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A symmetry of silting quivers

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Abstract
We investigate symmetry of the silting quiver of a given algebra which is induced by an anti-automorphism of the algebra. In particular, one shows that if there is a primitive idempotent fixed by the anti-automorphism, then the 2-silting quiver (= the support τ-tilting quiver) has a bisection. Consequently, in that case, we obtain that the cardinality of the 2-silting quiver is an even number (if it is finite).

1. Introduction

In this paper, we study symmetry of the silting quiver of a finite dimensional algebra Λ over an algebraically closed field; the *silting quiver* is a quiver whose vertices are (basic) silting objects and arrows $T \to U$ are drawn whenever U is an irreducible left mutation of T, and it coincides with the Hasse quiver of the poset $\mathsf{silt}\,\Lambda$ of silting objects [5].

The main theorem (Theorem 1.2) of this paper shows that an anti-automorphism of Λ (i.e., an algebra isomorphism $\Lambda^{\mathrm{op}} \simeq \Lambda$) induces a symmetry of $\mathsf{silt}\,\Lambda$. Here, Λ^{op} stands for the opposite algebra of Λ. Focusing on 2-term silting objects, which bijectively correspond to support τ-tilting modules [3], we obtain a bisection of the poset $\mathsf{2silt}\,\Lambda$ of 2-term silting objects if there is a fixed primitive idempotent by the anti-automorphism (Theorem 1.4). Thus, in that case, it turns out that the cardinality of $\mathsf{2silt}\,\Lambda$ is even (if it is finite).

When Λ is 2-silting finite (= τ-tilting finite); i.e., $|\,\mathsf{2silt}\,\Lambda| < \infty$, counting the number of elements in $\mathsf{2silt}\,\Lambda$ is one of the important problems in this area; see [1, 2, 4, 9, 15]. In this context, Theorem 1.4 gives a very useful method to reduce the whole pattern to half of $\mathsf{2silt}\,\Lambda$. Indeed, this may be applied to such works on Hecke algebras and Schur algebras, see [8], [17], etc.

For example, the following admit anti-automorphisms fixing a primitive idempotent:

- enveloping algebras (Theorem 2.1);
- preprojective algebras of Dynkin type (Theorem 2.5);
- cellular algebras (Theorem 2.6);
- symmetric algebras with radical cube zero, which contain multiplicity-free Brauer line/cycle algebras (Theorem 2.7);
- selfinjective Nakayama algebras, which contain Brauer star algebras with an exceptional vertex in the center (Theorem 2.8);
- group algebras (Theorem 2.10);
- the trivial extensions of algebras with an anti-automorphism fixing a primitive idempotent (Theorem 2.12).

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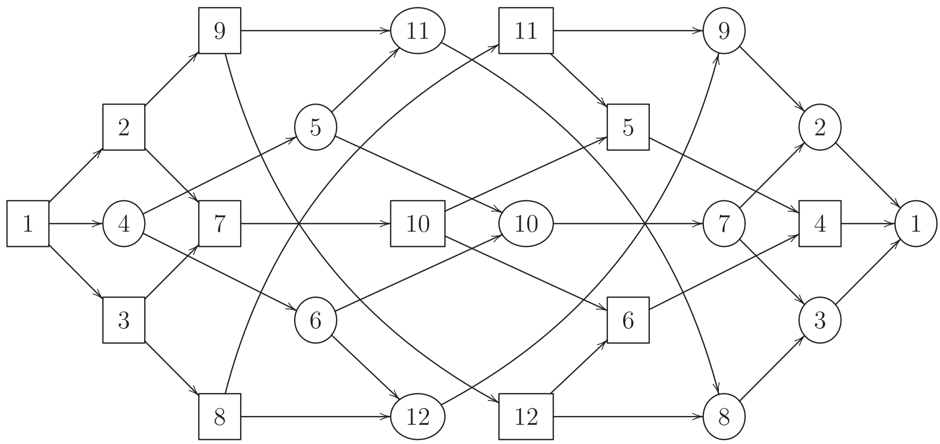
Here is an illustration of the symmetry of $\mathsf{2silt}\,\Lambda$ for the preprojective algebra Λ of Dynkin type A_3, in which $\boxed{-}$ and $\bigcirc\!\!\!\!-$ correspond:



Notation. *Throughout this paper, let Λ be a finite dimensional algebra over an algebraically closed field K, and $\mathcal{K}_\Lambda := \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ denote the perfect derived category of Λ. The Λ-dual is denoted by $(-)^* := \mathrm{Hom}_?(-, \Lambda)$ for $? = \mathcal{K}_\Lambda$ or $\mathcal{K}_{\Lambda^{\mathrm{op}}}$.*

2. Results

We say that an object T of $\mathcal{K}_\Lambda$ is *silting* (*tilting*) if it satisfies $\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, T[i]) = 0$ for every integer $i > 0$ ($i \neq 0$) and $\mathcal{K}_\Lambda = \mathsf{thick}\,T$. Here, $\mathsf{thick}\,T$ stands for the smallest thick subcategory of $\mathcal{K}_\Lambda$ containing T. We denote by $\mathsf{silt}\,\Lambda$ ($\mathsf{tilt}\,\Lambda$) the set of isomorphism classes of basic silting (tilting) objects in $\mathcal{K}_\Lambda$.

