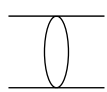
11
Reggeon branchings

We considered strong interactions in the framework of the Regge poles; this picture applies, by the way, to weak and electromagnetic interactions as well.

It turns out, however, that the Regge poles are not enough to describe consistently the high-energy behaviour of scattering amplitudes. To see why and how more complicated singularities – branch cuts – emerge in the ℓ-plane, let us recall, what we learned about the possible energy dependence of two-particle scattering amplitudes at various t.

At $t > 0$ the elastic amplitude may grow as a power $s^{\alpha(t)}$ with $\alpha(t) > 1$. What do we know about the vacuum trajectory $\alpha(t)$? Below the t-channel threshold, $t = 4\mu^2$, the trajectory is real and decreases with t decreasing. Moreover, the Froissart theorem taught us that at $t = 0$ the ampitude cannot grow with energy faster than

$A(s, t = 0) \;=$  $< s \ln^2 s.$

What happens at negative t? In non-relativistic quantum mechanics $\alpha(t)$ passes through integer points $-1, -2, \ldots$ and decreases indefinitely with t decreasing. Thus, in quantum mechanics we see no restrictions on the rate of the energy falloff of the amplitude: for large angles, $|t| \sim s$, it can fall arbitrarily fast with s increasing.

How about relativistic theory?

Recall that in fact we have already faced some difficulty with the Regge-pole picture. Indeed, we expected that the corrections to the leading vacuum pole (pomeron, $\mathbf{P}$) will be coming from other Regge poles and will be relatively suppressed as a *power* of s. However, having analysed contributions to the total cross section of various particle production topologies, we found in the r.h.s. of the equation for $\operatorname{Im}\mathbf{P}$ *logarithmically* behaving

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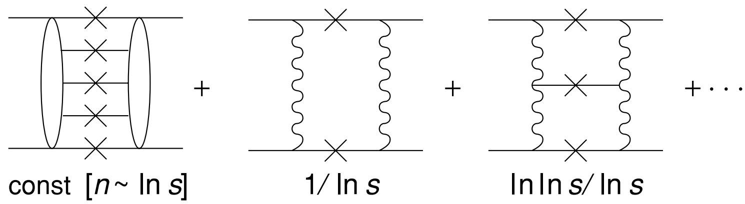
contributions due to production processes of a few particles with large rapidity intervals between them:



One cannot state a priori that these logarithmic terms will not sum up into a power. Still, this discrepancy is worrisome and worth bearing in mind.

Strangely enough, the apparent logarithmic nature of the corrections and the question of the possible falloff of the elastic amplitude turned out to be closely related.

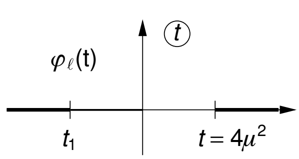
11.1 $\ell = -1$ and restriction on the amplitude falloff with energy

Let us return to the partial wave amplitude with positive signature:

$$\varphi_\ell(t) = \frac{2}{\pi}\int_{z_0}^{\infty} \frac{dz}{(t-4\mu^2)^\ell} Q_\ell(z) A_1(z,t).$$

We found simple analytic properties in t for $\varphi_\ell(t)$ at $t > 4\mu^2$; when $t < 0$, an additional complexity appeared:

$$\Delta\varphi_\ell = -\int_{z_0}^{-1} \frac{dz}{(t-4\mu^2)^\ell} P_\ell(z) A_1(z,t)\,; \qquad t_1 < t < 0.$$



Moving further down in t we hit the third spectral function, where the 'imaginary part' A_1 became itself complex (see Fig. 7.3 on page 167). Then the expression for the discontinuity changed:

$$\Delta\varphi_\ell = -\int_{z_0}^{-1} \frac{dz}{(t-4\mu^2)^\ell} P_\ell(z) A_1^*(z,t) + \frac{2}{\pi}\int_{z_1}^{z_2} \frac{dz}{(t-4\mu^2)^\ell} Q_\ell(z)\rho_{su}. \tag{11.1}$$

11.1.1 *Partial waves have poles at $\ell = -n$*

At the first glance the additional term that appeared in the r.h.s. of (11.1) looks innocent. Indeed, although Q_ℓ has *poles* at negative integer $\ell = -n$,

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$n > 0$, in the expression for the amplitude A,

$$A(s,t) = \frac{i}{4}\int \frac{d\ell}{\sin \pi\ell}\varphi_\ell(t)\cdot(t-4\mu^2)^\ell[P_\ell(-z)+P_\ell(z)], \tag{11.2}$$

the Legendre function P_ℓ has *zeros* in the same points,

$$P_\ell(z) \simeq \frac{\Gamma(\ell+\frac{1}{2})}{\sqrt{\pi}\Gamma(\ell+1)}(2z)^\ell, \quad |z|\to\infty. \tag{11.3}$$

In fact, if we chose the Mellin transformation instead of the Legendre one (with z^ℓ in place of P_ℓ), the poles would not appear at all. So these poles look to be artefacts of the passage from the amplitude $A(s,t)$ to the partial wave, $\varphi_\ell(t)$. It would have been the case if not for the t-channel unitarity: recall that the Legendre transformation was special in the sense that it diagonalized the *two-particle unitarity condition*.

11.1.2 Such poles contradict unitarity

In the interval between two- and four-pion thresholds, $4\mu^2 < t < 16\mu^2$, the partial wave is bounded from above by the unitarity condition:

$$\Delta\varphi_\ell = \rho_\ell\,\varphi_\ell\varphi_\ell^* \qquad \Longrightarrow \qquad |\varphi_\ell| < \frac{1}{2\rho_\ell}. \tag{11.4}$$

How could its discontinuity then become infinite?

In order to appreciate that it is not easy for the amplitude to do so, let us turn to the dispersion relation

$$\varphi_\ell(t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt'}{t'-t}\,\mathrm{Im}\,\varphi_\ell(t') + \frac{1}{\pi}\int_{-\infty}^{t_0}\frac{dt'}{t'-t}\delta\varphi_\ell(t'). \tag{11.5}$$

If $\delta\varphi_\ell\to\infty$ at some ℓ, the same is true for $\varphi_\ell(t)$ together with its discontinuity (unless there are special cancellations in the second integral in (11.5)).

Note that the contribution of the right cut (the first integral) is finite. Moreover, it could not possibly cancel the contribution of the left cut identically in t, since their analytic properties are different.

What could be a way out?

(1) In principle, the integral of $\rho(s,u)$ could turn to zero at $\ell = -n$. This is, however, not the case, since the double spectral function is positively definite, at least near the edge of the hyperbola.

(2) Taking into account multiparticle unitarity conditions would not help us either since, once again,the contribution coming from the

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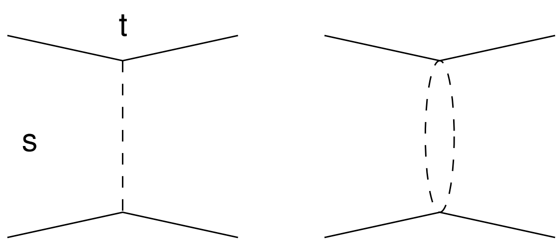
next threshold at $t = 16\mu^2$ has specific analytic features, and therefore it cannot cancel the pole identically, at arbitrary t, in the interval $4\mu^2 < t < 16\mu^2$.

Thus, the pole is there. Although the amplitude A itself is finite, the corresponding partial wave φ acquires the pole which would have been but an artefact of rewriting (P_ℓ in place of z^ℓ) if not for the *unitarity condition*.

