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A CENTRAL LIMIT THEOREM FOR REVERSIBLE PROCESSES WITH NONLINEAR GROWTH OF VARIANCE

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Abstract

Kipnis and Varadhan (1986) showed that, for an additive functional, S_n say, of a reversible Markov chain, the condition $\mathrm{E}[S_n^2]/n \to \kappa \in (0, \infty)$ implies the convergence of the conditional distribution of $S_n/\sqrt{\mathrm{E}[S_n^2]}$, given the starting point, to the standard normal distribution. We revisit this question under the weaker condition, $\mathrm{E}[S_n^2] = n\ell(n)$, where ℓ is a slowly varying function. It is shown by example that the conditional distributions of $S_n/\sqrt{\mathrm{E}[S_n^2]}$ need not converge to the standard normal distribution in this case; and sufficient conditions for convergence to a (possibly nonstandard) normal distribution are developed.

Keywords: Conditional distribution; Markov chain; self-adjoint operator; slowly varying function.

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1. Introduction

Consider a reversible Markov chain $\ldots, W_{-1}, W_0, W_1, \ldots$, defined on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$, with a Polish state space $\mathcal{W}$, transition function Q, and marginal distribution π. Thus, $\pi\{B\} = \mathrm{P}[W_n \in B]$, $Q(w; B) = \mathrm{P}[W_{n+1} \in B \mid W_n = w]$, and (the reversibility condition)

$$\int_A Q(w; B)\pi\{\mathrm{d}w\} = \int_B Q(w; A)\pi\{\mathrm{d}w\} \tag{1}$$

for Borel sets $A, B \subseteq \mathcal{W}$, $w \in \mathcal{W}$, and $n \in \mathbb{Z}$. Using (and abusing) notation in a standard manner, we write

$$Qf(w) = \int_{\mathcal{W}} f(z)Q(w; \mathrm{d}z) \quad \pi\text{-almost everywhere}$$

for $f \in L^1(\pi)$ and $Q^k = Q \circ \cdots \circ Q$ for the iterates of Q. In addition, let $L_0^p(\pi) = \{f \in L^p(\pi) : \int_{\mathcal{W}} f \,\mathrm{d}\pi = 0\}$,

$$V_n = I + Q + \cdots + Q^{n-1}, \quad \text{and} \quad \bar{V}_n = \frac{1}{n}(V_1 + \cdots + V_n),$$

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and let $\|\cdot\|$ denote the norm in an L^2 space, either $L^2(\pi)$ or $L^2(\mathrm{P})$. Finally, '$\xrightarrow{\mathrm{D}}$' denotes convergence in distribution and '$\xrightarrow{\mathrm{P}}$' denotes convergence in probability of conditional distributions; that is, if the $Z_n : \Omega \to \mathbb{R}$ are random variables and G is a distribution function, then $Z_n \mid W_0 \xrightarrow{\mathrm{P}} G$ means that the conditional distribution of Z_n given W_0 converges in probability to G.

The reversibility condition (1) is equivalent to requiring that (W_0, W_1) and (W_1, W_0) have the same distribution, since the left-hand side of (1) is $\mathrm{P}[W_0 \in A,\ W_1 \in B]$ and the right-hand side is $\mathrm{P}[W_0 \in B,\ W_1 \in A]$. An important consequence (also equivalent) is that the restriction of Q to $L^2(\pi)$ is a self-adjoint operator. For $\langle f, g\rangle = \int_{\mathcal{W}} fg\,\mathrm{d}\pi$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\pi)$, $\langle f, Qg\rangle = \mathrm{E}[f(W_0)g(W_1)] = \mathrm{E}[f(W_1)g(W_0)] = \langle Qf, g\rangle$ for all $f, g \in L^2(\pi)$.