Let us recall silting mutation and a partial order on $\mathsf{silt}\,\Lambda$.

Definition-Theorem 1.1. [5, Theorem 2.11, 2.31, 2.35 and Definition 2.41]

1. *Let T be a silting object of $\mathcal{K}_\Lambda$ with decomposition $T = X \oplus M$. Taking a minimal left $\mathsf{add}\,M$-approximation $f : X \to M'$ of X, we construct a new object $\mu_X^-(T) := Y \oplus M$, where Y is the mapping cone of f. Then $\mu_X^-(T)$ is also silting, and we call it the* left mutation *of T with respect to X. Dually, we define the* right mutation *$\mu_X^+(T)$ of T with respect to X.*
2. *For objects T and U of $\mathcal{K}_\Lambda$, we write $T \geq U$ if $\mathrm{Hom}_{\mathcal{K}_\Lambda}(T, U[i]) = 0$ for $i > 0$. Then $\geq$ gives a partial order on $\mathsf{silt}\,\Lambda$.*
3. *We construct the* silting quiver *$\mathcal{H}$ of $\mathcal{K}_\Lambda$ as follows.*
   - *The vertices of $\mathcal{H}$ are basic silting objects of $\mathcal{K}_\Lambda$;*
   - *We draw an arrow $T \to U$ if U is a left mutation of T with respect to an indecomposable direct summand.*

   Then $\mathcal{H}$ coincides with the Hasse quiver of the partially ordered set $\mathsf{silt}\,\Lambda$.

We define a subset of $\mathsf{silt}\,\Lambda$ by

$$\mathsf{2silt}\,\Lambda := \{T \in \mathsf{silt}\,\Lambda \mid \Lambda \geq T \geq \Lambda[1]\}.$$

This bijectively corresponds to the poset of support τ-tilting modules [3, Theorem 3.2].

We say that Λ *admits an anti-automorphism* if there is a K-linear automorphism $\varsigma : \Lambda \to \Lambda$ satisfying $\varsigma(xy) = \varsigma(y)\varsigma(x)$, or equivalently if an algebra isomorphism $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$ exists. Here, Λ^{op} stands for the opposite algebra of Λ. In this case, we obtain an equivalence $\mathcal{K}_{\Lambda^{\mathrm{op}}} \to \mathcal{K}_\Lambda$, also denoted by σ.

We now investigate that an anti-automorphism of Λ induces a symmetry of $\mathsf{silt}\,\Lambda/\mathsf{2silt}\,\Lambda$.

Theorem 1.2. *Assume Λ admits an anti-automorphism σ. Then we have the following.*

1. *The functor $\mathbb{S}_\sigma := \sigma \circ (-)^*$ induces an anti-automorphism of the poset $\mathsf{silt}\,\Lambda$.*
2. *Let T be a silting object. Then there is an algebra isomorphism $\mathrm{End}_{\mathcal{K}_\Lambda}(T)^{\mathrm{op}} \simeq \mathrm{End}_{\mathcal{K}_\Lambda}(\mathbb{S}_\sigma(T))$. Moreover, if Γ is derived equivalent to Λ, then so is Γ^{op}; hence, Γ and Γ^{op} are also derived equivalent.*
3. *The functor $S_\sigma := [1] \circ \mathbb{S}_\sigma$ induces an anti-automorphism of the poset $\mathsf{2silt}\,\Lambda$.*

Proof. (1)(3) It is evident that $(-)^*$ and σ yield an anti-isomorphism $\mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Lambda^{\mathrm{op}}$ and an isomorphism $\mathsf{silt}\,\Lambda^{\mathrm{op}} \to \mathsf{silt}\,\Lambda$, respectively. Composing them makes an anti-automorphism of $\mathsf{silt}\,\Lambda$. This immediately implies that $S_\sigma := [1] \circ \mathbb{S}_\sigma$ is also an anti-automorphism of $\mathsf{2silt}\,\Lambda$.

(2) Clearly, $\mathrm{End}_{\mathcal{K}_\Lambda}(\mathbb{S}_\sigma(T)) \simeq \mathrm{End}_{\mathcal{K}_{\Lambda^{\mathrm{op}}}}(T^*) \simeq \mathrm{End}_{\mathcal{K}_\Lambda}(T)^{\mathrm{op}}$. If Γ is derived equivalent to Λ, then there is a tilting object T of $\mathcal{K}_\Lambda$ with $\Gamma \simeq \mathrm{End}_{\mathcal{K}_\Lambda}(T)$. Since $\mathbb{S}_\sigma(T)$ is also tilting, it is seen by (1) that $\Gamma^{\mathrm{op}} \simeq \mathrm{End}_{\mathcal{K}_\Lambda}(\mathbb{S}_\sigma(T))$ is derived equivalent to Λ. □

We discuss a benefit derived from the symmetry S_σ of $\mathsf{2silt}\,\Lambda$.

Let P be an indecomposable projective Λ-module. We define subsets of $\mathsf{2silt}\,\Lambda$ by

$$\mathcal{T}_P^- := \{T \in \mathsf{2silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\} \text{ and}$$
$$\mathcal{T}_P^+ := \{T \in \mathsf{2silt}\,\Lambda \mid \Lambda \geq T \geq \mu_{P[1]}^+(\Lambda[1])\}.$$

Denote by X^i the ith term of a complex X. We make the following observation.