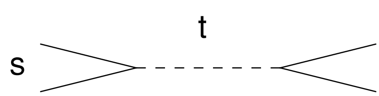
How could we resolve this real contradiction with unitarity?

11.1.3 *Condensing poles (quantum-mechanical analogy)*

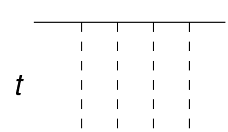
Let us recall the meaning of the singularities of the amplitude. As we have discussed in Lecture 7, singularities of t-channel particle exchange amplitudes determined the s-channel 'potential',



Correspondingly, singularities in s (and u)



have to be understood as determining the interaction potential in the t-channel. The existence of the third spectral function ρ_{su} reflects the relativistic nature of the t-channel 'potential'. It is because of ρ_{su} that our 'potential' becomes infinitely large when $\ell \to -1$.



In NQM, potential scattering in the t-channel is described by the diagrams with successive particle exchange. The lines do not cross since the non-relativistic interaction is instantaneous; there is no retardation. There is only elastic scattering so at positive t we will have a single cut starting from $t = 4\mu^2$. The partial wave $f_\ell(t)$ will have a left cut too, with a rather complicated structure, reflecting multiple singularities of the amplitude in s, mimicking the 'potential'. However, the term with Q_ℓ in $\Delta\varphi_\ell$ will be absent since in non-relativistic quantum mechanics $\rho_{su} \equiv 0$, so that the pole in ℓ will be absent as well. In spite of this, we shall stick to NQM and try to imitate the catastrophic

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growth of the contribution of the left cut in the $\ell \to -1$ limit simply by increasing the magnitude of the non-relativistic potential. In so doing, the contradiction has to disappear, since a potential in quantum mechanics is arbitrary, and the answer must remain reasonable.

When V increases (attraction), sooner or later a level (bound state) appears from beneath the right cut. With the increase of the potential the number of levels and their binding energies are growing. In general, to the expression (11.5) for φ_ℓ a sum over bound states has to be added:

$$\varphi_\ell = \frac{1}{\pi}\int \mathrm{Im}\,\varphi_\ell \;+\; \frac{1}{\pi}\int \delta\varphi_\ell \;+\; \sum \frac{C_n}{t_n - t}\,.$$

It is this sum that compensates the growing contribution of the left cut ($\delta\varphi_\ell$) while the contribution of the right cut ($\mathrm{Im}\,\varphi_\ell$) and φ_ℓ itself stay finite. In the limit $V \to \infty$ the number of poles becomes infinite, and they eventually fill the t-axis below the threshold ($t < t_0$). Replacing the sum over poles by an integral,

$$\sum_n \frac{C_n}{t_n - t} \;\stackrel{V\to\infty}{\longrightarrow}\; \int_{-\infty}^{t_0} \frac{\chi(t')\,dt'}{t' - t}\,,$$

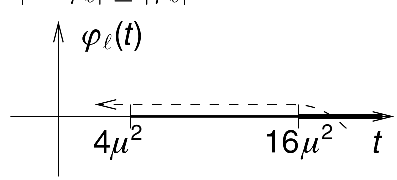
we will have, in the main part, $\chi(t') \approx -\delta\varphi(t')$ as $V \to \infty$.

This would have been the solution of the problem of infinitely large potential (which is our model for the singularity at $\ell \to -1$ of the left-cut contribution) if there were no *multiparticle thresholds* and related cuts.

11.1.4 Another possibility: moving branch points

In the relativistic theory there are inelastic sheets. Their existence provides another way out of the contradiction with unitarity. Instead of accumulating infinitely many poles, inelastic sheets may allow us to change the very analytic properties of the partial wave φ_ℓ instead. Indeed, we have arrived at the conclusion that the partial wave is bounded from above by equating in (11.4) its *discontinuity* $\Delta\varphi_\ell$ with the *imaginary part*, $\mathrm{Im}\,\varphi_\ell$:

$$\begin{aligned} \Delta_t\varphi_\ell(t) &= \rho_\ell|\varphi_\ell(t)|^2; \\ \Delta_t\varphi_\ell(t) &= \mathrm{Im}\,\varphi_\ell(t) \quad\Longrightarrow\quad |\mathrm{Im}\,\varphi_\ell| \le |\varphi_\ell| < \text{const.} \end{aligned} \tag{11.6}$$



Imagine now that from beneath a multiparticle sheet a *branch point* singularity emerged and moved to the left along the real axis. Then in the two-particle threshold region $4\mu^2 < t < 16\mu^2$ the discontinuity does not coincide with the imaginary part anymore, $\Delta\varphi \neq \mathrm{Im}\,\varphi$, and no restriction on the magnitude of φ_ℓ would follow from (11.6).

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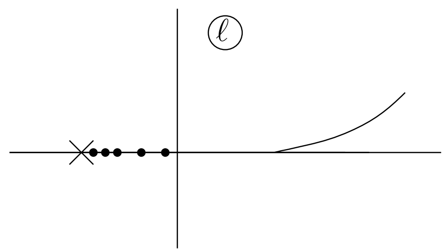
We shall explore this possibility later and will see that in the interesting cases it is moving branchings that resolve our contradiction. For the time being let us return to the first – pole – scenario: with $\ell \to -1$ we have more and more poles emerging on the physical sheet and passing through a given point t,

$$t_n(\ell) < t\,, \quad n = 1, 2, \ldots, N; \qquad N \to \infty \text{ with } \ell \to -1.$$

In the ℓ-plane this picture corresponds to the poles condensing towards $\ell = -1$ for any fixed t:



Thus, due to the unitarity condition, the *pole* in the partial wave φ_ℓ at $\ell = -1$ transforms into an *essential singularity.* I have demonstrated this, using quantum-mechanical analogy. Instead, one could just solve the integral equation for the unitarity condition (presuming that the inelastic cuts are not 'catastrophic') to come to the same conclusion rigourously.

From the fact that the point $\ell = -1$ is an essential singularity in the Sommerfeld–Watson integral,

$$A(s,t) \propto \int \frac{d\ell}{\sin \pi\ell} \varphi_\ell(t) \left[(-s)^\ell + (s)^\ell\right],$$

it immediately follows that for any t

$$|A(s,t)| > s^{-1-\varepsilon}\,, \quad \epsilon > 0. \tag{11.7}$$

We conclude that the amplitude $A(s,t)$ cannot decrease faster than $1/s$.

11.1.5 The origin of the $\ell = -1$ singularity

Let us have a closer look: how did it happen that $\ell = -1$ turned out to be a singular point and what was the rôle of the third spectral function ρ_{su} in this.

Consider the original amplitude $A(s,t)$ at negative t-values. Since I am interested in the possibility of A decreasing, I can write the dispersion relation without subtractions:

$$A(s,t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{ds'}{s'-s} A_1(s',t) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{du'}{u'-u} A_2(u',t). \tag{11.8}$$

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Let us consider a 'good' amplitude such that

$$A_1,\ A_2,\ A_3 < \frac{1}{s}.$$

For $t > t_1$ the imaginary part $\operatorname{Im} A$ was due to $(i\varepsilon)$ in the denominators in (11.8): $\operatorname{Im} A = A_1$ for $s > 4\mu^2$ ($\operatorname{Im} A = A_2$ for $u > 4\mu^2$).

