < t_0 < t$ such that $F(t_0) > 0$. Set $f_1 = \Gamma(\mathrm{e}^{-t_0/2})f$, $g_1 = \Gamma(\mathrm{e}^{-t_0/2})g$. Then $f_1, g_1 \geq 0$ and $(f_1|g_1) = F(t_0) > 0$. Therefore, $f_1 g_1 \neq 0$ and $h = \min(f_1, g_1) \neq 0$. This yields

$$\begin{aligned}(f|\Gamma(\mathrm{e}^{-t})g) &= (f_1|\Gamma(\mathrm{e}^{-(t-t_0)})g_1) \\ &\geq (h|\Gamma(\mathrm{e}^{-(t-t_0)})g_1) \geq (h|\Gamma(\mathrm{e}^{-(t-t_0)})h) \\ &= \|\Gamma(\mathrm{e}^{-(t-t_0)/2})h\|^2 > 0,\end{aligned}$$

which completes the proof of (2). □

Below we recall Nelson's famous hyper-contractivity theorem.

Theorem 9.30 *Let $a \in B(\mathcal{X})$ and $1 < p < q < \infty$. If*

$$\|a\| \leq (p-1)^{\frac{1}{2}}(q-1)^{-\frac{1}{2}},$$

then $\Gamma_{\mathrm{rw}}(a)$ is a contraction from $L^p(Q, \mathrm{d}\mu)$ to $L^q(Q, \mathrm{d}\mu)$.

9.4 Wick and anti-Wick bosonic quantization

As elsewhere in this chapter, $\mathcal{Z}$ is a Hilbert space, $\mathcal{Y} = \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, $\mathcal{Y}^\# = \mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$, $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ and $\mathbb{C}\mathcal{Y}^\# = \overline{\mathcal{Z}} \oplus \mathcal{Z}$. We recall from Subsect. 3.5.6 that $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^\#)$ is identified with $\mathrm{Pol}_{\mathrm{s}}(\mathbb{C}\mathcal{Y}^\#)$. We can go from one representation to the other by analytic continuation/restriction. Thus we will freely switch between a polynomial in $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^\#)$ and $\mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$:

$$\begin{aligned}\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z}) \ni (\overline{z}, z) &\mapsto b(\overline{z}, z), \\ \overline{\mathcal{Z}} \oplus \mathcal{Z} \ni (\overline{z}_1, z_2) &\mapsto b(\overline{z}_1, z_2).\end{aligned}$$

We consider the Fock CCR representation

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big).$$

Recall that $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ is the $*$-algebra generated by $\phi(y)$, $y \in \mathcal{Y}$. It can be faithfully represented by operators on the space $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

We will define and study the bosonic Wick and anti-Wick quantization. The Wick quantization is the most frequently used quantization in quantum field theory and many-body quantum physics.

9.4.1 Wick and anti-Wick ordering

Let $b \in \mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}})$. Recall that in Subsect. 3.4.4 we defined the multiple creation and annihilation operators $a^*(b)$ and $a(b)$. Note that the possibility of unambiguously

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

defining $a^*(b)$ and $a(b)$ follows from the fact that $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ are isotropic subspaces of $\mathbb{C}\mathcal{Y}$ for $\omega_{\mathbb{C}}$.

Definition 9.31 *For $b_1, b_2 \in \mathrm{Pol_s}(\overline{\mathcal{Z}})$ we set*

$$\mathrm{Op}^{a^*,a}(b_1\overline{b_2}) := a^*(b_1)a(b_2),$$

$$\mathrm{Op}^{a,a^*}(\overline{b_2}b_1) := a(b_2)a^*(b_1).$$

These maps extend by linearity to maps

$$\begin{aligned} \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}^{a^*,a}(b) \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}), \\ \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#}) \ni b \mapsto \mathrm{Op}^{a,a^*}(b) \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}), \end{aligned} \tag{9.35}$$

called the Wick *and* anti-Wick bosonic quantizations*.*

Definition 9.32 *The inverse maps to (9.35) will be denoted by*

$$\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a^*,a} \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#}),$$

$$\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y}) \ni B \mapsto \mathrm{s}_B^{a,a^*} \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#}).$$

The polynomial $\mathrm{s}_B^{a^*,a}$, resp. s_B^{a,a^*} is called the Wick*, resp.* anti-Wick symbol of the operator B*.*

Remark 9.33 *Suppose that we fix an o.n. basis $\{e_i \ : \ i \in I\}$ in $\mathcal{Z}$. Every polynomial $b \in \mathrm{Pol_s}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ can be written as*

$$\sum_{\nu,\beta} b_{\nu,\beta}\overline{z}^{\nu}z^{\beta},$$

where ν, β are multi-indices, that is, elements of $\{0, 1, 2, \dots\}^I$. Then

$$\mathrm{Op}^{a^*,a}(b) = \sum_{\nu,\beta} b_{\nu,\beta}a^{*\nu}a^{\beta}, \tag{9.36}$$

$$\mathrm{Op}^{a,a^*}(b) = \sum_{\nu,\beta} b_{\nu,\beta}a^{\beta}a^{*\nu}. \tag{9.37}$$

The r.h.s. of (9.36), resp. (9.37) is probably the most straightforward, even if often somewhat heavy, notation for the Wick, resp. anti-Wick quantization.