Lemma 1.3. *We have $\mathcal{T}_P^- = \{T \in \mathsf{2silt}\,\Lambda \mid P \in \mathsf{add}\,T^{-1}\}$ and $\mathcal{T}_P^+ = \{T \in \mathsf{2silt}\,\Lambda \mid P \in \mathsf{add}\,T^0\}$. In particular, $\mathcal{T}_P^- \sqcup \mathcal{T}_P^+ = \mathsf{2silt}\,\Lambda$.*

Proof. Let $T \in \mathsf{2silt}\,\Lambda$. We know that T is of the form $[T^{-1} \to T^0]$ with $T^{-1}, T^0 \in \mathsf{add}\,\Lambda$. By [5, Lemma 2.25], we have $\mathsf{add}\,T^{-1} \cap \mathsf{add}\,T^0 = 0$. It is easily seen that $\mathsf{add}\,(T^{-1} \oplus T^0) = \mathsf{add}\,\Lambda$. Now, we obtain from [5, Theorem 2.35] (and its dual) that:

(i) $P \in \mathsf{add}\,T^{-1} \iff \mu_P^-(\Lambda) \geq T$;
(ii) $P \in \mathsf{add}\,T^0 \iff T \geq \mu_{P[1]}^+(\Lambda[1])$.

This completes the proof. □

The symmetry S_σ is useful to analyze the cardinality of $\mathsf{2silt}\,\Lambda$ as follows.

Theorem 1.4. *Let e be a primitive idempotent of Λ and put $P := e\Lambda$. Assume that Λ admits an anti-automorphism σ. If $\sigma(e) = e$, then we have a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$, i.e., $\mathcal{T}_P^- \overset{\text{anti}}{\simeq} \mathcal{T}_P^+$. In particular, $|\mathsf{2silt}\,\Lambda| = 2 \cdot |\mathcal{T}_P^-| = 2 \cdot |\mathcal{T}_P^+|$.*

Proof. We see that S_σ gives a one-to-one correspondence between $\mathcal{T}_P^-$ and $\mathcal{T}_{\mathbb{S}_\sigma(P)}^+$. As $\mathbb{S}_\sigma(P) \simeq P$ by assumption, the assertion follows from Lemma 1.3. □

Let $T := [T_1 \to T_0]$ be a 2-term silting object of $\mathcal{K}_\Lambda$; i.e., $T \in \mathsf{2silt}\,\Lambda$, and $\mathcal{E}$ denote a complete list of pairwise orthogonal primitive idempotents of Λ. Recall that the *g-vector* g_T of T is the vector $(g_e)_{e\in\mathcal{E}}$ which is given by $g_e := c_0^e - c_1^e$. Here, c_i^e stands for the multiplicity of $e\Lambda$ in T_i.

We immediately obtain the following corollary.

Corollary 1.5. *Suppose that Λ admits an anti-automorphism σ satisfying $\sigma(e) = e$ for every primitive idempotent e of Λ. Then S_σ reverses the directions of the g-vectors of all 2-term silting objects in $\mathcal{K}_\Lambda$.*

Proof. Let $T := [e_1\Lambda \to e_0\Lambda]$ be a 2-term silting object of $\mathcal{K}_\Lambda$, where e_0 and e_1 are idempotents of Λ. Since any idempotent is fixed by σ, we observe that S_σ sends T to the 2-term silting object $[e_0\Lambda \to e_1\Lambda]$, which immediately tells us the fact that $g_{S_\sigma(T)} = -g_T$. □

3. Applications and examples

We explore when Λ admits an anti-automorphism σ with $\sigma(e) = e$ for some primitive idempotent e of Λ, and give applications and examples of Theorem 1.4.

Let us start with enveloping algebras.

Theorem 2.1. *The enveloping algebra $\Lambda^{\mathrm{op}} \otimes_K \Lambda$ has an anti-automorphism $(a \otimes b \mapsto b \otimes a)$ fixing the primitive idempotent $e \otimes e$ for a primitive idempotent e of Λ. In particular, there is a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$, where $P := (e \otimes e)\Lambda^{\mathrm{op}} \otimes_K \Lambda$.*

Let $Q := (Q_0, Q_1)$ be a (finite) quiver, where Q_0 and Q_1 are the sets of vertices and arrows, respectively. For a vertex v of Q, we denote by e_v the primitive idempotent of KQ corresponding to v. The opposite quiver of Q is denoted by Q^{op}; that is, it consists of the same vertices as Q and reversed arrows a^* for arrows a of Q, i.e., a^* is obtained by swapping the source and target of a. For an admissible ideal I of KQ, reversing arrows makes the admissible ideal I^{op} of KQ^{op}; for example, $ab \in I$ implies $b^*a^* \in I^{\mathrm{op}}$.

We consider the case that an isomorphism $\iota : Q^{\mathrm{op}} \to Q$ of quivers exists; ι gives rise to an algebra isomorphism $KQ^{\mathrm{op}} \to KQ$, which will be also written by ι.