Decreasing t, we reach t_1 where A_1 and A_2 become complex and our amplitude seems to acquire an additional complexity

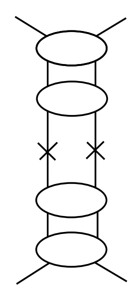
$$\delta \operatorname{Im} A(s,t) = \frac{1}{\pi}\int \frac{ds'}{s'-s}\rho_{su}(s',t) + \frac{1}{\pi}\int \frac{du'}{u'-u}\rho_{su}(u',t). \tag{11.9}$$

How could this be? Isn't A_1, *by definition*, the full imaginary part of the amplitude in the physical region of the s-channel? There is no contradiction, of course, since $(s'-s)+(u'-u)=0$.

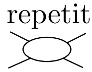
At the same time we observe that, taken separately, neither of the contributions of the right and left cuts decreases faster than $1/s$:

$$\operatorname{Im} A_{\text{right}} = \frac{1}{\pi}\int \frac{ds'}{s'-s}\rho_{su}(s') \;=\; -\frac{1}{\pi}\int \frac{ds'\rho_{su}(s')}{s} \;\sim\; \frac{1}{s}. \tag{11.10}$$

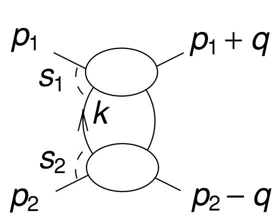
Let us recall that we *were forced* to treat separately the contributions of the right and left cuts when we continued partial waves φ_ℓ onto the complex ℓ-plane. Hence, the nature of the pole in Q_ℓ is related to the $1/s$ falloff of the contribution of each cut.



The most important point remains to be understood. Namely, where did the essential singularity come from, when the pole itself seemed to be of "kinematical" nature? It is t-channel unitarity which is responsible.

Unitarity means that repetitions are needed. Let us take a scattering block $f =$  (which does not include a two-particle intermediate state in the t channel) and see how the amplitude will behave when we start repeating it in the t-channel. We are going to demonstrate that the contributions of the right and left cuts of the block f enter separately, *one by one*, the asymptotics of the iterated amplitude.

Consider the high-energy limit of the amplitude

$A \;=$ 

In terms of the Sudakov vectors,

$$k = \alpha p_1 + \beta p_2 + \mathbf{k}_\perp \qquad (p_+ \simeq p_1,\ p_- \simeq p_2),$$

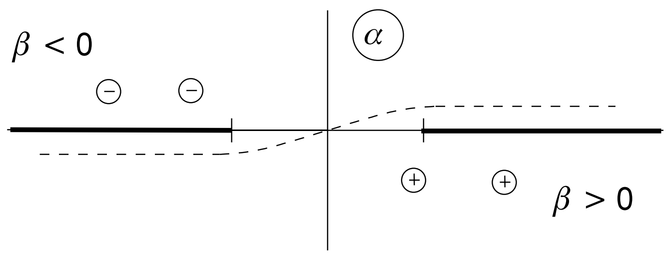
we may write

$$A \sim \int \frac{d^4k}{(2\pi)^4 i} \frac{f(s_1; k_\perp, q_\perp) f(s_2; k_\perp, q_\perp)}{[m^2 - \alpha\beta s + \mathbf{k}_\perp^2 - i\varepsilon]\,[m^2 - \alpha\beta s + (\mathbf{q}-\mathbf{k})_\perp^2 - i\varepsilon]}. \quad (11.11)$$

For simplicity we omitted in the second propagator, $(\alpha-\alpha_q)(\beta-\beta_q)$, the small longitudinal Sudakov components of the total momentum transfer, $\alpha_q, \beta_q \sim m^2/s$; as always, we consider small momentum transfer $-q^2 \simeq \mathbf{q}_\perp^2 = \mathcal{O}(m^2)$. (In fact we have already analysed this integral when we calculated the box diagram in Section 9.2.3, see page 224.) The block energies are

$$s_1 \simeq 2p_1 k \simeq \beta s, \quad s_2 \simeq -2p_2 k \simeq -\alpha s.$$

Consider first the integral over α. In the α plane we have cuts of the lower block amplitude $f(s_2)$,



Depending on the sign of β, the poles of the propagators in α are either both below ($\beta > 0$) or above ($\beta < 0$) the real axis. If $\beta > 0$, the contour can be closed on the left, that is around the *right* cut in the invariant energy $s_2 = -\alpha s$ of the lower block:

$$A = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} \int_0^\infty d\beta f(\beta s) \int_{s_0}^\infty \frac{ds_2}{2\pi} \frac{\operatorname{Im} f_{\text{right}}(s_2)}{[\ \ ][\ \ ]} + \int_{\beta<0} \{\operatorname{Im} f_{\text{left}}(s_2)\}.$$

Each integral is complex due to a (right/left) cut of $f(\beta s)$. Evaluating the imaginary part we have

$$\operatorname{Im} A = \int \frac{d^2\mathbf{k}_\perp}{2\pi\, s} \int_{s_0}^\infty \frac{ds_1}{2\pi} \int_{s_0}^\infty \frac{ds_2}{2\pi} \frac{\operatorname{Im} f(s_1) \operatorname{Im} f(s_2)}{[\ \ ][\ \ ]} + \left\{ \begin{array}{ll} s_1 & \to -s_1 \\ s_2 & \to -s_2 \end{array} \right\}.$$

While the integrand of A in (11.11) contains the full block amplitudes, symbolically,

$$A \sim f \otimes f \sim (f_{\text{right}} + f_{\text{left}}) \otimes (f_{\text{right}} + f_{\text{left}}), \quad (11.12a)$$

in the imaginary part the cross-terms between right and left cuts cancel:

$$\operatorname{Im} A \sim \frac{1}{s\,m^2}\left[\left(\int ds' \operatorname{Im} f_{\text{right}}(s')\right)^2 + \left(\int ds' \operatorname{Im} f_{\text{left}}(s')\right)^2\right]. \quad (11.12\text{b})$$

We see that although each block amplitude in (11.12a) may be falling fast with energy due to cancellation between its two cuts, the imaginary part of the *iterated* amplitude (and therefore $A(s)$ itself) cannot decrease faster than $1/s$.

The expectation that an amplitude can fall arbitrarily fast owing to the cancellation between right- and left-cut contributions (each of which falls as $1/s$) turns out to be incorrect: it does not stand confrontation with the t-channel unitarity relation.

To understand better what is so special about the point $\ell = -1$ in the unitarity relation, let us return to the representation (11.2) for the amplitude and redefine, once more, the partial wave by embedding into it the ℓ-dependent normalization factor from the asymptotic expression for the Legendre function (11.3):

$$A(s,t) \simeq \frac{i}{4}\int \frac{d\ell}{\sin \pi\ell}\,\psi_\ell(t)\cdot\left[(-s)^\ell + s^\ell\right], \quad s\to\infty; \quad (11.13\text{a})$$

$$\psi_\ell(t) = \frac{\Gamma(\ell+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\ell+1)}\cdot\varphi_\ell(t). \quad (11.13\text{b})$$

Now that (11.13a) is 'clean', the normalization will reappear in the unitarity relation. Expressed in terms of new partial waves ψ_ℓ it now reads

$$\begin{gathered}\Delta\psi_\ell(t) = \rho_\ell(t)\cdot\psi_\ell(t)(\psi_\ell(t))^*, \quad 4\mu^2 < t < 16\mu^2,\\ \rho_\ell(t) = \frac{\sqrt{\pi}\,\Gamma(\ell+1)}{\Gamma(\ell+\frac{1}{2})}C_\ell(t), \qquad C_\ell = \frac{1}{8\pi}\frac{k_t^{2\ell+1}}{\sqrt{t}},\end{gathered} \quad (11.14)$$

with C_ℓ given in (7.30). The factor ρ_ℓ can be looked upon as the phase space volume, continued to non-integer ℓ. We see that at negative integer values of ℓ it turns to infinity, making them singular points for the unitarity condition.