Given $g \in L_0^2(\pi)$, let $X_k = g(W_k)$, $S_n = X_1 + \cdots + X_n$, and $\sigma_n^2 = \mathrm{E}[S_n^2]$. Kipnis and Varadhan [7] showed that if

$$\lim_{n\to\infty} \frac{\sigma_n^2}{n} = \kappa \in [0, \infty) \tag{2}$$

then the conditional distribution of $S_n/\sqrt{n}$ given W_0 converges in probability to the normal distribution with mean 0 and variance κ. We show in Proposition 1 that $\kappa > 0$ except for trivial special cases; then $\sigma_n^{-1} S_n \mid W_0 \xrightarrow{\mathrm{P}} \mathrm{Normal}[0, 1]$. Kipnis and Varadhan showed that S_n could be written in the form $S_n = M_n + R_n$, where $0 = M_0, M_1, M_2, \ldots$ is a square integrable martingale with (strictly) stationary increments $D_k = M_k - M_{k-1}$ and $\|R_n\| = o(\sqrt{n})$. The result has applications to Monte Carlo Markov chains (see, for instance, [12]), since many algorithms lead to reversible chains, and, to interacting particle systems [6], [7].

Here we consider the case in which (2) is weakened to

$$\sigma_n^2 = n\ell(n), \tag{3}$$

where ℓ is a slowly varying function, as defined in Chapter 1 of [2]. An example will show that the main result from [7] does not extend completely. Some features do extend, however. For the remainder of the paper, reversibility is assumed along with $g \in L_0^2(\pi)$, and ℓ is defined by (3).

Further developments under condition (2) may be found in [3], and [10] is a recent article on asymptotic normality of sums of stationary processes with nonlinear growth of variance.

2. Generalities

In the first proposition below, we show that the case $\lim_{n\to\infty} \ell(n) = \infty$ only needs to be considered. The relation

$$\sigma_n^2 = [2\langle g, \bar{V}_n g\rangle - \|g\|^2]n \tag{4}$$

is used in its proof.

Proposition 1. *If* $\liminf_{n\to\infty} \ell(n) < \infty$ *then (2) holds, and if* $\liminf_{n\to\infty} \ell(n) = 0$ *then* $S_n = \frac{1}{2}[1 + (-1)^{n-1}]X_1$ *with probability* 1.

Proof. Since Q is self-adjoint, we may write $Q = \int_\Lambda \lambda\,\mathrm{d}M(\lambda)$, where $\Lambda \subseteq [-1, 1]$ is the spectrum of Q and M is a countably additive, projection-valued set function defined on the Borel sets of Λ. Then $Q^k = \int_\Lambda \lambda^k\,\mathrm{d}M(\lambda)$ for all $k \geq 1$. See [5, Chapter 2]. Let $\mu_g(B) = \langle g, M(B)g\rangle$. Then μ_g is a measure, and

$$\langle g, \bar{V}_n g\rangle = \int_\Lambda \left(1 - \frac{\lambda}{n}\frac{1-\lambda^n}{1-\lambda}\right)\frac{\mu_g(\mathrm{d}\lambda)}{1-\lambda}. \tag{5}$$

Observe that the integrand on the right-hand side of (5) is nonnegative. So, if $\liminf_{n\to\infty} \ell(n) < \infty$ then the limit inferior of the left-hand side of (5) is finite and, therefore,

$$\int_{\Lambda} \frac{\mu_g(\mathrm{d}\lambda)}{1-\lambda} < \infty \tag{6}$$

by Fatou's lemma. It is clear the integrands on the right-hand side of (5) are dominated by an integrable function; hence, the integral converges to that on the left-hand side of (6), and (2) holds with

$$\kappa = 2\int_{\Lambda} \frac{\mu_g(\mathrm{d}\lambda)}{1-\lambda} - \|g\|^2 = \int_{\Lambda} \frac{1+\lambda}{1-\lambda}\mu_g(\mathrm{d}\lambda).$$

If $\liminf_{n\to\infty} \ell(n) = 0$ then the last integral is 0 and, therefore, μ_g is a point mass at $\{-1\}$. It follows that $Qg = -g$, $\mathrm{E}[(X_0 + X_1)^2] = 0$, $X_n = (-1)X_{n-1}$ with probability 1, and $S_n = \frac{1}{2}[1 + (-1)^{n-1}]X_1$ with probability 1.

As a consequence, there is no loss of generality in supposing that $\ell(n) \to \infty$, which we do where convenient. For if $\liminf_{n\to\infty} \ell(n) < \infty$ then the Kipnis–Varadhan result is applicable.