More generally, one can assume that $\mathcal{Z} = L^2(\Xi, \mathrm{d}\xi)$, where $(\Xi, \mathrm{d}\xi)$ is a measure space. Then polynomials on $\mathcal{Z}$ can be written as

$$\sum_{n,m} \int \cdots \int b(\xi_1, \dots \xi_n; \xi'_m, \dots, \xi'_1)\overline{z}_{\xi_1} \cdots \overline{z}_{\xi_n} z_{\xi'_m} \cdots z_{\xi'_1},$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

and one writes

$$\sum_{n,m} b(\xi_1,\dots\xi_n;\xi_m',\dots,\xi_1')a_{\xi_1}^*\cdots a_{\xi_n}^* a_{\xi_m'}\cdots a_{\xi_1'} \quad \textit{instead of} \quad \mathrm{Op}^{a^*,a}(b),$$
$$\sum_{n,m} b(\xi_1,\dots\xi_n;\xi_m',\dots,\xi_1')a_{\xi_m'}\cdots a_{\xi_1'} a_{\xi_1}^*\cdots a_{\xi_n}^* \quad \textit{instead of} \quad \mathrm{Op}^{a,a^*}(b).$$

Thus a_ξ^* and a_ξ are treated as "operator-valued measures", which acquire their meaning after being "smeared out" with "test functions".

The following theorem is the analog of Thm. 4.38 devoted to the x, D- and D, x-quantizations.

Theorem 9.34 *Let $b, b_-, b_+, b_1, b_2 \in \mathrm{Pol_s}(\overline{\mathcal{Z}}, \mathcal{Z})$.*

(1) $\mathrm{Op}^{a,a^*}(b)^* = \mathrm{Op}^{a,a^*}(\overline{b})$ *and* $\mathrm{Op}^{a^*,a}(b)^* = \mathrm{Op}^{a^*,a}(\overline{b})$.

(2) *For $w \in \mathcal{Z}$,*

$$\begin{gathered}\mathrm{Op}^{a^*,a}(wb) = a^*(w)\mathrm{Op}^{a^*,a}(b), \quad \mathrm{Op}^{a^*,a}(\overline{w}b) = \mathrm{Op}^{a^*,a}(b)a(w),\\ [\mathrm{Op}^{a^*,a}(b), a^*(w)] = \mathrm{Op}^{a^*,a}(w\nabla_z b), \quad [a(w), \mathrm{Op}^{a^*,a}(b)] = \mathrm{Op}^{a^*,a}(\overline{w}\nabla_{\overline{z}} b).\\ (\Omega|\mathrm{Op}^{a^*,a}(b)\Omega) = b(0). \end{gathered} \tag{9.38}$$

(3) *If $\mathrm{Op}^{a,a^*}(b_-) = \mathrm{Op}^{a^*,a}(b_+)$, then*

$$\begin{aligned} b_+(\overline{z}, z) &= \mathrm{e}^{\nabla_{\overline{z}}\nabla_z} b_-(\overline{z}, z)\\ &= (2\pi\mathrm{i})^{-d}\int \mathrm{e}^{-(\overline{z}-\overline{z_1})(z-z_1)} b_+(\overline{z_1}, z_1)\mathrm{d}\overline{z}_1\mathrm{d}z_1, \quad \textit{if} \quad \dim\mathcal{Z} = d. \end{aligned}$$

(4) *If $\mathrm{Op}^{a^*,a}(b_1)\mathrm{Op}^{a^*,a}(b_2) = \mathrm{Op}^{a^*,a}(b)$, then*

$$\begin{aligned} b(\overline{z}, z) &= \mathrm{e}^{\nabla_{z_1}\nabla_{\overline{z_1}}} b_1(\overline{z}, z_1)b_2(\overline{z}_1, z)\big|_{z_1=z}\\ &= (2\pi\mathrm{i})^{-d}\int \mathrm{e}^{-(\overline{z}-\overline{z}_1)(z-z_1)} b_1(\overline{z}, z_1)b_2(\overline{z}_1, z)\mathrm{d}z_1\mathrm{d}\overline{z_1}, \quad \textit{if} \ \dim\mathcal{Z} = d. \end{aligned}$$

If $\mathrm{Op}^{a,a^*}(b_1)\mathrm{Op}^{a,a^*}(b_2) = \mathrm{Op}^{a,a^*}(b)$, then

$$b(\overline{z}, z) = \mathrm{e}^{-\nabla_{\overline{z}_1}\nabla_{z_1}} b_1(\overline{z}_1, z)b_2(\overline{z}, z_1)\big|_{z_1=z}.$$

Proof If we use the complex-wave representation, we see that the Wick, resp. anti-Wick quantization can be viewed as the $\overline{z}, \nabla_{\overline{z}}$, resp. $\nabla_{\overline{z}}, \overline{z}$ quantization. Therefore, we can apply the same combinatorial arguments as in the proof of Thm. 4.38. □

Remark 9.35 *The exponentials of differential operators in the above formulas can always be understood as finite sums of differential operators, since we consider polynomial symbols. Note also that in the expression for the anti-Wick symbol of a product of two operators there is no integral formula.*

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

The theorem that we state below is what is usually meant by *Wick's theorem.* We will discuss its diagrammatic interpretation in Chap. 20. It is an analog of Thm. 4.39.

Theorem 9.36 *Let* $b_1, \dots, b_n, b \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#})$ *and*

$$\mathrm{Op}^{a^*,a}(b) = \mathrm{Op}^{a^*,a}(b_1)\cdots\mathrm{Op}^{a^*,a}(b_n).$$

Then

$$\begin{aligned}&b(\overline{z}, z) &(9.39)\\ &= \exp\Big(\sum_{i<j}\nabla_{\overline{z}_i}\cdot\nabla_{z_j}\Big)b_1(\overline{z}_1, z_1)\cdots b_n(\overline{z}_n, z_n)\big|_{z=z_1=\cdots=z_n},\\ &(\Omega|\mathrm{Op}^{a^*,a}(b)\Omega) &(9.40)\\ &= \exp\Big(\sum_{i<j}\nabla_{\overline{z}_i}\cdot\nabla_{z_j}\Big)b_1(\overline{z}_1, z_1)\cdots b_n(\overline{z}_n, z_n)\big|_{0=z_1=\cdots=z_n}.\end{aligned}$$

Proof (9.39) is shown by the same arguments as Thm. 4.39. (9.40) follows from (9.39) and (9.38). □

9.4.2 Relation between Wick, anti-Wick and Weyl–Wigner quantizations

Let us assume that $\dim \mathcal{Z} < \infty$, so that the Weyl–Wigner quantization of a polynomial in $\mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#})$ is well defined.