Conclusion: if there are only poles,

(1) the amplitude cannot fall faster than s^{-1}; and

(2) in the ℓ-plane the poles get closer and closer to each other as they approach $\ell = -1$.

This itself does not contradict the Regge-pole hypothesis, but in fact, as we will see now, the solution lies elsewhere.

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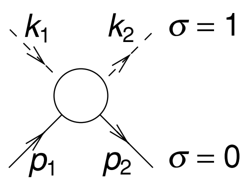
11.2 Scattering of particles with non-zero spin

Up to now, we considered spinless particles; including spins does not modify essentially our previous considerations. The conclusion, however, turns out to be far more dramatic.

11.2.1 Energy behaviour of scattering amplitudes

Consider the scattering of a vector and a scalar particles:



$$= A_{\mu\nu}(p_1 + p_2, k_1, k_2).$$

The amplitude now carries vector indices and is built of Lorentz tensors,

$$A^{\mu\nu} \; = \; A_0 p^\mu p^\nu + A_1 g^{\mu\nu} + A_2(p^\mu k_1^\nu - k_2^\mu p_\nu) + \cdots \quad (p \equiv p_1 + p_2)$$

(we have taken into account the symmetry with respect to $k_1 \leftrightarrow -k_2$). Now we have to write dispersion relations for each invariant amplitude A_i (one has only to be careful not to include artificial singularities which may emerge while rewriting momenta in terms of each other). Repeating the above analysis, we would obtain again

$$A_0^{\text{right}} \sim A_0^{\text{left}} \sim s^{-1}.$$

Let us see, e.g. what gives the *longitudinal polarization* vector,

$$e_\mu^{(0)}(k) = (k_z; k_0, \mathbf{0}_\perp)\frac{1}{\sqrt{k^2}},$$

for the invariant matrix element

$$M^{\lambda_1\lambda_2} = e_{\mu_1}^{\lambda_1} e_{\mu_2}^{\lambda_2} A^{\mu_1\mu_2}.$$

For forward scattering, $k_1 = k_2$, in the laboratory frame $(p_1 = p_2 = (m_0; 0, \mathbf{0}_\perp))$ we obtain

$$M^{00} = (A_0 p^\mu p^\nu + \cdots) e_\nu^0 e_\mu^0 \sim A_0 \left(2m_0 \frac{k_{1z}}{m_1}\right)^2 \simeq A_0 \frac{s^2}{m_1^2}. \qquad (11.15)$$

This means that considering, as before, contributions of each cut, M_{left}^{00} and M_{right}^{00}, separately, we will come to the conclusion that the scalar–vector scattering amplitude must *grow* in the large-s limit at least as

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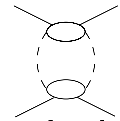


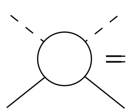
$M \gtrsim s$! It is easy to arrive at the same conclusion analysing an interaction of spinless particles via vector particle exchange, $A = \mathcal{O}(s)$. That is, introducing particles with spin σ and following the same path, we can prove that the amplitude cannot be smaller than

$$A > s^{-1-\varepsilon+2\sigma}\,.$$

Already for $\sigma = 2$ this contradicts the Froissart bound. (In principle, some tricky cancellations cannot be excluded, but so far there are no indications for that.) One can interpret this conclusion as an observation of the contradictory character of the hypothesis of the existence of Regge poles *only*.

11.2.2 Azimov shift in terms of spiral amplitudes

This phenomenon known as 'the Azimov shift' can be explicitly seen in the t-channel as well. To parametrize the amplitude of the transition between two scalar and two vector particles, it is convenient to employ the formalism of spiral amplitudes. These are amplitudes with definite values of *helicities* of participating particles with $\sigma \neq 0$. Helicity is the projection of spin onto the direction of the particle momentum. In the cms of the t-channel, momenta of two vector particles are opposite, so that the *difference* of their helicities represents the projection of the total spin. The generalization of the t-channel partial-wave expansion reads



$$= A_{\lambda_1\lambda_2} = \sum_j (2j+1) Y_{j\lambda}(\theta,\varphi) f_{j\lambda}(t); \quad \lambda = \lambda_1 - \lambda_2. \tag{11.16}$$

The total angular momentum vector, $\mathbf{j} = \boldsymbol{\ell} + \boldsymbol{\sigma}$, is a sum of the orbital momentum $\boldsymbol{\ell}$ and the total spin $\boldsymbol{\sigma}$. Since $\boldsymbol{\ell} = [\mathbf{r} \times \mathbf{p}]$ is orthogonal to the direction of the cms momentum $\mathbf{z}$, the helicity parameter λ in (11.16) equals $\lambda = \sigma_z = j_z$. Physically, the angular momentum projection is restricted, $|j_z| \leq j$. Indeed, the boundary $|\lambda| \leq j$ is contained in the normalization of the *spherical harmonics*:

$$Y_{j\lambda}(\theta,\varphi) \equiv \sqrt{\frac{\Gamma(j-\lambda+1)}{\Gamma(j+\lambda+1)}} P_{j\lambda}(z)\, \mathrm{e}^{i\lambda\varphi}, \quad z = \cos\theta, \tag{11.17a}$$

where $P_{j\lambda}$ are the associated Legendre functions. At large $|z|$ we have

$$Y_{j\lambda} \simeq \frac{\Gamma(j+\frac{1}{2})}{\sqrt{\pi\Gamma(j+\lambda+1)\Gamma(j-\lambda+1)}} (2z)^j, \tag{11.17b}$$

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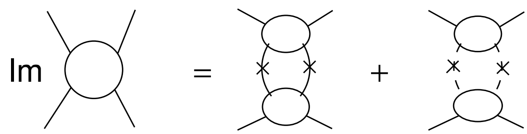
where we have dropped the trivial dependence on the azimuth angle by setting $\varphi = 0$. We see that indeed, owing to the Γ factors, $Y = 0$ for $|\lambda| > j$. Mark that for $\lambda = 0$ the normalization factor in (11.17b) coincides with that of the spinless case, (11.13b).

The continuation of the series (11.16) to complex j with the help of the Sommerfeld–Watson integral does not pose difficulties:

$$A_\lambda(s,t) \simeq \frac{i}{4}\int \frac{dj}{\sin \pi j}\psi_{j\lambda}(t)\cdot\left[(-s)^j + s^j\right], \quad s \to \infty; \tag{11.18a}$$

$$\psi_{j\lambda}(t) = \frac{\Gamma(j+\frac{1}{2})}{\sqrt{\pi\Gamma(j+\lambda+1)\Gamma(j-\lambda+1)}}\cdot k_t^{-2j}\cdot f_{j\lambda}(t). \tag{11.18b}$$

The changes will affect the unitarity condition.



The first term in the r.h.s. with exchange of scalar particles will stay as before, so we concentrate on the new contribution describing particles with spins σ_1 and σ_2 in the intermediate state:

$$\Delta f_j(t) = \cdots + \sum_\lambda \tau(t) f_{j\lambda}(t+i0) f_{j\lambda}(t-i0).$$

The sum over λ has emerged since for a given j we can still have different helicities in the intermediate state. The sum runs up to $\lambda_{\max} = \sigma_1 + \sigma_2$. Due to conservation of parity, it is sufficient to sum over $\lambda \geq 0$, doubling the contributions of $\lambda \geq 1$.