The proof of the next proposition uses (4) and

$$\bar{V}_n = \sum_{k=0}^{n-1}\left(1 - \frac{k}{n}\right)Q^k. \tag{7}$$

Proposition 2. *If ℓ varies slowly in (3) then $\|V_n g\| = o(\sigma_n)$.*

Proof. Using the reversibility and (7),

$$\begin{aligned}
\|V_n g\|_2^2 &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\langle g, Q^{k+j} g\rangle \\
&= \sum_{i=0}^{n-1}(i+1)\langle g, Q^i g\rangle + \sum_{i=n}^{2n-2}(2n-1-i)\langle g, Q^i g\rangle \\
&= \sum_{i=0}^{2n-2}(2n-1-i)\langle g, Q^i g\rangle - 2\sum_{i=0}^{n-1}(n-1-i)\langle g, Q^i g\rangle \\
&= \tfrac{1}{2}[\sigma_{2n-1}^2 + (2n-1)\|g\|^2] - [\sigma_{n-1}^2 + (n-1)\|g\|^2] \\
&= \tfrac{1}{2}\sigma_{2n-1}^2 - \sigma_{n-1}^2 + \tfrac{1}{2}\|g\|^2.
\end{aligned}$$

The proposition then follows directly from (3) and the slow variation of ℓ.

Corollary 1. *If ℓ varies slowly then there is a sequence of square integrable martingales $0 = M_{n,1}, M_{n,2}, \dots$ with stationary increments $D_{n,k} = M_{n,k} - M_{n,k-1}$, $k \geq 1$, for which $\max_{k\leq n}\|S_k - M_{n,k}\| = o(\sigma_n)$.*

Proof. The result follows from Proposition 2 and Theorem 1 of [13]. It is relevant that

$$D_{n,k} = \bar{V}_n g(W_k) - Q\bar{V}_n g(W_{k-1})$$

and $M_{n,k} = D_{n,1} + \cdots + D_{n,k}$ in the proof of the latter result.

Corollary 2. *If ℓ varies slowly and there is a $\lambda \geq 0$ for which*

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathrm{E}[D_{n,k}^2 \mid W_{k-1}] \to^{\mathrm{P}} \lambda \tag{8}$$

and

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathrm{E}[D_{n,k}^2 \mathbf{1}_{\{|D_{n,k}|>\epsilon\sigma_n\}} \mid W_{k-1}] \to^{\mathrm{P}} 0 \tag{9}$$

for every $\epsilon > 0$, then

$$\left.\frac{S_n}{\sigma_n}\,\right|\, W_0 \xrightarrow{\mathrm{P}} \mathrm{Normal}[0, \lambda]. \tag{10}$$

Proof. The result follows from the martingale central limit theorem (see, for example, [1, pp. 475–478]) applied conditionally given $\mathcal{F}_0 := \sigma(\dots, W_{-1}, W_0)$. For $\lambda = 1$, the proof is detailed in [13], and the extension to $\lambda \neq 1$ presents no difficulty.

In the next proposition we write $S_n = S_n(g)$ and $\sigma_n = \sigma_n(g)$ to emphasize the dependence on g. We also use the following result.

Lemma 1. *If $Z_n \mid W_0 \xrightarrow{\mathrm{P}} G$ and $Z'_n - Z_n \to^{\mathrm{P}} 0$, then $Z'_n \mid W_0 \xrightarrow{\mathrm{P}} G$.*

Proof. The result follows from the unconditional version of Slutzky's theorem, by considering subsequences along which convergence in probability can be replaced by almost-sure convergence.

Proposition 3. *If $\ell(n) \to \infty$ and (10) holds for a given g, then, for any $j \geq 1$, $\sigma_n(Q^j g) \sim \sigma_n(g)$ and (10) holds with the same λ when g is replaced by $Q^j g$.*

Proof. It suffices to prove the result for $j = 1$. In this case, the proof follows from $S_n(g) - S_n(Qg) = \sum_{k=1}^{n}[g(W_k) - Qg(W_{k-1})] + Qg(W_0) - Qg(W_n)$, which implies that

$$\begin{aligned} |\sigma_n(g) - \sigma_n(Qg)| &\leq \|S_n(g) - S_n(Qg)\| \\ &\leq \|g(W_1) - Qg(W_0)\|\sqrt{n} + 2\|Qg(W_0)\| \\ &= o[\sigma_n(g)], \end{aligned}$$

and Lemma 1.

Remark 1. The proof of Proposition 3 does not use the reversibility and, therefore, is valid for any stationary process.