The following theorem gives the connection between the Weyl–Wigner and the Wick and the anti-Wick quantizations. We express these connections using two alternative notations: either we treat them as functions of the complex variables $(\overline{z}_1, z_2) \in \overline{\mathcal{Z}} \oplus \mathcal{Z}$, or we treat the symbols as functions of the real variable $v \in \mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$.

Theorem 9.37 *Let* $b_-, b, b_+ \in \mathbb{C}\mathrm{Pol_s}(\mathcal{Y}^{\#})$. *Let*

$$\mathrm{Op}^{a^*,a}(b_+) = \mathrm{Op}(b) = \mathrm{Op}^{a,a^*}(b_-).$$

(1) *One can express the Wick symbol in terms of the Weyl–Wigner symbol:*

$$\begin{aligned}b_+(\overline{z}, z) &= \mathrm{e}^{\frac{1}{2}\nabla_{\overline{z}}\cdot\nabla_z}b(\overline{z}, z)\\ &= (\pi\mathrm{i})^{-d}\int \mathrm{e}^{-2(\overline{z}-\overline{z}_1)\cdot(z-z_1)}b(\overline{z}_1, z_1)\mathrm{d}\overline{z}_1\mathrm{d}z_1,\\ b_+(v) &= \mathrm{e}^{\frac{1}{4}\nabla_v^2}b(v)\\ &= \pi^{-d}\int \mathrm{e}^{-(v-v_1)^2}b(v_1)\mathrm{d}v_1.\end{aligned}$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

(2) *One can express the Weyl–Wigner symbol in terms of the anti-Wick symbol:*

$$\begin{aligned} b(\overline{z},z) &= \mathrm{e}^{\frac{1}{2}\nabla_{\overline{z}}\cdot\nabla_z} b_-(\overline{z},z) \\ &= (\pi\mathrm{i})^{-d}\int \mathrm{e}^{-2(\overline{z}-\overline{z}_1)\cdot(z-z_1)} b_-(\overline{z}_1,z_1)\mathrm{d}\overline{z}_1\mathrm{d}z_1, \\ b(v) &= \mathrm{e}^{\frac{1}{4}\nabla_v^2} b_-(v) \\ &= \pi^{-d}\int \mathrm{e}^{-(v-v_1)^2} b_-(v_1)\mathrm{d}v_1. \end{aligned}$$

Proof Let $b_1, b_2 \in \mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}})$, $b_+(\overline{z},z) = b_1(\overline{z})\overline{b}_2(z)$. We have

$$\begin{aligned} \mathrm{Op}^{a^*,a}(b_+) &= a^*(b_1)a(b_2) \\ &= \mathrm{Op}(b_1)\mathrm{Op}(\overline{b}_2) = \mathrm{Op}(b). \end{aligned}$$

Using the formula for the product of two Weyl–Wigner quantized operators, we obtain

$$\begin{aligned} b(\overline{z},z) &= \mathrm{e}^{\frac{\mathrm{i}}{2}(\nabla_{\overline{z}_1},\nabla_{z_1})\cdot\omega(\nabla_{\overline{z}_2},\nabla_{z_2})} b_1(\overline{z}_1)\overline{b}_2(z_2)\big|_{(\overline{z},z)=(\overline{z}_1,z_2)} \\ &= \mathrm{e}^{-\frac{1}{2}(\nabla_{\overline{z}_1}\cdot\nabla_{z_2}-\nabla_{\overline{z}_2}\cdot\nabla_{z_1})} b_1(\overline{z}_1)\overline{b}_2(z_2)\big|_{(\overline{z},z)=(\overline{z}_1,z_2)} \\ &= \mathrm{e}^{-\frac{1}{2}\nabla_{\overline{z}}\cdot\nabla_z} b_1(\overline{z})\overline{b}_2(z), \end{aligned}$$

where in the second line we use the definition (9.6) of the symplectic form ω. This proves the first formula of (1). The second follows from the first, using the identities of Subsect. 4.1.9. (2) follows from (1) and Thm. 9.34 (3). □

9.4.3 Wick and anti-Wick quantization as covariant and contravariant quantization

For $z \in \mathcal{Z}$, we consider the Gaussian coherent vectors Ω_z and the corresponding projections P_z in $\Gamma_{\mathrm{s}}(\mathcal{Z})$, defined in Def. 9.9. We will show that the Wick, resp. anti-Wick quantizations coincide with the covariant, resp. contravariant quantization for Gaussian coherent vectors.

Theorem 9.38 (1) *Let $B \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$. Then for all $z \in \mathcal{Z}$, $\Omega_z \in \mathrm{Dom}\, B$ and*

$$\mathrm{s}_B^{a^*,a}(\overline{z},z) = (\Omega_z|B\Omega_z), \quad z \in \mathcal{Z}. \tag{9.41}$$

(2) *Let $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$. Let the dimension of $\mathcal{Z}$ be finite. Then*

$$\mathrm{Op}^{a,a^*}(b) = (2\pi\mathrm{i})^{-d}\int b(z,\overline{z})P_z\mathrm{d}z\mathrm{d}\overline{z}. \tag{9.42}$$

(The integral should be understood in terms of a sesquilinear form on an appropriate domain.)