Let us express the unitarity relation in terms of the partial waves ψ_j and $\psi_{j\lambda}$ defined by (11.13b) and (11.18b), correspondingly. Collecting the normalization factors, we get

$$\Delta\psi_j = \sum_\lambda^{\sigma_1+\sigma_2} C_j(t)\,\frac{\sqrt{\pi}\,\Gamma(j-\lambda+1)\Gamma(j+\lambda+1)}{\Gamma(j+1)\Gamma(j+\frac{1}{2})}\,\psi_{j\lambda}\psi^*_{j\lambda}. \tag{11.19}$$

In the spinless case, $\sigma_1 = \sigma_2 = 0$, we recover (11.14).

Decreasing j, we hit the first pole at $j = \sigma_1 + \sigma_2 - 1$,

$$\Delta\psi_j \propto \psi_{j\lambda}\frac{1}{j+1-\sigma_1-\sigma_2}\psi^*_{j\lambda}, \qquad \lambda = \sigma_1 + \sigma_2. \tag{11.20}$$

In the first unphysical point in j (when the angular momentum is taken *smaller* than its projection), $j = \lambda - 1$, the phase space volume of the intermediate state becomes infinite. As we have discussed, this results in the lower limit of the high-energy behaviour of the scattering amplitude,

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$A(s) \propto s^{\sigma_1+\sigma_2-1}$. Taking vector particles in the intermediate state, $\sigma_1 = \sigma_2 = 1$, we obtain $A \propto s$, the result that we have explicitly derived before. In general, (11.20) permits *elementary particles* to have spins 0, $\frac{1}{2}$ and 1, and no more.

What to do with particles with higher spins which exist, can be produced in a t-channel reaction and therefore participate in the unitarity relation? The problem was solved by Mandelstam. It turns out that when a *reggeized* particle is present in the intermediate state, a new singularity – a moving branch cut – appears from an unphysical sheet related to a multi-particle threshold, which *exactly compensates* the pole (11.20).

Imagine that we have a particle with spin σ which is a *bound state* of two scalar particles (with mass μ) and lies on the Regge trajectory $\alpha(t)$. Let m denote its mass, $\alpha(m^2) = \sigma$.

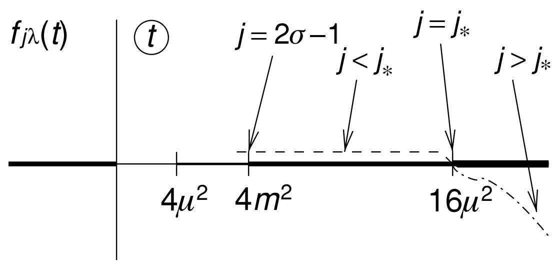
As we will demonstrate shortly, the position of the Mandelstam branch cut derives from the trajectory of the pole and reads

$$j \;=\; j_2(t) = 2\alpha\left(\frac{t}{4}\right) - 1. \tag{11.21}$$

The unitarity condition (11.20) holds above the two-particle threshold, $t \geq 4m^2$. Let us show that the contribution of this branching to the unitarity relation,

$$\Delta\psi_j(t) \;=\; \psi_{j,2\sigma}\frac{c}{j+1-2\sigma}\psi^*_{j,2\sigma} + \delta_{\text{branch}}\psi_j(t),$$

fully screens the unphysical pole due to exchange of two particles.



The trajectory $\alpha(t)$ is complex above the two-particle threshold, $t > 4\mu^2$. Therefore $\operatorname{Im} j_2(t) > 0$ for $t > 16\mu^2$. This means that for real j larger than $j_* \equiv 2\alpha(4\mu^2) - 1$, the position $t_2(j)$ of the branch singularity (11.21) in the t-plane is complex. The branching then is on the unphysical sheet, since complex singularities on the physical one are forbidden by causality. It emerges on the physical sheet through the tip of the four-particle threshold, $t = (4\mu)^2$, and moves to the left with j decreasing. When we approach the troubling point $j = 2\sigma - 1$, the branching arrives precisely

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to $t = 4m^2$,

$$j = 2\alpha\left(\frac{t}{4}\right) - 1 \;\to\; 2\sigma - 1 \qquad \Longrightarrow \qquad \alpha\left(\frac{t}{4}\right) \to \sigma = \alpha(m^2),$$

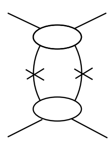
covers fully the two-particle cut due to the $\sigma\sigma$ exchange and cancels the unphysical pole at $j < |j_z|$.

At the same time, in the *physical* integer values of j the branch cut contribution *vanishes*, leaving the two-particle exchange unperturbed. So, the Mandelstam branching is rather sophisticated. It is interesting that all this occurs automatically as soon as we suppose that the particle is reggeized.

11.2.3 A model for a moving branch-point singularity

Before we turn to the derivation of the reggeon branchings, let us try to guess the answer. Near the pole due to the exchange of particles with spins σ_1 and σ_2, the unitarity relation has the form

$$2\,\mathrm{Im}\, f_j =$$



$$+ \;\; \tau(t)\frac{c}{j+1-\sigma_1-\sigma_2}\tilde{f}_{j\sigma}\tilde{f}^*_{j\sigma}, \tag{11.22}$$

where we have extracted the singularity from the partial wave amplitudes $f_{j\sigma}$ describing the $0+0 \to \sigma_1 + \sigma_2$ transition, $f_{j\sigma} \to \tilde{f}_{j\sigma}$. It contains the phase space volume $\tau(t)$ which for particles with masses m_1 and m_2 reads

$$\tau(t, m_1^2, m_2^2) = \frac{k_c}{16\pi\omega_c} = \frac{\sqrt{t^2 - 2t(m_1^2+m_2^2) + (m_1^2-m_2^2)^2}}{16\pi\sqrt{t}}. \tag{11.23}$$

When the particles are reggeized, $\sigma_i = \sigma_i(m_i^2)$, their masses become 'variable', suggesting to include into (11.22) integrals over masses, $t_1 = m_1^2$ and $t_2 = m_2^2$:

$$\int \frac{dt_1\, dt_2\, \tau(t, t_1, t_2)}{j+1-\sigma_1(t_1)-\sigma_2(t_2)}\tilde{f}_{j\sigma}\tilde{f}^*_{j\sigma}. \tag{11.24}$$

Given the additional integrations, (11.24) would no longer be a pole but a branch cut in j. Adding the propagators of reggeized particles, either $[\sin\frac{\pi}{2}\sigma_i(t_i)]^{-1}$, or $[\cos\frac{\pi}{2}\sigma_i(t_i)]^{-1}$, depending on the signature, we would get a natural model of a moving branching. Included into the unitarity relation (11.22), this expression could compensate the 'elementary' pole.

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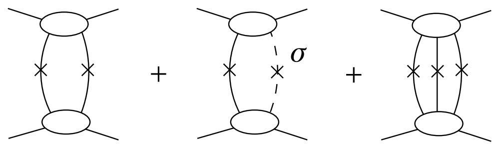
11.3 Multiparticle unitarity and Mandelstam singularities

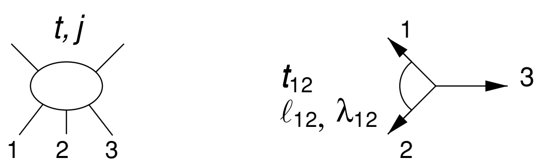
To understand how the reggeon branchings appear, we have to study for the first time multi-particle unitarity conditions.

11.3.1 Three-particle unitarity condition for partial waves

Let us consider the simplest example of $2 \to 2$ scattering of scalar particles above the three-particle threshold, $9\mu^2 < t < 16\mu^2$.