Proposition 2.2. *Let Λ be an algebra presented by a quiver Q and an admissible ideal I of KQ. Suppose that there is an isomorphism $\iota : Q^{\mathrm{op}} \to Q$ of quivers satisfying $I^{\mathrm{op}} = \iota^{-1}(I)$ and fixing a vertex v; put $P := e_v\Lambda$. Then we have a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$. In particular, $|\mathsf{2silt}\,\Lambda| = 2 \cdot |\mathcal{T}_P^-|$.*

Proof. As $I^{\mathrm{op}} = \iota^{-1}(I)$, we get isomorphisms

$$\Lambda^{\mathrm{op}} = (KQ/I)^{\mathrm{op}} \simeq KQ^{\mathrm{op}}/I^{\mathrm{op}} \overset{\iota}{\simeq} KQ/I = \Lambda;$$

write the composition by $\sigma : \Lambda^{\mathrm{op}} \to \Lambda$. Since $\iota(v) = v$ by assumption, we have $\sigma(e_v) = e_v$. Thus, the assertion follows from Theorem 1.4. □

Example 2.3. *Let Λ be the algebra given by the A_n-quiver $Q : 1 \xrightarrow{x} 2 \xrightarrow{x} \cdots \xrightarrow{x} n$ and the admissible ideal $I = 0$ or $I := \langle x^r \rangle$ for some $r > 0$. We have an isomorphism $Q^{\mathrm{op}} \to Q$ of quivers which assigns $i \mapsto n - i + 1$ $(i \in Q_0)$ and $x^* \mapsto x$ $(x \in Q_1)$. The equalities $I^{\mathrm{op}} = \langle (x^*)^r \rangle = \iota^{-1}(I)$ imply that Λ admits an anti-automorphism σ. If n is even, then we apply Theorem 1.2. If n is odd, then the vertex $v := \frac{n+1}{2}$ is fixed by σ, whence we can apply Proposition 2.2; we get $\mathcal{T}_{e_v\Lambda}^- \overset{\mathrm{anti}}{\simeq} \mathcal{T}_{e_v\Lambda}^+$. The following are the Hasse quivers of $\mathsf{2silt}\,\Lambda$ for $n = 2$ and $n = 3$, in which $\boxed{-}$ and $-$ correspond and $\bullet$ is stable by S_σ.*

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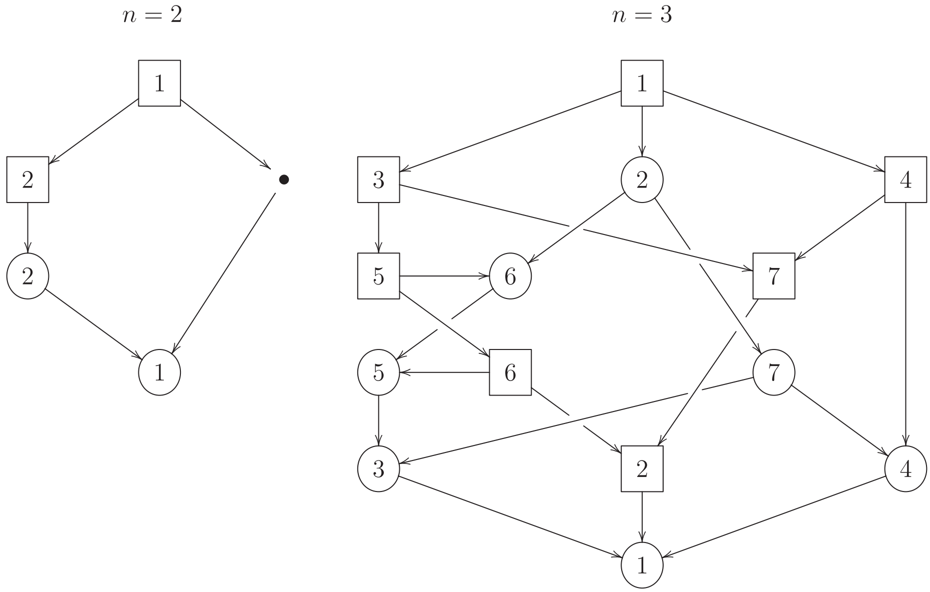

Here, in the RHS, $\boxed{-}$ *and* $\bigcirc$ *are the members of* $\mathcal{T}^+_{P_2}$ *and* $\mathcal{T}^-_{P_2}$*, respectively.*

3.1. Algebras presented by double quivers

Recall that the *double* quiver $\overline{Q}$ of Q is the quiver constructed by $\overline{Q}_0 := Q_0$ and $\overline{Q}_1 := Q_1 \sqcup \{a^* \mid a \in Q_1\}$, where a^* is obtained by swapping the source and target of a. Clearly, the assignments $v \mapsto v$ ($v \in Q_0$), $a^* \mapsto a^*$ and $(a^*)^* \mapsto a$ ($a \in Q_1$) make an isomorphism $\iota : \overline{Q}^{\mathrm{op}} \to \overline{Q}$ of quivers; note that ι fixes all vertices.

Let us give examples of algebras presented by a double quiver.

Example 2.4.