Proof Let $b_1, b_2 \in \mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}})$. Set

$$b(\overline{z},z) := b_1(\overline{z})\overline{b_2(\overline{z})} \in \mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}}\oplus\mathcal{Z}).$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Then

$$
\begin{aligned}
(\Omega_z|\mathrm{Op}^{a^*,a}(b)\Omega_z) &= (\Omega_z|a^*(b_1)a(b_2)\Omega_z)\\
&= (\Omega|W(\mathrm{i}z,-\mathrm{i}\overline{z})a^*(b_1)a(b_2)W(-\mathrm{i}z,\mathrm{i}\overline{z})\Omega)\\
&= \Big(\Omega|(a^*(b_1)+b_1(\overline{z}))(a(b_2)+\overline{b_2(\overline{z})})\Omega\Big)\\
&= b_1(\overline{z})\overline{b_2(\overline{z})} \quad = \quad b(\overline{z},z).
\end{aligned}
$$

This proves (9.41). Next, we compute

$$
\begin{aligned}
\mathrm{Op}^{a,a^*}(b) &= a(b_2)a^*(b_1)\\
&= (2\pi\mathrm{i})^{-d}\int a(b_2)P_z a^*(b_1)\mathrm{d}\overline{z}\mathrm{d}z\\
&= (2\pi\mathrm{i})^{-d}\int W(\mathrm{i}z-\mathrm{i}\overline{z})(a(b_2)+\overline{b_2(\overline{z})})P_0(a^*(b_1)\\
&\quad\ + b_1(\overline{z}))W(-\mathrm{i}z+\mathrm{i}\overline{z})\mathrm{d}\overline{z}\mathrm{d}z\\
&= (2\pi\mathrm{i})^{-d}\int \overline{b_2(\overline{z})}b_1(\overline{z})P_z\mathrm{d}\overline{z}\mathrm{d}z = (2\pi\mathrm{i})^{-d}\int b(\overline{z},z)P_z\mathrm{d}\overline{z}\mathrm{d}z.
\end{aligned}
$$

This proves (9.42). □

Remark 9.39 *Thm. 9.38 (1) says that the Wick symbol coincides with the covariant symbol defined with the help of Gaussian coherent states. Thus, using the notation of Sect. 8.5, (9.41) can be denoted* $\mathrm{s}_B^{\mathrm{cv}}(\overline{z}+z)$*. (Strictly speaking, however, operators in* $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$ *are usually unbounded, so they do not belong to the class considered in Sect. 8.5.)*

Thm. 9.38 (2) says that the anti-Wick quantization coincides with the contravariant quantization for Gaussian coherent states. Thus, using the notation of Sect. 8.5, (9.42) can be denoted $\mathrm{Op}^{\mathrm{ct}}(b)$*. (Strictly speaking, however, functions in* $\mathbb{C}\mathrm{Pol}(\mathcal{Y}^{\#})$ *usually do not belong to* $\mathrm{Meas}^1(\mathcal{Y}^{\#})+L^\infty(\mathcal{Y}^{\#})$*, so they do not belong to the class considered in Sect. 8.5.)*

9.4.4 Wick symbols on Fock spaces

So far, we have defined the Wick symbol only for operators in $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$. In this case, it is a polynomial on $\mathrm{Re}(\overline{\mathcal{Z}}\oplus\mathcal{Z})$.

We will now extend the definition of the Wick symbol to a rather large class of quadratic forms on $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

Definition 9.40 *Let B be a quadratic form on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ such that Ω_z belongs to its domain for any $z\in\mathcal{Z}$. We define the* Wick symbol of B *as*

$$
s_B^{a^*,a}(\overline{z},z) := (\Omega_z|B\Omega_z). \tag{9.43}
$$

By Thm. 9.38 (1), the above definition of the Wick symbol agrees with Def. 9.32 for $B\in\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$. In (9.43), the Wick symbol is viewed as a function

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

on $\mathcal{Y}^{\#}=\mathrm{Re}(\overline{\mathcal{Z}}\oplus\mathcal{Z})$. An alternative point of view on the Wick symbol uses holomorphic functions on $\overline{\mathcal{Z}}\oplus\mathcal{Z}$.

Proposition 9.41 *The holomorphic extension of (9.43) to $\overline{\mathcal{Z}}\oplus\mathcal{Z}$ (see Def. 5.81) is*

$$\mathrm{s}_B^{a^*,a}(\overline{z_1},z_2)=\mathrm{e}^{-\overline{z}_1\cdot z_2+\frac{1}{2}\overline{z}_1\cdot z_1+\frac{1}{2}\overline{z}_2\cdot z_2}(\Omega_{z_1}|B\Omega_{z_2}).$$

Proposition 9.42 *Let B be a positive closed quadratic form such that $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})\subset$ $\mathrm{Dom}\,B$ and for each $z\in\mathcal{Z}$ the series*

$$\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!}}(z^{\otimes n}|Bz^{\otimes m})\frac{1}{\sqrt{m!}}$$

is absolutely convergent. Then the Wick symbol of B and its holomorphic extension are

$$s_B^{a^*,a}(\overline{z},z)=\mathrm{e}^{-\overline{z}\cdot z}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!}}(z^{\otimes n}|Bz^{\otimes m})\frac{1}{\sqrt{m!}},\tag{9.44}$$

$$\mathrm{s}_B^{a^*,a}(\overline{z_1},z_2)=\mathrm{e}^{-\overline{z_1}\cdot z_2}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!}}(z_1^{\otimes n}|Bz_2^{\otimes m})\frac{1}{\sqrt{m!}}.\tag{9.45}$$