$2\,\mathrm{Im}\, f_j =$  (11.25)

We suppose that a particle with spin σ shown by the dashed line is a bound state of two scalar ones and can be produced in the intermediate state together with a scalar. First of all, let us write the three-particle term in the unitarity condition. We can describe a three-particle system in two steps. First we group two particles and treat them as a composite object with an invariant mass $t_{12} = (k_1 + k_2)^2$ and an internal orbital momentum $\ell = \ell_{12}$ and its projection, $\lambda = \lambda_{12}$. Then we combine the pair, which looks as a particle with an arbitrary 'mass' and 'spin', with the remaining scalar particle k_3 into the system with the total angular momentum j and energy t. Such a representation can be rigorously derived by parameterizing by Euler angles the internal geometry and the orientation of the plane formed by three particles in the t-channel cms.



This makes five independent variables characterizing a five-point amplitude, The structure of the three-particle term is rather similar to that of the case of the unitarity condition with spin,

$$\Delta f_j^{(3)} = \sum_{\ell,\lambda} \int dt_{12}\, K(t, t_{12}) f_{j\ell\lambda}(t, t_{12}) f^*_{j\ell\lambda}(t, t_{12}), \tag{11.26a}$$

with the only difference that now the 'mass' t_{12} of the composite particle (12) varies in the interval

$$4\mu^2 < t_{12} < (\sqrt{t} - \mu)^2. \tag{11.26b}$$

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The function K in (11.26a) is given by the product of the phase space volume functions (11.23),

$$K(t, t_{12}) = \tau(t, t_{12}, \mu^2)\tau(t_{12}, \mu^2, \mu^2). \tag{11.26c}$$

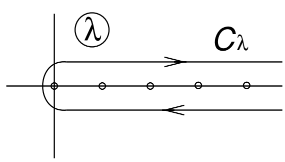
Introducing the partial waves ψ_j and $\psi_{j\lambda}$ in order to extract the angular momentum singularities as before, we get

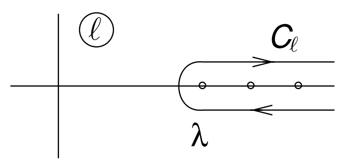
$$\Delta\psi_j^{(3)} = \sum_{\ell,\lambda} \int dt_{12}\, K(t, t_{12})\, \frac{c^2}{j+1-\lambda} \psi_{j\ell\lambda}\psi^*_{j\ell\lambda}, \tag{11.27}$$

where we have explicitly extracted the main singularity from the Γ factors, cf. (11.19). Unlike (11.19), the sum over λ in (11.27) is not limited from above since the 'spin' ℓ of the composite object (12) can be arbitrary. We must replace the sum by an integral; otherwise we have poles at arbitrarily large j and the continuation to complex j is impossible. Let us write

$$\Delta\psi_j^{(3)} = \int dt_{12} \frac{K(t, t_{12})}{(2i)^2} \int_{\mathcal{C}_\lambda} \frac{d\lambda}{\tan \pi\lambda} \int_{\mathcal{C}_\ell} \frac{d\ell}{\tan \pi(\ell-\lambda)} \frac{c^2\psi_{j\ell\lambda}\psi^*_{j\ell\lambda}}{j+1-\lambda}. \tag{11.28}$$

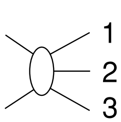
To correspond to the physical sum, $\sum_{\lambda=0}^{\infty}\sum_{\ell=\lambda}^{\infty}$, the contours have to be drawn as follows,



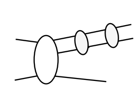


We will discuss the question of convergence of this representation later. In any case, as soon as the integration contours are deformed as in the standard Sommerfeld–Watson case, there are no problems with the poles at large j values anymore and the continuation can be carried out. Unfortunately, it is not unique. We could have added a function vanishing in integer points (for example, we could put sin instead of tan in the denominator). Let us suppose, nevertheless, that this continuation is reasonable and see what will be the structure of singularities in j. The pole is there in (11.28); it could not just disappear. However, some other singularities, not so apparent at the first glance, may emerge.

Let us examine the three-particle system in the intermediate state.



Particles 1 and 2 will interact in the final state and can give a resonance which will manifest itself as a pole in $\ell = \ell_{12}$ of the partial wave amplitude $f^{(2\to 3)}_{j\ell\lambda}$.



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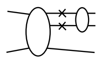
How can this possibility be extracted in a model-independent way?

The $2 \to 3$ amplitude has to satisfy the unitarity condition not only in t but also in t_{12}:

$$\Delta_{12} f^{(2\to3)}_{j\ell\lambda} \equiv \frac{1}{2i}[f^{(2\to3)}(t+i\varepsilon, t_{12}+i\varepsilon) - f^{(2\to3)}(t+i\varepsilon, t_{12}-i\varepsilon)]$$



$$= f^{(2\to3)}_{j\ell\lambda}(t^+, t_{12})\tau(t_{12}, \mu^2, \mu^2) f^{(2)*}_{\ell}(t_{12}). \quad (11.29)$$

In the region $4\mu^2 < t_{12} < 9\mu^2$ the two-particle unitarity condition in t_{12} is valid. This makes (11.29) a linear equation for $f^{(2\to3)}$. Its solution is simple:

$$f^{(2\to3)}_{j\ell\lambda} = G_{j\ell\lambda}(t, t_{12}) f^{(2)}_{\ell}(t_{12}),$$

where G does not have a two-particle cut in t_{12} (two-particle irreducible amplitude). Indeed, evaluating the discontinuity in t_{12} and using the two-particle unitarity condition for the elastic amplitude $f^{(2)}_{\ell}$ we have

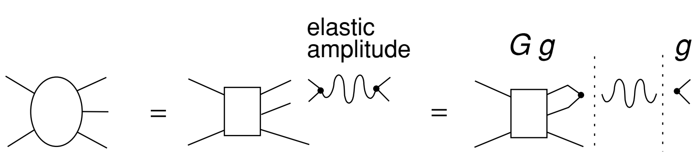
$$\Delta f^{(2\to3)} = G \cdot \Delta f^{(2)}_{\ell}(t_{12}) = G \cdot (f^{(2)}_{\ell}\, \tau\, f^{(2)*}_{\ell}) = f^{(2\to3)} \cdot \tau f^{(2)*}_{\ell},$$

which coincides with (11.29).

We know that $f^{(2)}_{\ell}$ can have a Regge pole,

$$f^{(2)}_{\ell}(t_{12}) \simeq \frac{g^2(t_{12})}{\ell - \alpha(t_{12})};$$
$$f^{(2\to3)}_{j\ell\lambda}(t, t_{12}) \simeq \Big[G_{j\ell\lambda}(t, t_{12}) g(t_{12})\Big] \cdot \frac{1}{\ell - \alpha(t_{12})} \cdot g(t_{12}). \quad (11.30)$$

This expression has a clear diagrammatical meaning:



with the combination $N = Gg$ playing the rôle of the reggeon production amplitude, and g – the amplitude of its decay.