Remark 2. Proposition 3 illustrates an important difference between the cases $\ell(n) \to \infty$ and $\ell(n) \to \kappa$, considered in [7]. For if (2) holds then

$$\kappa = \kappa(g) = 2 \lim_{n\to\infty} \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) \langle g, Q^k g\rangle - \|g\|^2. \tag{11}$$

It is then not difficult to see that (11) holds when g is replaced by $Q^j g$, and that $[\kappa(g) + \cdots + \kappa(Q^n g)]/n$ approaches 0 as $n \to \infty$, by Theorem 2 of [14].

Remark 3. Kipnis and Varadhan [7] showed that if (2) holds then $D_{n,k}$ converges in $L^2(P)$ for every k. Clearly, this is impossible if $\ell(n) \to \infty$. However, if $D_{n,1}/\sqrt{\ell(n)}$ converged in $L^2(P)$, then (8) and (9) would follow easily with $\lambda = 1$, and the conditional distributions of S_n/σ_n would converge to the standard normal distribution, as noted in [13]. This hope cannot be realized either, however, if $\lim_{n\to\infty} \ell(n) = \infty$. For, $D_{n,1}/\sqrt{\ell(n)}$ cannot be a Cauchy sequence, in this case. To see this, first observe that

$$\left\| \frac{D_{n,1}}{\sqrt{\ell(n)}} - \frac{D_{m,1}}{\sqrt{\ell(m)}} \right\|^2 = \frac{1}{\ell(n)} \|D_{n,1}\|^2 + \frac{1}{\ell(m)} \|D_{m,1}\|^2 - \frac{2}{\sqrt{\ell(m)\ell(n)}} \langle D_{m,1}, D_{n,1} \rangle$$

and

$$\begin{aligned}
\langle D_{m,1}, D_{n,1} \rangle &= \langle \bar{V}_n g(w_1) - Q\bar{V}_n g(w_0), \bar{V}_m g(w_1) - Q\bar{V}_m g(w_0) \rangle \\
&= \langle \bar{V}_n g, \bar{V}_m g \rangle - \langle Q\bar{V}_n g, Q\bar{V}_m g \rangle \\
&= \langle \bar{V}_n g, \bar{V}_m g \rangle - \langle Q^2 \bar{V}_n g, \bar{V}_m g \rangle \\
&= \langle (I - Q^2)\bar{V}_n g, \bar{V}_m g \rangle \\
&= \left\langle \left(V_2 - \frac{1}{n} Q V_n V_2 \right) g, \bar{V}_m g \right\rangle.
\end{aligned}$$

So, for any fixed m,

$$\lim_{n\to\infty} \left\| \frac{D_{n,1}}{\sqrt{\ell(n)}} - \frac{D_{m,1}}{\sqrt{\ell(m)}} \right\|^2 = 1 + \frac{1}{\ell(m)} \|D_{m,1}\|^2,$$

and, therefore,

$$\lim_{m\to\infty} \lim_{n\to\infty} \left\| \frac{D_{n,1}}{\sqrt{\ell(n)}} - \frac{D_{m,1}}{\sqrt{\ell(m)}} \right\|^2 = 2.$$

3. Examples

For a simple reversible chain, let ν be a probability measure on the Borel sets of $\mathbb{R}$, let $p\colon \mathbb{R} \to (0, 1)$ be a measurable function for which

$$\theta = \int_{\mathbb{R}} \frac{\mathrm{d}\nu}{1 - p} < \infty,$$

and let

$$Q(w; B) = p(w)\mathbf{1}_B(w) + [1 - p(w)]\nu\{B\} \tag{12}$$

for Borel sets $B \subseteq \mathbb{R}$ and $w \in \mathbb{R}$. Then Q is a stochastic transition function with stationary distribution

$$\mathrm{d}\pi = \frac{\mathrm{d}\nu}{\theta(1 - p)},$$

and (1) is satisfied. Thus, there is a reversible Markov chain $\ldots, W_{-1}, W_0, W_1, \ldots$ with transition function Q and marginal distribution π. This construction is classical, and is described in [11, pp. 134–135].