Proof Recalling that

$$\Omega_z=\mathrm{e}^{-\frac{1}{2}\overline{z}\cdot z}\sum_{n=0}^{\infty}\frac{z^{\otimes n}}{\sqrt{n!}},$$

and using that B is closed, we see that $\Omega_z\in\mathrm{Dom}\,B$ and $(\Omega_z|B\Omega_z)$ is given by the convergent series in (9.44). Applying the Cauchy–Schwarz inequality, we obtain that the series in the r.h.s. of (9.45) is absolutely convergent. Then we use Prop. 9.41. □

In the following proposition we compute the Wick symbol of various operators in the sense of Def. 9.40:

Proposition 9.43 (1) *For $h\in B(\mathcal{Z})$, we have $\mathrm{s}_{\mathrm{d}\Gamma(h)}^{a^*,a}(\overline{z},z)=\overline{z}\cdot hz$.*

(2) *If p is a contraction on $\mathcal{Z}$, we have $\mathrm{s}_{\Gamma(p)}^{a^*,a}(\overline{z},z)=\mathrm{e}^{-\overline{z}\cdot z+\overline{z}\cdot pz}$.*

Example 9.44 *The anti-Wick, Weyl–Wigner and Wick symbols of $P_0=|\Omega)(\Omega|$ (the projection onto Ω) are given below (compare with Examples 4.42 and 8.74):*

$$\begin{aligned}\mathrm{s}_{P_0}^{a,a^*}(\overline{z},z)&=(2\pi)^d\delta_0,\\ \mathrm{s}_{P_0}(\overline{z},z)&=2^d\mathrm{e}^{-2\overline{z}\cdot z},\\ \mathrm{s}_{P_0}^{a^*,a}(\overline{z},z)&=\mathrm{e}^{-\overline{z}\cdot z}.\end{aligned}$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

9.4.5 Wick quantization: the operator formalism

Recall from Subsect. 8.5.3 that in general it is easier to find the covariant symbol of an operator than to compute the covariant quantization of a symbol. This remark applies to the Wick quantization. In this subsection we will describe this more difficult direction.

It is convenient to represent Wick symbols as operators acting on the Fock space. We need, however, to restrict ourselves to a rather small class of such operators.

Recall that N is the number operator and $\mathbb{1}_{\{n\}}(N)$ is the orthogonal projection from $\Gamma_{\mathrm{s}}(\mathcal{Z})$ onto $\Gamma_{\mathrm{s}}^{n}(\mathcal{Z})$.

Definition 9.45 *For $b \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, set $b_{n,m} := \mathbb{1}_{\{n\}}(N) b \mathbb{1}_{\{m\}}(N)$. Let*

$$\begin{aligned}&B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\\ &:= \big\{b \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big) \ : \ \textit{there exists } \ n_0 \ \textit{ such that } \ b_{n,m} = 0 \ \textit{ for } \ n, m > n_0\big\}.\end{aligned}$$

Definition 9.46 *Let $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. Then we define its* Wick quantization, *denoted by $\mathrm{Op}^{a^*,a}(b)$, as the quadratic form on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ defined for $\Phi, \Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ as*

$$\begin{aligned}(\Phi|\mathrm{Op}^{a^*,a}(b)\Psi) &= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\min(m,n)} \frac{\sqrt{n!m!}}{k!} (\Phi| b_{n-k,m-k} \otimes \mathbb{1}_{\mathcal{Z}}^{\otimes k} \Psi),\\ &= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{(n+k)!(m+k)!}}{k!} (\Phi| b_{n,m} \otimes \mathbb{1}_{\mathcal{Z}}^{\otimes k} \Psi).\end{aligned}$$

The above definition is essentially an extension of Def. 9.31.

Proposition 9.47 *Let $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. Set $B = \mathrm{Op}^{a^*,a}(b)$, with the Wick quantization defined as in Def. 9.46. Then the Wick symbol of B in the sense of Def. 9.40 and its holomorphic extension are*

$$\mathrm{s}_B^{a^*,a}(\overline{z}, z) = \sum_{n,m=0}^{\infty} (z^{\otimes n} | b z^{\otimes m}), \tag{9.46}$$

$$\mathrm{s}_B^{a^*,a}(\overline{z}_1, z_2) = \sum_{n,m=0}^{\infty} (z_1^{\otimes n} | b z_2^{\otimes m}). \tag{9.47}$$

Consequently, if $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#}) \simeq \mathrm{Pol}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ is identified with $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ with the help of (9.46) or (9.47), then Def. 9.31 coincides with Def. 9.40.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Proof B clearly satisfies the hypotheses of Def. 9.40, since $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$. Using (9.44), we obtain

$$\begin{aligned}
\mathrm{s}_B^{a^*,a}(\overline{z},z)\mathrm{e}^{\overline{z}\cdot z} &= \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}}(z^{\otimes n}|Bz^{\otimes m})\frac{1}{\sqrt{m!}} \\
&= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\min(n,m)} \frac{1}{k!}(z^{\otimes n}|b_{n-k,m-k}\otimes 1\!\!1_{\mathcal{Z}}^{\otimes k}\, z^{\otimes m}) \\
&= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\min(n,m)} \frac{1}{k!}(\overline{z}\cdot z)^k(z^{\otimes(n-k)}|b_{n-k,m-k}z^{\otimes(m-k)}) \\
&= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\infty}\frac{1}{k!}(\overline{z}\cdot z)^k(z^{\otimes n}|b_{n,m}z^{\otimes m}) = \sum_{n,m=0}^{\infty}(z^{\otimes n}|b_{n,m}z^{\otimes m})\mathrm{e}^{\overline{z}\cdot z}.
\end{aligned}$$

$\square$

In the following identities it is convenient to use the new, more general definition of the Wick quantization:

Proposition 9.48 *In the following identities* $b \in B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, $h \in B(\mathcal{Z}) \subset B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, $p \in B(\mathcal{Z}_1,\mathcal{Z}_2)$.

$$\begin{aligned}
\mathrm{Op}^{a^*,a}(h) &= \mathrm{d}\Gamma(h); \\
[\mathrm{d}\Gamma(h),\mathrm{Op}^{a^*,a}(b)] &= \mathrm{Op}^{a^*,a}([\mathrm{d}\Gamma(h),b]); \\
\Gamma(p)\mathrm{Op}^{a^*,a}(b\Gamma(p)) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b)\Gamma(p); \\
\Gamma(p)\mathrm{Op}^{a^*,a}(b) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b\Gamma(p^*))\,\Gamma(p), \qquad \textit{if } p \textit{ is isometric}; \\
\Gamma(p)\mathrm{Op}^{a^*,a}(b)\Gamma(p^*) &= \mathrm{Op}^{a^*,a}(\Gamma(p)b\Gamma(p^*)), \qquad \textit{if } p \textit{ is unitary.}
\end{aligned}$$

The following proposition describes the special class of particle preserving operators:

Proposition 9.49 *If* $b \in B\big(\Gamma_{\mathrm{s}}^m(\mathcal{Z})\big)$, *then*

$$\begin{aligned}
\tfrac{1}{m!}\big(\Phi|\mathrm{Op}^{a^*,a}(b)\Psi\big) &= \sum_{k=1}^{\infty}\tfrac{(m+k)!}{m!k!}(\Phi|b\otimes 1\!\!1_{\mathcal{Z}}^{k}\Psi) \\
&= \sum_{k=1}^{\infty}\sum_{1\le i_1<\cdots<i_m\le m+k}(\Phi|b_{i_1,\dots,i_m}^{m+k}\Psi).
\end{aligned}$$

The operators $b_{i_1,\dots,i_m}^{m+k} \in B\big(\Gamma_{\mathrm{s}}^{m+k}(\mathcal{Z})\big)$ *are defined as follows:*

$$b_{i_1,\dots,i_m}^{m+k} := \Theta(\sigma)\; b\otimes 1\!\!1_{\mathcal{Z}}^{\otimes k}\;\Theta(\sigma)^{-1} \in B\big(\Gamma_{\mathrm{s}}^{m+k}(\mathcal{Z})\big),$$

where $\sigma \in S_n$ *is any permutation that transforms* $(1,\dots,m)$ *onto* $(i_1,\dots,i_m)$. *Thus* $b_{i_1,\dots,i_m}^{m+k}$ *is the "m-body interaction" acting on the* i_1*-th,.. through* i_m*-th particles.*

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

9.4.6 Estimates on Wick polynomials

Let $b \in B\big(\Gamma_{\mathrm{s}}^q(\mathcal{Z}), \Gamma_{\mathrm{s}}^p(\mathcal{Z})\big) \subset B^{\mathrm{fin}}\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ for $p, q \in \mathbb{N}$. The following estimates are known as N_τ *estimates.*

Proposition 9.50 *Let $m > 0$ be a self-adjoint operator on $\mathcal{Z}$. Then for all Ψ_1, $\Psi_2 \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ one has*

$$\begin{aligned}&\Big|\Big(\mathrm{d}\Gamma(m)^{-p/2}\Psi_1|\mathrm{Op}^{a^*,a}(b)\mathrm{d}\Gamma(m)^{-q/2}\Psi_2\Big)\Big| \\ &\leq \|\Gamma(m)^{-\frac{1}{2}}b\Gamma(m)^{-\frac{1}{2}}\|\|\Psi_1\|\|\Psi_2\|.\end{aligned} \tag{9.48}$$

In particular, $\mathrm{Op}^{a^*,a}(b)$ *extends to an operator on* $\Gamma_{\mathrm{s}}(\mathcal{Z})$ *with domain* $\mathrm{Dom}\, N^{(p+q)/2}$.

Proof Noting that $N\mathrm{Op}^{a^*,a}(b) = \mathrm{Op}^{a^*,a}(b)(N+p-q)$, we see that the second statement follows from the first for $m = \mathbb{1}$.

To prove the first statement, we will assume for simplicity that $\mathcal{Z}$ is separable (the non-separable case can be treated by the same arguments, replacing sequences by nets). It clearly suffices to prove (9.48) for Ψ_1, Ψ_2 such that $\Psi_i = \Gamma(\pi)\Psi_i$, where π is a finite rank projection. Moreover, if (π_n) is an increasing sequence of orthogonal projections with $\mathrm{s}-\lim \pi_n = \mathbb{1}$, and if $b_n = \Gamma(\pi_n)b\Gamma(\pi_n)$, it suffices to prove (9.48) for $\mathrm{Op}^{a^*,a}(b_n)$. Therefore, we may assume that $\mathcal{Z}$ is finite-dimensional. Let $(e_1, \dots, e_n)$ be an o.n. basis of eigenvectors for m and $m_k = (e_k|me_k)$. For $\vec{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we define $e_{\vec{k}}$ as in Subsect. 3.3.5. We set

$$f_{\vec{k}} := \frac{\sqrt{|k|!}}{\sqrt{\vec{k}!}}e_{\vec{k}}.$$

Let us consider the operator

$$A : \begin{array}{l}\Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \mathcal{Z}, \\ \Psi \mapsto \sum_{i=1}^n a(e_i)\Psi \otimes e_i,\end{array}$$

and define by induction

$$A_q : \begin{array}{l}\Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \otimes_{\mathrm{s}}^q \mathcal{Z}, \\ A_q := \Big(A \otimes \mathbb{1}_{\otimes_{\mathrm{s}}^{q-1}\mathcal{Z}}\Big)A_{q-1}.\end{array}$$