1. [11] *The* preprojective algebra Π_Q *of a Dynkin quiver Q is defined as the quotient $K\overline{Q}/\overline{I}$ of $K\overline{Q}$ by $\overline{I} := \langle aa^* - a^*a \mid a \in Q_1\rangle$. Then, it is finite dimensional and selfinjective.*
2. [18, Example 1.6] *Let Q be a quiver and I an admissible ideal of KQ. For a path $p = a_1a_2 \cdots a_\ell$ in Q, write $p^* := a^*_\ell \cdots a^*_2 a^*_1$; extending it linearly, we also use the terminology p^* for a linear combination p in KQ. We define an ideal $\overline{I}$ of $K\overline{Q}$ which is generated by p, p^* ($p \in I$) and ab^* ($a, b \in Q_1$). Then the algebra $\Lambda(Q, I) := K\overline{Q}/\overline{I}$ is finite dimensional. If Q contains no oriented cycle, then $\Lambda(Q, I)$ is a quasi-hereditary algebra with a duality.*

Now, an application of Proposition 2.2 is obtained.

Theorem 2.5. *Let $\Lambda = \Pi_Q$ for a Dynkin quiver Q or $\Lambda(Q, I)$ for a quiver Q and an admissible ideal I of KQ. Then we have a bijection between $\mathcal{T}^-_P$ and $\mathcal{T}^+_P$ for any indecomposable projective module P of Λ. In particular,* $|\,\mathsf{2silt}\ \Lambda| = 2 \cdot |\mathcal{T}^-_P|$.

Proof. We can easily check the equality $\overline{I}^{\mathrm{op}} = \iota^{-1}(\overline{I})$ holds, and apply Proposition 2.2. □

3.2. Cellular algebras

Cellular algebras were introduced by Graham and Lehrer [10]. An algebra Λ is called *cellular* if it admits a cellular basis; that is, a basis with certain nice multiplicative properties. We refer to

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[14] for more details. By the definition, each cellular basis of Λ admits an involution σ; i.e., an anti-automorphism σ of Λ with $\sigma^2 = 1$. It is shown in [14, Proposition 5.1] that the involution σ fixes all simples of a cellular algebra. Hence, we have the following result.

Theorem 2.6. *Let Λ be a cellular algebra. Then there exists a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$ for any indecomposable projective module P of Λ. In particular, $|\,\mathsf{2silt}\,\Lambda| = 2 \cdot |\mathcal{T}_P^-|$.*

Nowadays, a lot of interesting algebras have been found to be cellular, for example, Ariki–Koike algebras, (q-)Schur algebras as well as various generalizations, block algebras of category $\mathcal{O}$, and various diagram algebras. We hope that Theorem 2.6 will be useful to verify the finiteness of $|\,\mathsf{2silt}\,\Lambda|$ for the aforementioned algebras, especially for Hecke algebras [18], Schur algebras [17], etc.

3.3. Symmetric algebras with radical cube zero

We get the following result.

Theorem 2.7. *Let Λ be a symmetric algebra with radical cube zero. Then there exists a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$ for any indecomposable projective module P of Λ. In particular, $|\,\mathsf{2silt}\,\Lambda| = 2 \cdot |\mathcal{T}_P^-|$.*

Proof. By [2, Proposition 3.3], it turns out that the Gabriel quiver of Λ is given by adding loops to the double quiver of a quiver Q; denote by $\widehat{Q}$ the quiver of Λ. We also observe that $aa^* \neq 0 \neq a^*a$ for any arrow a of $\widehat{Q}$ and $ab = 0$ unless $b = a^*$ and $a = b^*$; if a is an added loop, write $a^* = a$. Thus, we get an isomorphism $\iota : \widehat{Q}^{\mathrm{op}} \to \widehat{Q}$ of quivers which fixes all vertices.

Let i be a vertex of $\widehat{Q}$ and a an arrow starting at i. Since Λ is symmetric, it is seen that aa^* spans the socle of $P_i := e_i\Lambda$ as a vector space. Applying changes of basis, we have $aa^* = bb^*$ for every arrow b of $\widehat{Q}$ starting from i. Let I denote the ideal of $K\widehat{Q}$ consisting of such relations; so $\Lambda \simeq K\widehat{Q}/I$. Then, we obtain the equality $I^{\mathrm{op}} = \iota^{-1}(I)$, whence the assertion follows from Proposition 2.2. $\square$

3.4. Selfinjective Nakayama algebras

It is well known that a self-injective Nakayama algebra is presented by a cycle quiver $\begin{tikzcd} \bullet \arrow[r, "x"'] & \bullet \arrow[r, "x"'] & \cdots \arrow[r, "x"'] & \bullet \arrow[lll, bend right, "x"'] \end{tikzcd}$ with relations $x^r = 0$ for some $r > 0$. Here is an easy application of Proposition 2.2.

Theorem 2.8. *Let Λ be a self-injective Nakayama algebra and P an indecomposable projective module of Λ. Then we have a bijection between $\mathcal{T}_P^-$ and $\mathcal{T}_P^+$. In particular, $|\,\mathsf{2silt}\,\Lambda| = 2 \cdot |\mathcal{T}_P^-|$.*

Remark 2.9. *Let Λ be a self-injective Nakayama algebra given by a cycle quiver Q. Whenever we choose a vertex i of Q, one gets an isomorphism $Q^{\mathrm{op}} \to Q$ of quivers fixing i. So, a bijection between $\mathcal{T}_{e_i\Lambda}^-$ and $\mathcal{T}_{e_i\Lambda}^+$ depends on the choice of vertices.*

3.5. Group algebras

Let G be a finite group and p the characteristic of K. While the group algebra KG is, in general, neither basic nor ring-indecomposable,[1] it admits an anti-automorphism by $g \mapsto g^{-1}$; we can then apply Theorem 1.2 to KG.