Now let $\tau_0, \tau_1, \tau_2, \ldots$ be the times before the process jumps, where

$$\tau_0 = \max\{n \geq 0\colon W_n = W_0\} \quad \text{and} \quad \tau_k = \max\{n > \tau_{k-1}\colon W_n = W_{\tau_{k-1}+1}\}.$$

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Then $W_{\tau_k} = W_{\tau_{k-1}+1}$, and

$$S_{\tau_m} = \tau_0 X_0 + (\tau_1 - \tau_0)X_{\tau_1} + \cdots + (\tau_m - \tau_{m-1})X_{\tau_m}.$$

By the Markov property, (τ_0, W_0) and $[(\tau_j - \tau_{j-1}), W_{\tau_j}]$, $j \geq 1$, are independent random vectors for which $W_{\tau_j} \sim \nu$ and

$$\mathrm{P}[\tau_j - \tau_{j-1} \geq k \mid W_{\tau_j} = w] = p(w)^{k-1}$$

for all $w \in \mathcal{W}$, $k \geq 1$, and $j \geq 1$. It follows that $\mathrm{E}[\tau_j - \tau_{j-1} \mid W_{\tau_j} = w] = 1/[1 - p(w)]$ and

$$\mathrm{E}[\tau_j - \tau_{j-1}] = \int_{\mathcal{W}} \frac{\mathrm{d}\nu}{1 - p} = \theta.$$

By way of contrast, $W_{\tau_0} = W_0 \sim \pi$, and $\mathrm{E}[\tau_0] = \int p \, \mathrm{d}\pi/(1-p)$, possibly infinite. Let $Y_j = (\tau_j - \tau_{j-1})X_{\tau_j}$ and $T_m = Y_1 + \cdots + Y_m$, so that $S_{\tau_m} = \tau_0 W_0 + T_m$. Then $Y_1, Y_2, \ldots$ are independent and identically distributed (i.i.d.); moreover, $\mathrm{E}[Y_j] = 0$, since

$$\mathrm{E}[(\tau_j - \tau_{j-1})X_{\tau_j}] = \mathrm{E}\left[\frac{g(W_{\tau_j})}{1 - p(W_{\tau_j})}\right] = \int_{\mathcal{W}} \frac{g}{1-p} \, \mathrm{d}\nu = \theta \int_{\mathcal{W}} g \, \mathrm{d}\pi,$$

and $g \in L_0^2(\pi)$. Let

$$H(y) = \int_{|Y_j| \leq y} Y_j^2 \mathrm{dP},$$

and recall the following version of the central limit theorem for i.i.d. variables (with possibly infinite variances); see, for example, [4, pp. 576–578]. If $Y_1, Y_2, \ldots$ are (any) i.i.d. random variables for which $\mathrm{E}[Y_j] = 0$ and $H(y)$ varies slowly at ∞, then there are γ_m for which

$$\gamma_m^2 \sim mH(\gamma_m) \quad \text{and} \quad \frac{T_m}{\gamma_m} \xrightarrow{\mathrm{D}} \mathrm{Normal}[0, 1].$$

The following lemma is intuitive, and the proof is presented after Proposition 4 is established. To state it, define integer-valued random variables m_n such that $\tau_{m_n} \leq n < \tau_{m_n+1}$ for $n = 1, 2 \ldots$.

Lemma 2. *As $n \to \infty$, $S_n - T_{m_n} = O_p(1)$. If H varies slowly at ∞ then $T_{m_n} - T_{\lfloor n/\theta \rfloor} = o_p(\gamma_n)$.*

Proposition 4. *If $H(y)$ varies slowly and $\gamma_m^2 \sim mH(\gamma_m)$, then*

$$\frac{S_n}{\gamma_n} \,\bigg|\, W_0 \xrightarrow{\mathrm{P}} \mathrm{Normal}\left[0, \frac{1}{\theta}\right].$$

Proof. That $T_m/\gamma_m \xrightarrow{\mathrm{D}} \mathrm{Normal}[0, 1]$ was noted above. So, since $\gamma_{\lfloor n/\theta \rfloor} \sim \gamma_n/\sqrt{\theta}$, $T_{\lfloor n/\theta \rfloor}/\gamma_n \xrightarrow{\mathrm{D}} \mathrm{Normal}[0, 1/\theta]$, and since W_0 and T_m are independent for all m, the conditional distributions have the same limit. The proposition now follows directly from Lemmas 1 and 2.