It is easy to verify that

$$A_q\Psi = \sum_{|\vec{l}|=q} \frac{|\vec{l}|!}{\vec{l}!}a(e_{\vec{l}})\Psi \otimes e_{\vec{l}} = \sum_{|\vec{l}|=q} a(f_{\vec{l}})\Psi \otimes f_{\vec{l}}. \tag{9.49}$$

Since $\{f_{\vec{l}}\}_{|\vec{l}|=q}$ is an o.n. basis of $\otimes_{\mathrm{s}}^q\mathcal{Z}$, we have

$$b = \sum_{|\vec{k}|=p, |\vec{l}|=q} b_{\vec{k},\vec{l}}|f_{\vec{k}})(f_{\vec{l}}|, \ b_{\vec{k},\vec{l}} = (f_{\vec{k}}|bf_{\vec{l}}),$$

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

and hence

$$\mathrm{Op}^{a^*,a}(b) = \sum_{|\vec{k}|=p,|\vec{l}|=q} b_{\vec{k},\vec{l}}\, a^*\big(|f_{\vec{k}})\big)a\big((f_{\vec{l}}|\big). \tag{9.50}$$

From (9.49) and (9.50), we get that

$$\mathrm{Op}^{a^*,a}(b) = A_p^* \left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes b\right) A_q. \tag{9.51}$$

Inserting factors of $\Gamma(m)^{\frac12}$, we see that (9.48) follows if we prove that

$$\|1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes \Gamma(m)^{\frac12} A_p \mathrm{d}\Gamma(m)^{-q/2}\| \leq 1. \tag{9.52}$$

To prove (9.52), we note that, first for $\alpha = 1$ and then for any $\alpha \in \mathbb{R}$, one has

$$A\mathrm{d}\Gamma(m)^\alpha = \left(\mathrm{d}\Gamma(m) \otimes 1\!\!1_{\mathcal{Z}} + 1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes m\right)^\alpha A. \tag{9.53}$$

Applying (9.53) for $\alpha = -\frac12$, we obtain by induction on q that

$$\begin{aligned}
&A_q \mathrm{d}\Gamma(m)^{-q/2} \\
&= \left(A \otimes 1\!\!1_{\otimes_{\mathrm{s}}^{q-1}\mathcal{Z}}\right) \left(\mathrm{d}\Gamma(m) \otimes 1\!\!1_{\otimes_{\mathrm{s}}^{q-1}\mathcal{Z}} + 1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes \mathrm{d}\Gamma^q(m)\right)^{-\frac12} A_{q-1}\mathrm{d}\Gamma(m)^{-(q-1)/2},
\end{aligned}$$

and hence

$$\begin{aligned}
&\left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes \Gamma^q(m)^{\frac12}\right) A_q \mathrm{d}\Gamma(m)^{-q/2} \\
&= \left(\left(\left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes m^{\frac12}\right)A\right) \otimes 1\!\!1_{\otimes_{\mathrm{s}}^{q-1}\mathcal{Z}}\right) \left(\mathrm{d}\Gamma(m) \otimes 1\!\!1_{\otimes_{\mathrm{s}}^{q-1}\mathcal{Z}} + 1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes \mathrm{d}\Gamma^{q-1}(m)\right)^{-\frac12} \\
&\quad \times \left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes \Gamma^{q-1}(m)^{\frac12}\right) A_{q-1}\mathrm{d}\Gamma(m)^{-(q-1)/2}.
\end{aligned} \tag{9.54}$$

As a special case of (9.51), we have

$$\mathrm{d}\Gamma(b) = A^* \left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes m\right) A,$$

which implies that

$$\left\|\left(1\!\!1_{\Gamma_{\mathrm{s}}(\mathcal{Z})} \otimes m^{\frac12}\right) A \mathrm{d}\Gamma(m)^{-\frac12}\right\| \leq 1.$$

Clearly, this implies that the first factor in the r.h.s. of (9.54) has norm less than 1, which implies (9.52). □

9.4.7 Bargmann kernel of an operator

Recall that in Def. 9.12 for any $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ we defined its complex-wave transform $T^{\mathrm{cw}}\Psi \in \mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. In the context of the complex-wave transformation one sometimes introduces the so-called Bargmann kernel of an operator, which can be used as an alternative to its distributional kernel, and also to its Wick symbol.

For simplicity, in (2) and (3) of Prop. 9.52 below we assume that the dimension of $\mathcal{Z}$ is finite.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Definition 9.51 *Let* $B \in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$*. We define the* Bargmann *or* complex-wave kernel of B *as*

$$\overline{\mathcal{Z}} \oplus \mathcal{Z} \ni (\overline{z_1}, z_2) \mapsto B^{\mathrm{Bar}}(\overline{z_1}, z_2) := \sum_{n,m=0}^{\infty} (z_1^{\otimes n} \frac{1}{\sqrt{n!}} | B \frac{1}{\sqrt{m!}} z_2^{\otimes m})$$
$$= (\mathrm{e}^{a^*(z_1)}\Omega | B \mathrm{e}^{a^*(z_2)}\Omega).$$

Proposition 9.52 (1) *The relationship between the Bargmann kernel and the Wick symbol of an operator B on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is given by the following identity:*

$$B^{\mathrm{Bar}}(\overline{z_1}, z_2) = \mathrm{e}^{\overline{z_1}\cdot z_2} s_B^{a^*,a}(\overline{z_1}, z_2) = \mathrm{e}^{\frac{1}{2}\overline{z}_1\cdot z_1 + \frac{1}{2}\overline{z}_2\cdot z_2}(\Omega_{z_1} | B \Omega_{z_2}).$$