Before we substitute the Regge pole (11.30) into the unitarity relation (11.28) let us make the following simplifying observation. When we single out the final state interaction,

$$\psi_{j\ell\lambda} \implies G_{j\ell\lambda} \cdot f^{(2)}_{\ell}(t_{12}),$$

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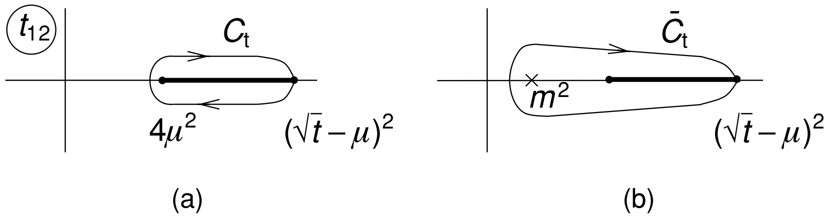


Fig. 11.1

the r.h.s. of (11.28) acquires the following structure,

$$K\psi_{j\ell\lambda}\psi^*_{j\ell\lambda} \to \tau(t,t_{12},\mu^2)GG^* \cdot \big(f_\ell^{(2)}\,\tau(t_{12},\mu^2,\mu^2)\,f_\ell^{(2)*}\big).$$

Here we have extracted from (11.26c) the phase space volume factor $\tau(t_{12})$ to form the *discontinuity* of the elastic scattering amplitude,

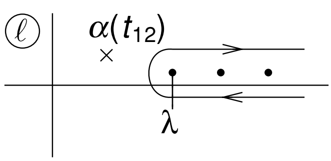
$$\begin{aligned} K\psi\psi^* &= \tau(t,t_{12},\mu^2)G\cdot\Big\{\Delta_{12}f_\ell^{(2)}(t_{12})\Big\}\cdot G^* \\ &= \Delta_{12}\Big\{\tau(t,t_{12},\mu^2)Gf_\ell^{(2)}(t_{12})G^*\Big\}. \end{aligned} \tag{11.31}$$

This allows us to replace the integral over t_{12} along the real interval (11.26b) by a contour integration as shown in Fig. 11.1(a) and write

$$\int_{4\mu^2}^{(\sqrt{t}-\mu)^2} dt_{12}\;\Delta_{12}\{\tau Gf_\ell^{(2)}(t_{12})G^*\} \Longrightarrow \int_{\mathcal{C}_t}\frac{dt_{12}}{2i}\{\tau Gf_\ell^{(2)}(t_{12})G^*\}.$$

Now we can substitute the Regge pole (11.30) in the elastic final state scattering amplitude,

$$\Delta\psi_j^{(3)} = \int_{\mathcal{C}_t}\frac{dt_{12}}{2i}\frac{\tau(t,t_{12},\mu^2)}{(2i)^2}\int_{\mathcal{C}_\lambda}\frac{d\lambda}{\tan\pi\lambda}\frac{1}{j+1-\lambda}\int_{\mathcal{C}_\ell}\frac{d\ell}{\tan\pi(\ell-\lambda)}\frac{N^2}{\ell-\alpha(t_{12})}.$$



First we look at the integral over ℓ. Since $t_{12} > 4\mu^2$, the trajectory $\alpha(t_{12})$ has an imaginary part. When we change λ, the integral becomes singular when λ hits the point $\alpha(t_{12})$ and the two poles pinch the contour:

$$\Delta\psi_j^{(3)} = \int_{\mathcal{C}_t}\frac{dt_{12}}{2i}\frac{\tau(t,t_{12},\mu^2)}{\tan\pi\alpha(t_{12})}\frac{N(t+)N(t-)}{j-\alpha(t_{12})+1}. \tag{11.32a}$$

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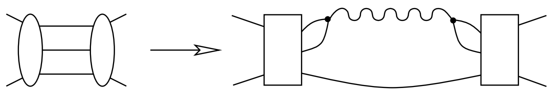
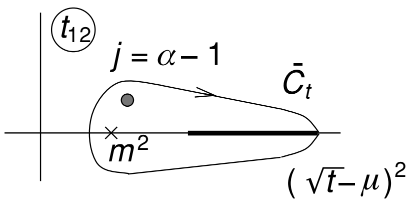
This is just our model for a branching singularity (11.24), only for one spinless particle, and one with spin,



This three-particle contribution has to be compared with the second term in the unitarity condition (11.25), due to the exchange of two particles, one of which, σ, we suppose to be a bound state belonging to the trajectory,

$$\Delta\psi_j^{(\sigma)} = \tau(t, m^2, \mu^2)\frac{\psi(t+)\,\psi(t-)}{j-\sigma+1}. \tag{11.32b}$$

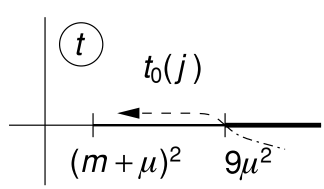
In fact, the latter is nothing but the *residue* of the former integral (11.32a) at the pole $\tan\pi\alpha = 0$, $\alpha(t_{12}) = \sigma$. Therefore, we can take into account both contributions (11.32) by simply modifying the integration contour to include the particle pole, as shown in Fig. 11.1(b). By derivation, the pole at $\alpha(t_{12}) = j+1$ lies *inside* the contour $\bar{\mathcal{C}}_t$. As a result, the point $j = \sigma - 1$ where the two poles collide, turns out to be a *regular* point in the j-plane: the three-particle singularity compensates the two-particle one.



The integral (11.32a) develops a singularity in j at $j = j_0$ such that the pole of the integrand hits the immobile endpoint of the integration contour:

$$t_{12} = (\sqrt{t} - \mu)^2,\ \ \alpha(t_{12}) = j+1 \quad\Longrightarrow\quad j_0(t) = \alpha((\sqrt{t}-\mu)^2) - 1. \tag{11.33}$$

This is the new branching singularity we are looking for. Actually, for t above the three-particle threshold, $t > (3\mu)^2$, the argument of α in (11.33) exceeds $(2\mu)^2$, so that the trajectory is complex, and the new singularity is hidden on the unphysical sheet beneath the three-particle unitary cut. For $t < 9\mu^2$ it emerges on the physical sheet. Its position in the t-plane, $t_0(j)$, moves on the left with j decreasing. If we take $j = \sigma - 1$, then from (11.33) follows $\sigma = \alpha((\sqrt{t} - \mu)^2)$, $t = (m+\mu)^2$, showing that the branching arrives at the tip of the two-particle threshold, to rescue the unitarity relation for the partial wave in spite of the '$\ell = -1$' singularity phenomenon.



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11.3.2 Four-particle unitarity

The analysis of the four-particle unitarity condition proceeds along the same lines. We group intermediate state particles into two pairs,

and get the singularity $1/((j+1-\lambda_1-\lambda_2)$ at the first unphysical value of their orbital angular momentum $L=-1$. Introducing Regge poles into the elastic final state rescattering amplitudes, we arrive at the integration over the pair masses,

$$\iint \frac{dt_1\, dt_2\, \tau(t,t_1,t_2)}{j-\alpha(t_1)-\alpha(t_2)+1}\, \frac{\psi(t_1,t_2)\, \psi^*(t_1,t_2)}{\tan\pi\alpha(t_1)\tan\pi\alpha(t_2)}. \tag{11.34}$$

To extract singularities from the double integral is more difficult. In fact, there can be many singularities. Among them there is, however, one that does not depend on specific features of the Regge pole trajectory and does not contain masses explicitly. Let us outline the main steps of the derivation of this singularity which is the one we are looking for.