Proof of Lemma 2. First observe that $S_n - T_{m_n} = \tau_0 W_0 + (n - \tau_{m_n})W_{\tau_m+1}$. It is clear that $\tau_0 W_0$ is stochastically bounded and that $|(n - \tau_{m_n})W_{\tau_{m_n+1}}| \leq (\tau_{m_n+1} - \tau_{m_n})|W_{\tau_{m_n+1}}|$. To see that the latter term is stochastically bounded, let f denote the marginal mass function of $\tau_j - \tau_{j-1}$, $j = 1, 2, \ldots$. Then the asymptotic distribution of $\tau_{m_n+1} - \tau_{m_n}$ has probability mass function $\tilde{f}(k) = kf(k)/\theta$, by the renewal theorem, [4, p. 271], and the conditional distribution of $W_{\tau_{m_n+1}}$ given $\tau_{m_n+1} - \tau_{m_n}$ does not depend on n. That $(\tau_{m_n+1} - \tau_{m_n})|W_{\tau_{m_n+1}}| = O_P(1)$ follows easily.

The proof of the second assertion uses the following version of Lévy's inequality [9, p. 259]: if H varies slowly at ∞ then $K^{-1} := \inf\{\min(\mathrm{P}[T_k < 0], \mathrm{P}[T_k > 0]) \colon k \geq 1\} > 0$, and

$$\mathrm{P}\Big[\max_{k \leq n} |T_k| > t\Big] \leq K\,\mathrm{P}[|T_n| > t] \tag{13}$$

for all $t > 0$. Observe that

$$\begin{aligned}\mathrm{P}[|T_{m_n} - T_{\lfloor n/\theta \rfloor}| \geq \epsilon\gamma_n] \leq \mathrm{P}\bigg[\bigg|m_n - \bigg\lfloor \frac{n}{\theta} \bigg\rfloor\bigg| \geq \delta n\bigg] \\ + \mathrm{P}\Big[\max_{|k\theta - n| \leq \theta\delta n + \theta} |T_k - T_{\lfloor n/\theta \rfloor}| \geq \epsilon\gamma_n\Big].\end{aligned} \tag{14}$$

The first term on the right-hand side approaches 0 for any $\delta > 0$ by the law of large numbers. Letting $N_n = \lfloor n\delta/\theta \rfloor + 4$ and using (13), the second term is at most $2K\,\mathrm{P}[|T_{N_n}| \geq \epsilon\gamma_n]$. So, by the central limit theorem, the limit superior of the right-hand side of (14) is at most $4K[1 - \Phi(\epsilon/\sqrt{\delta})]$, which approaches 0 as $\delta \to 0$.

For the example below, observe that if $f \in L^1(\pi)$ then $Qf(w) = p(w)f(w) + [1 - p(w)]\int_{\mathcal{W}} f\,\mathrm{d}\nu$. So, if $\mathcal{W} = \mathbb{R}$, ν is a symmetric measure, p is a symmetric function, and f is an odd function, then $Q^n f = p^n \times f$.

Example 1. Consider (12) with $p(w) = \mathrm{e}^{-1/|w|}$ and

$$\nu\{\mathrm{d}z\} = \mathrm{e}\frac{[1 - p(z)]\mathrm{d}z}{2z^2} \quad \text{for } |z| \geq 1. \tag{15}$$

In which case $\theta = \mathrm{e}$ and $\pi\{\mathrm{d}w\} = \mathrm{d}w/2w^2$ for $|w| \geq 1$. Let $g(w) = \operatorname{sgn}(w)$. Then $g \in L_0^\infty(\pi)$, $Q^n g = p^n \times g$, and

$$\langle g, Q^n g \rangle = \int_{\mathbb{R}} p^n \mathrm{d}\pi = \int_1^\infty \mathrm{e}^{-n/w} \frac{\mathrm{d}w}{w^2} = \frac{1}{n} \int_0^n \mathrm{e}^{-x} \mathrm{d}x \sim \frac{1}{n}.$$

It follows that $\langle g, \bar{V}_n g \rangle \sim \langle g, V_n g \rangle \sim \log(n)$ and $\sigma_n^2 = [2\langle g, \bar{V}_n g \rangle - \|g\|^2]n \sim 2n\log(n)$. So, (3) is satisfied with $\ell(n) \sim 2\log(n)$.