(2) *Let $B \in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$, $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. Then one has*

$$(T^{\mathrm{cw}} B\Psi)(\overline{z}_1) = (2\pi\mathrm{i})^{-d} \int B^{\mathrm{Bar}}(\overline{z}_1, z_2) T^{\mathrm{cw}}\Psi(z_2) \mathrm{e}^{-\overline{z}_2\cdot z_2} \mathrm{d}\overline{z}_2 \mathrm{d}z_2. \tag{9.55}$$

(3) *Let $B_1, B_2 \in B^{\mathrm{fin}}(\Gamma_{\mathrm{s}}(\mathcal{Z}))$. Then*

$$(B_1 B_2)^{\mathrm{Bar}}(\overline{z}_1, z_2) = (2\pi\mathrm{i})^{-d} \int B_1^{\mathrm{Bar}}(\overline{z}_1, z_0) B_2^{\mathrm{Bar}}(\overline{z}_0, z_2) \mathrm{e}^{-\overline{z}_0\cdot z_0} \mathrm{d}\overline{z}_0 \mathrm{d}z_0. \tag{9.56}$$

Proof (1) is obvious. To prove (2) and (3) we use

$$\mathbb{1} = (2\pi\mathrm{i})^{-d} \int P_z \mathrm{d}\overline{z}\mathrm{d}z. \tag{9.57}$$

We obtain

$$(\Omega_{z_1} | B\Psi) = (2\pi\mathrm{i})^{-d} \int (\Omega_{z_1} | B\Omega_{z_2})(\Omega_{z_2} | \Psi) \mathrm{d}z_2 \mathrm{d}\overline{z_2}, \tag{9.58}$$

$$(\Omega_{z_1} | B_1 B_2 \Omega_{z_2}) = (2\pi\mathrm{i})^{-d} \int (\Omega_{z_1} | B_1 \Omega_{z_0})(\Omega_{z_0} | B_2 \Omega_{z_2}) \mathrm{d}z_0 \mathrm{d}\overline{z_0}. \tag{9.59}$$

Now (9.58) implies (2) and (9.59) implies (3). □

9.4.8 Link between the two Wick operations

In this subsection we use the conventions of Subsect. 9.3.1. In particular, we consider a real Hilbert space $\mathcal{X}$ equipped with a positive operator c. We consider the Kähler space with involution $(2c)^{-\frac{1}{2}}\mathcal{X} \oplus (2c)^{\frac{1}{2}}\mathcal{X}$ equipped with the Kähler anti-involution $\mathrm{j} = \begin{bmatrix} 0 & -(2c)^{-1} \\ 2c & 0 \end{bmatrix}$ (see (9.22)). Recall that the Wick transformation w.r.t. the covariance c is given by

$$:G(x): = \mathrm{e}^{-\frac{1}{2}\nabla_x\cdot c\nabla_x} G(x).$$

The following proposition explains the link between the Wick transformation on functions on $\mathcal{X}$ and the Wick ordering of operators.

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press

Proposition 9.53 *Let $F \in \mathbb{C}\mathrm{Pol_s}(\mathcal{X})$. Then*

$$\mathrm{Op}^{a^*,a}(F) = :F(x_{\mathrm{rw}}):,$$

where on the r.h.s. we use the functional calculus, as explained in Remark 8.27.

Proof From Thm. 9.37, we have $\mathrm{Op}^{a^*,a}(F) = \mathrm{Op}(\mathrm{e}^{-\frac{1}{4}\nabla_v^2}F)$. Setting $\nabla_v = (\nabla_x, \nabla_\xi)$, we have

$$\nabla_v^2 = \nabla_x \cdot 2c\nabla_x + \nabla_\xi \cdot (2c)^{-1}\nabla_\xi.$$

Thus, on a function that depends only on x, we have

$$\mathrm{e}^{-\frac{1}{4}\nabla_v^2}F(x) = \mathrm{e}^{-\frac{1}{2}\nabla_x\cdot c\nabla_x}F(x) = :F(x):.$$

Furthermore, for such functions the Weyl–Wigner quantization coincides with the functional calculus. □

It is often convenient to use multiplication operators expressed as $:F(x_{\mathrm{rw}}):$, as explained in Prop. 9.53. In particular, let $w \in \mathbb{C}\mathcal{X}$. Recall that

$$w \cdot x_{\mathrm{rw}} = a^*(w) + a(\overline{w}).$$

For later use let us note the identity

$$:(w \cdot x_{\mathrm{rw}})^p: = \sum_{r=0}^{p} \binom{p}{r} a^*(w)^r a(\overline{w})^{p-r}. \tag{9.60}$$

9.5 Notes

The essential self-adjointness of bosonic field operators was established by Cook (1953).

A modern exposition of the mathematical formalism of second quantization can also be found e.g. in Glimm–Jaffe (1987) and Baez–Segal–Zhou (1991).

The complex-wave representation goes back to the work of Bargmann (1961) and Segal (1963). Therefore, it is often called the Bargmann or Bargmann–Segal representation. The name "complex-wave representation" was coined by Segal (1978); see also Baez–Segal–Zhou (1991).

The name "real-wave representation" also comes from Baez–Segal–Zhou (1991). The properties of second quantized operators in the real-wave representation were first established by Nelson (1973). The proof of Prop. 9.29 (1) follows Nelson (1973), and that of Prop. 9.29 (2) follows Simon (1974). Nelson's hyper-contractivity theorem, Thm. 9.30, is proven in Nelson (1973).

The Wick theorem goes back to a paper of Wick (1950) about the evaluation of the S-matrix.

The "N_τ estimates" were used in constructive quantum field theory and are due to Glimm–Jaffe (1985).

Wick quantization in the context of particle preserving Hamiltonians is used, for example, in Dereziński (1998).

https://doi.org/10.1017/9781009290876.010 Published online by Cambridge University Press