The following situation enables us to apply Theorem 1.4.

[1] It is well known that if there is a normal p-subgroup of G containing its centralizer, then KG is ring-indecomposable; see [16, Exercise V. 2. 10] for example.

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Theorem 2.10. *Let G be a semidirect product $E \ltimes D$ of a p'-group E (i.e., $p \nmid |E|$) on a p-group D. Then there exists a primitive idempotent e of $\Lambda := KG$ such that $\mathcal{T}^{-}_{e\Lambda}$ bijectively corresponds to $\mathcal{T}^{+}_{e\Lambda}$. In particular, $|\,\mathsf{2silt}\,\Lambda|$ is double $|\mathcal{T}^{-}_{e\Lambda}|$.*

Proof. As the argument above, we know that Λ admits an anti-automorphism σ ($g \mapsto g^{-1}$). Since $|E|$ is invertible in K, we put $e := \dfrac{1}{|E|}\sum_{g\in E} g$; clearly, it is an idempotent fixed by σ. It is seen that $e\Lambda = eKG = eKD \simeq KD$ (as KD-modules), which implies that e is primitive. Thus, we deduce the assertion from Theorem 1.4. □

We obtain an interesting observation.

Corollary 2.11. *Let Λ be a p-block of KG with a normal defect group D and E its inertial quotient. If E has trivial Schur multiplier (i.e., $H^2(E, K^{\times}) = 1$), then the number of 2-term silting objects is even if it is finite.*

Proof. Thanks to Külshammer's theorem [13, Theorem A], we see that Λ is Morita equivalent to the twisted group algebra $K^{\alpha}[E \ltimes D]$ for some 2-cocycle α, which is just $K[E \ltimes D]$ by assumption. Thus, we find out that $|\,\mathsf{2silt}\,\Lambda|$ is even by Theorem 2.10. □

It is known that groups of deficiency zero have the trivial Schur multiplier; see [12]. Here, the *deficiency* of a group G is defined to be the maximum of the integers $|X| - |R|$ for all presentations $G = \langle X \mid R\rangle$ of G, which is nonpositive if G is a finite group. Typical examples of deficiency-zero finite groups are cyclic groups $\langle g \mid g^n = 1\rangle$ and quaternion groups $\langle a, b \mid a^{2n} = 1, a^n = b^2, ba = a^{-1}b\rangle = \langle a, b \mid bab = a^{n-1}, aba = b\rangle$. Thus, the first example of Corollary 2.11 should be the case that D is cyclic; then, E is automatically cyclic, Λ is a symmetric Nakayama algebra [6, Theorem 17.2], and so $|\,\mathsf{2silt}\,\Lambda| = \binom{2n}{n}$ (even), where $n := |E|$ [1, Corollary 2.29]. Moreover, the equality $|\,\mathsf{2silt}\,\Lambda| = \binom{2n}{n}$ holds even if we drop the assumption of D being normal in G; then, Λ is still a Brauer tree algebra [6, Theorem 17.1], whence the equality is obtained from [7, Theorem 5.1].

3.6. *Trivial extension algebras*

The *trivial extension* $T(\Lambda)$ of an algebra Λ (by its minimal cogenerator $D\Lambda$) is defined to be $\Lambda \oplus D\Lambda$ as a K-vector space with multiplication given by $(a, f) \cdot (b, g) := (ab, ag + fb)$. Here, D denotes the K-dual. We can easily verify that there is a one-to-one correspondence between simple modules of Λ and $T(\Lambda)$; so we use the same symbol e as a primitive idempotent of Λ and $T(\Lambda)$ (via the correspondence).

We state that a bisection of $\mathsf{2silt}\,\Lambda$ can be extended to that of $\mathsf{2silt}\,T(\Lambda)$.

Theorem 2.12. *An anti-automorphism σ of Λ induces one on $T(\Lambda)$, say $\overline{\sigma}$. If σ fixes a primitive idempotent e of Λ, then the corresponding idempotent e of $T(\Lambda)$ is stable by $\overline{\sigma}$. In the case, we have a bisection of $\mathsf{2silt}\,T(\Lambda)$ with respect to $P := eT(\Lambda)$.*

Proof. Note that $T(\Lambda)^{\mathrm{op}} = T(\Lambda^{\mathrm{op}})$. Since $\sigma^{-1} : \Lambda \to \Lambda^{\mathrm{op}}$ is an algebra isomorphism, we have a K-linear automorphism $t_\sigma := \mathrm{Hom}_K(\sigma^{-1}, K) : D(\Lambda^{\mathrm{op}}) \to D\Lambda$ of $D\Lambda$. For any $a, b \in \Lambda^{\mathrm{op}}$ and $f \in D(\Lambda^{\mathrm{op}})$, we get equalities

$$\begin{aligned} t_\sigma(a \bullet f \bullet b)(x) &= (a \bullet f \bullet b)(\sigma^{-1}(x)) = f(b \bullet \sigma^{-1}(x) \bullet a) = f(\sigma^{-1}(\sigma(b)x\sigma(a))) \\ &= t_\sigma(f)(\sigma(b)x\sigma(a)) = (\sigma(a)t_\sigma(f)\sigma(b))(x). \end{aligned}$$