Recall the definition of the τ_j and the distribution of $[\tau_j - \tau_{j-1}, W_{\tau_j}]$. Then

$$\mathrm{P}[|(\tau_j - \tau_{j-1})X_{\tau_j}| > k] = \mathrm{P}[(\tau_j - \tau_{j-1}) > k] = \int_{\mathbb{R}} p^k \mathrm{d}\nu = \mathrm{e}\int_{\mathbb{R}} (1 - p)p^k \mathrm{d}\pi.$$

The last integral in the previous display is just

$$\int_1^\infty (1 - \mathrm{e}^{-1/z})\mathrm{e}^{-k/z} \frac{\mathrm{d}z}{z^2} = \frac{1}{k} \int_0^k (1 - \mathrm{e}^{-y/k})\mathrm{e}^{-y} \mathrm{d}y \sim \frac{1}{k^2} \int_0^\infty y\mathrm{e}^{-y} \mathrm{d}y = \frac{1}{k^2};$$

thus,

$$\mathrm{P}[|(\tau_j - \tau_{j-1})X_{\tau_j}| \geq k] \sim \frac{\mathrm{e}}{k^2}. \tag{16}$$

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It follows easily that $H(y) \sim 2\mathrm{e}\log(y) = \mathrm{e}\ell(y)$, $\gamma_n^2 = 2\mathrm{e}n\log(\gamma_n) \sim \mathrm{e}n\log(n) = \frac{1}{2}\mathrm{e}\sigma_n^2$, and, therefore,

$$\frac{S_n}{\sigma_n} \xrightarrow{\mathrm{D}} \mathrm{Normal}\left[0, \frac{1}{2}\right]$$

(a nonstandard normal distribution).

Since $\mathrm{E}[\sigma_n^{-2}S_n^2]$ is bounded, it follows that $\mathrm{E}\,|S_n| \sim \pi^{-1/2}\sigma_n$ and, therefore, that $S_n/\,\mathrm{E}\,|S_n| \xrightarrow{\mathrm{D}} \mathrm{Normal}[0, \frac{1}{2}\pi]$. The latter convergence can also be deduced from Theorem 4 of [10]. To this end, it suffices to verify Equation (3.2) of that paper. Since $|g| \le 1$, it is not difficult to see that the term whose limit is taken in [10, Equation (3.2)] is at most $\sigma_n^{-2}\sum_{k=1}^{n} k\beta_k$, where β_k is the coefficient of absolute regularity. So, it suffices to show that β_n is of order $1/n$, and this may be deduced from the equation at the top of page 136 of [11] together with the relation $\mathrm{P}[\tau_0 > n] = \int_{\mathbb{R}} p^n \mathrm{d}\pi \sim 1/n$. (The τ in [11] is our $\tau_0 + 1$.) Conditional convergence is not asserted in Theorem 4 of [10], but is implicit in the proof; $\mathrm{E}\,|S_n| \sim \pi^{-1/2}\sigma_n$ is not deducible from that theorem, however, because S_n is not normalized by σ_n there.

Example 2. A slight modification of Example 1 produces a very simple bounded stationary sequence whose normalized partial sums converge in distribution to a stable distribution. Other examples may be found in [8]. If (15) is changed to

$$\nu\{\mathrm{d}z\} = \frac{[1 - p(z)]\mathrm{d}z}{2\gamma_\alpha |z|^\alpha}$$

for $|z| \ge 1$, where $1 < \alpha < 2$ and $\gamma_\alpha = \int_0^1 y^{\alpha-2}(1 - \mathrm{e}^{-y})\mathrm{d}y$, then $\pi\{\mathrm{d}z\} = (\alpha - 1)/(2|z|^\alpha)\mathrm{d}z$ for $|z| \ge 1$, and

$$\mathrm{P}[Y > y] \sim \frac{\Gamma(\alpha)}{\gamma_\alpha y^\alpha}$$

as $y \to \infty$. It then follows that $n^{-1/\alpha}S_n \xrightarrow{\mathrm{D}} Z$, where Z has a symmetric stable distribution with characteristic function $\mathrm{e}^{-c_\alpha|t|^\alpha}$ and $c_\alpha = (\alpha - 1)\Gamma(\alpha)\int_0^\infty x^{-\alpha}\sin(x)\mathrm{d}x$.

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