Here, $\bullet$ stands for the multiplication or the action of Λ^{op}. It turns out that

$$t_\sigma(a \bullet f \bullet b) = \sigma(a)t_\sigma(f)\sigma(b).$$

Now, we define a K-linear automorphism $\overline{\sigma}: T(\Lambda^{\mathrm{op}}) \to T(\Lambda)$ by $(a,f) \mapsto (\sigma(a), t_\sigma(f))$. Let us check that $\overline{\sigma}$ is an anti-automorphism of $T(\Lambda)$; for any $a, b \in \Lambda^{\mathrm{op}}$ and $f, g \in D(\Lambda^{\mathrm{op}})$,

$$\begin{aligned}
\overline{\sigma}((a,f) \bullet (b,g)) &= \overline{\sigma}(a \bullet b, a \bullet g + f \bullet b) = (\sigma(a \bullet b), t_\sigma(a \bullet g + f \bullet b)) \\
&= (\sigma(a)\sigma(b), \sigma(a)t_\sigma(g) + t_\sigma(f)\sigma(b)) \\
&= (\sigma(a), t_\sigma(f)) \cdot (\sigma(b), t_\sigma(g)) \\
&= \overline{\sigma}(a,f) \cdot \overline{\sigma}(b,g).
\end{aligned}$$

Thus, the first assertion holds. As the second assertion is clear, the last one immediately follows from Theorem 1.4. $\square$

Remark 2.13. *Theorem 2.12 does not imply that taking trivial extensions transmits the τ-tilting finiteness. In fact, the radical-square-zero self-injective Nakayama algebra with 2 simple modules is τ-tilting finite, but its trivial extension is not so.*

3.7. Applying the main theorem twice

In this subsection, we try applying Theorem 1.4 twice in a row. Let us show the following.

Theorem 2.14. *Assume that Λ is basic and admits an anti-automorphism σ fixing a primitive idempotent e of Λ; write $P := e\Lambda$. Let P' be the mapping cone of a minimal left $\mathsf{add}(\Lambda/P)$-approximation of P; that is, $\mu_P^-(\Lambda) = P' \oplus \Lambda/P$. Putting $\Gamma := \mathrm{End}_{\mathcal{K}_\Lambda}(\mu_P^-(\Lambda))$, e' denotes the idempotent of Γ corresponding to P'. Assume that the following hold:*

1. *$\mu_P^-(\Lambda)$ is tilting;*
2. *There is an anti-automorphism σ' of Γ satisfying $\sigma'(e') = e'$.*

Then, we have a poset isomorphism $\mathcal{T}_P^- \simeq \mathcal{T}_{e'\Gamma}^+$ and $|\,\mathsf{2silt}\,\Lambda| = |\,\mathsf{2silt}\,\Gamma|$.

Proof. As $\mu_P^-(\Lambda)$ is tilting, we identify $\mathsf{2silt}\,\Gamma$ with $\{T \in \mathsf{silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\}$. By Lemma 1.3, we have an equality:

$$\begin{aligned}
&\{T \in \mathsf{silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\} \\
&= \{T \in \mathsf{silt}\,\Lambda \mid \mu_{P'}^-\mu_P^-(\Lambda) \geq T \geq \mu_P^-(\Lambda)[1]\} \sqcup \{T \in \mathsf{silt}\,\Lambda \mid \mu_P^-(\Lambda) \geq T \geq \Lambda[1]\},
\end{aligned}$$

in which the components of RHS have the same cardinality by Theorem 1.4. Thus, the cardinality of LHS in the equality is the double of that of $\mathcal{T}_P^-$, which is equal to the cardinality of $\mathsf{2silt}\,\Lambda$. $\square$

We give two examples; one illustrates Theorem 2.14, and the other explains that a derived equivalence does not necessarily preserve the cardinality of the poset $\mathsf{2silt}\,(-)$ even if a given algebra is a symmetric algebra which admits an anti-automorphism fixing a primitive idempotent.

Example 2.15. *Let Λ be the algebra presented by the quiver with relations as follows:*

$$\begin{tikzcd}
 & 1 \arrow[dr, "\beta"] & \\
2 \arrow[ur, "\alpha"] \arrow[rr, shift left, "\gamma^*"] & & 3 \arrow[ll, shift left, "\gamma"]
\end{tikzcd}
\qquad
\begin{cases}
\beta\gamma\alpha = 0 = \gamma(\gamma^*\gamma)^3 \\
\alpha\beta = \gamma^*\gamma\gamma^*
\end{cases}$$

Note that Λ is symmetric and admits an anti-automorphism which fixes the vertex 1 and switches the vertices 2 and 3. Set $P_i := e_i\Lambda$.

1. *Let T_1 be the left mutation of Λ with respect to P_1. By hand, we can check that the endomorphism algebra Γ_1 of T_1 is given by the quiver with relations:*

$$\begin{tikzcd} 2 \arrow[r, shift left, "\alpha"] & 1 \arrow[l, shift left, "\alpha^*"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\beta^*"] \end{tikzcd} \qquad \begin{cases} \alpha\beta\beta^*\beta = \beta^*\beta\beta^*\alpha^* = \alpha\alpha^* = 0 \\ \alpha^*\alpha = (\beta\beta^*)^2 \end{cases}$$

It is obtained that Γ_1 has an anti-automorphism fixing the vertex 1. Thus, we derive from Theorem 2.14 that $\mathsf{2silt}\,\Lambda$ and $\mathsf{2silt}\,\Gamma_1$ has the same cardinality; it is illustrated by (anti-) isomorphisms $\mathcal{T}^+_{(P_1)_\Lambda} \overset{\text{anti}}{\simeq} \mathcal{T}^-_{(P_1)_\Lambda} \simeq \mathcal{T}^+_{(P_1)_{\Gamma_1}} \overset{\text{anti}}{\simeq} \mathcal{T}^-_{(P_1)_{\Gamma_1}}$. Actually, $\mathsf{2silt}\,\Lambda$ and $\mathsf{2silt}\,\Gamma_1$ are finite sets and the numbers are 32 [4, Theorem 2].

2. *Let T_2 be the left mutation of Λ with respect to P_2. We have the endomorphism algebra Γ_2 presented by the quiver with relations:*

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\beta^*"] \arrow[loop above, "\alpha"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\gamma^*"] \end{tikzcd} \qquad \begin{cases} \beta\gamma = \beta\beta^* = 0 = \gamma^*\beta^* = \gamma^*\alpha \\ \alpha\gamma = 0 \\ \alpha^2 = \beta^*\beta \\ \alpha^3 = \gamma\gamma^* \end{cases}$$

Unfortunately, the cardinality of $\mathsf{2silt}\,\Gamma_2$ is 28 by [4, Theorem 2]. *Since Γ_2 admits an anti-automorphism fixing the vertex 2, a similar argument as the proof of Theorem 2.14 explains that $\mathcal{T}^-_{(P_2)_\Lambda} \simeq \mathcal{T}^+_{(P_2)_{\Gamma_2}} \overset{\text{anti}}{\simeq} \mathcal{T}^-_{(P_2)_{\Gamma_2}}$, and so we obtain $|\mathcal{T}^-_{(P_2)_\Lambda}| = 14$ and $|\mathcal{T}^+_{(P_2)_\Lambda}| = 18$. (Note that $\mathcal{T}^-_{(P_2)_\Lambda} \overset{\text{anti}}{\simeq} \mathcal{T}^+_{(P_3)_\Lambda}$; so, $|\mathcal{T}^-_{(P_3)_\Lambda}| = 18$ and $|\mathcal{T}^+_{(P_3)_\Lambda}| = 14$.) When U_3 is the left mutation of Γ_2 with respect to P_3, the endomorphism algebra of U_3 is isomorphic to Λ. This says that a derived equivalence does not necessarily preserve the number of $\mathsf{2silt}\,(-)$, although Γ_2 is symmetric and admits an anti-automorphism fixing all vertices.*

There are some special derived equivalence classes of algebras for which the cardinalities of $2\,\mathsf{silt}\,(-)$ are constant, but the proofs are case by case for each algebra. Using Theorem 2.14, we may give an explicit example of such classes.

Example 2.16. *Let Λ be the multiplicity-free Brauer triangle algebra; that is, it is given by the quiver with relations as follows.*

$$\begin{tikzcd} & 1 \arrow[dl, shift right, "a"'] \arrow[dr, shift left, "a^*"] & \\ 2 \arrow[ur, shift right, "a^*"'] \arrow[rr, shift right, "a"'] & & 3 \arrow[ll, shift right, "a^*"'] \arrow[ul, shift left, "a"] \end{tikzcd} \qquad \begin{cases} aa^* = a^*a \\ a^2 = 0 = (a^*)^2 \end{cases}$$

We see that Λ admits an anti-automorphism fixing every vertex; cf. Theorem 2.7.

Let $P := e_1\Lambda$ and Γ denote the endomorphism algebra of the left mutation $\mu^-_P(\Lambda)$; note that $\mu^-_P(\Lambda)$ is a tilting object in $\mathcal{K}_\Lambda$, and so Λ and Γ are derived equivalent. By hand, we obtain that Γ is presented by the quiver with relations:

$$\begin{tikzcd} 2 \arrow[r, shift left, "a"] & 1 \arrow[l, shift left, "a^*"] \arrow[r, shift left, "b^*"] & 3 \arrow[l, shift left, "b"] \end{tikzcd} \qquad \begin{cases} aa^* = 0 = bb^* \\ a^*ab^*b = b^*ba^*a \end{cases}$$

Observe that Γ admits an anti-automorphism fixing all vertices.

Thus, it turns out by Theorem 2.14 that $\mathsf{2silt}\,\Lambda$ *and* $\mathsf{2silt}\,\Gamma$ *have the same cardinality; actually, they are finite sets and the numbers are 32. See* $D(3K)$ *and* $D(3A)_1$ *in Table 1 of* [9]*. Moreover, the class* $\{\Lambda, \Gamma\}$ *forms a derived equivalence class.*

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