The integral over t_2 is singular in t_1 when the pole collides with the endpoint $t_2 \le (t_2)_{\max} = (\sqrt{t}-\sqrt{t_1})^2$, as before:

$$\Longrightarrow \quad j = \alpha(t_1)+\alpha((\sqrt{t}-\sqrt{t_1})^2)-1. \tag{11.35}$$

This is a non-linear equation for t_1. A singularity in j will appear when two solutions of (11.35) coincide, pinching the integration contour in the t_1-plane. The condition for having a multiple zero,

$$0=\frac{d}{dt_1}\big[\alpha(t_1)+\alpha((\sqrt{t}-\sqrt{t_1})^2)\big] = \alpha'(t_1)-\alpha'((\sqrt{t}-\sqrt{t_1})^2)\cdot\frac{\sqrt{t}-\sqrt{t_1}}{\sqrt{t_1}},$$

has an obvious solution $\sqrt{t}-\sqrt{t_1}=\sqrt{t_1}=\frac{1}{2}\sqrt{t}$. Substituting it into (11.35) we get the branch singularity in the j-plane appearing at

$$j_2(t) \;=\; 2\alpha\left(\frac{t}{4}\right)-1. \tag{11.36}$$

Restoring the signature factors that we have ignored in the preceding discussion, the discontinuity across the branch cut of the partial wave amplitude can be derived,

$$\begin{aligned}\delta_j f_j^{(2)}(t) &= \frac{\pi}{2}\int \frac{dt_1\, dt_2}{2i}\tau(t,t_1,t_2)N_+^{(2)}\delta\big(j+1-\alpha_1-\alpha_2\big)N_-^{(2)}\bar{\xi}_j\,,\\ \bar{\xi}_j &= \xi_j\,\xi_{\alpha_1(t_1)}\xi_{\alpha_2(t_2)}.\end{aligned} \tag{11.37}$$

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We supposed that there was only one reggeon in the two-particle scattering amplitude. A generalization is straightforward. An inclusion of different Regge poles does not make the analysis much more complicated. If we insert two Regge poles α and β in the four-particle unitarity condition (11.34), the position of the two-reggeon branch-cut singularity,

$$j + 1 - \alpha(t_1) - \beta(t_2) = 0, \quad \sqrt{t_1} + \sqrt{t_2} = \sqrt{t},$$

will be determined by the extremum of the function

$$\alpha(t_1) + \beta(t_2) + \kappa[\sqrt{t_1} + \sqrt{t_2}\,]$$

(with κ the Lagrange multiplier), resulting in

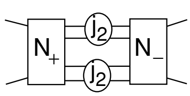
$$j_2 = \alpha\left(\left[\frac{\beta'}{\alpha'+\beta'}\right]^2 t\right) + \beta\left(\left[\frac{\alpha'}{\alpha'+\beta'}\right]^2 t\right) - 1.$$

Taking α to be the pomeron, $\alpha(0) = 1$, in the linear approximation we have

$$j_{R+\mathbf{P}}(t) \;\simeq\; \beta(0) + \frac{\alpha'\beta'}{\alpha'+\beta'}\cdot t \;\simeq\; \beta(t) - \frac{\beta'^2}{\alpha'+\beta'}\cdot t, \tag{11.38}$$

showing that the branch point lies *between* $\beta(t)$ and $\beta(0)$.

Even if the trajectory is unique, there appear many singularities in the j-plane. Indeed, having obtained the branching singularity j_2 of (11.36), we can iterate it anew,



$$\Longrightarrow \quad j_4(t) = 2j_2\left(\frac{t}{4}\right) - 1 = 4\alpha\left(\frac{t}{4}\right) - 3.$$

This way an infinite series of *Mandelstam branchings* is generated by a single Regge pole,

$$j_n(t) \;=\; n\alpha\left(\frac{t}{n^2}\right) - n + 1. \tag{11.39}$$

The corresponding discontinuity across the n-reggeon branch is given by

$$\delta_j f_j^{(n)}(t) = \frac{\pi}{n!}\int d\Gamma\tau\, N_+^{(n)}\delta\left(j - 1 - \sum_{i=1}^{n}[\alpha(t_i) - 1]\right) N_-^{(n)}\gamma_j^{(n)}. \tag{11.40}$$

As we have discussed in the previous lecture, the reggeon branchings are essential for the high-energy asymptotics in one case only, namely, that of the Pomeranchuk poles $\mathbf{P}$ with $\alpha_{\mathbf{P}}(0) = 1$, when the branchings condense to the point $j_n(t) = 1$ in the $n \to \infty$ limit. In the physical region of the s-channel, $t < 0$, this puts pomeron branchings in the *dominant* position, on the right of the pole $\alpha_{\mathbf{P}}(t)$, changing the asymptotic character of the elastic diffraction,

$$s^{\alpha(t)} \;\to\; s \cdot f_\alpha(\ln s, t).$$

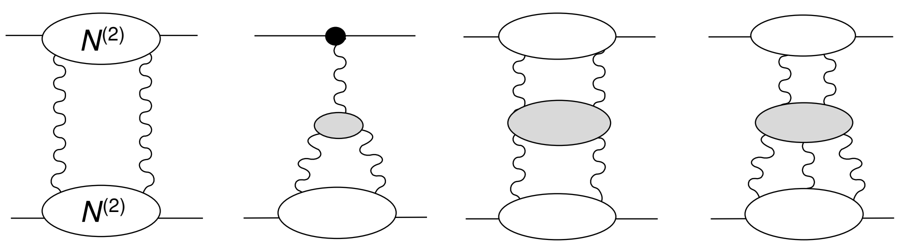
Analogously, high pomeron branching corrections to *non-vacuum* poles,

$$j_{R+n\mathbf{P}}(t) \;\simeq\; \beta(0) + \frac{\alpha'\beta'}{\alpha' + n\beta'} \cdot t,$$

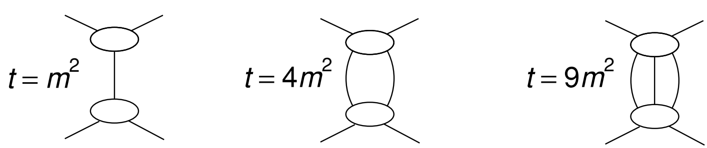
accumulate towards $j_n(t) \to \beta(0)$, and slow down the energy falloff of the 'charge exchange' cross sections,

$$s^{\beta(t)} \;\to\; s^{\beta(0)} \cdot f_\beta(\ln s, t).$$

We see that in order to describe these phenomena we need to know how to calculate and take into account multi-pomeron branchings. The problem is complicated by the fact that, according to (11.39), all the branch-point singularities are sitting at small t in one place, near $j = 1$. In this situation we cannot approximate the reggeon production block N by a constant since near $j = 1$ all the amplitudes are changing rapidly. We must localize and iterate all singular contributions which will lead us to the picture of *interacting reggeons*,



For ordinary hadrons all singularities were separated,



This permitted us to use the unitarity as a tool for *calculating* the interaction amplitudes in the case of a small coupling constant. Even if the coupling is large, iterating the unitarity conditions allowed us to extract some valuable information. Now the situation looks much more difficult. From the unitarity viewpoint, our reggeons behave rather as *massless* objects (like photons).

In fact such an analogy can be drawn explicitly. The emerging reggeon picture is similar to a non-relativistic multi-body problem of statistical physics. Imagine a system of particles with the 'dispersion law' $\omega_i = \epsilon(\mathbf{k}_i)$ describing the dependence of the particle energy on the momentum. The

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amplitude of scattering via, e.g. a two-particle intermediate state n will have the structure

$$f_{a,b}(\omega, \mathbf{k}^2) \propto \sum_n \frac{V_{a,n}V_{n,b}}{\epsilon_1 + \epsilon_2 - \omega}.$$

It has a cut in energy running from the point

$$\omega_2(\mathbf{k}^2) = \min_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \{\epsilon(\mathbf{k}_1) + \epsilon(\mathbf{k}_2)\}.$$

The discontinuity in energy reads

$$\delta_\omega f_{a,b}(\omega, \mathbf{k}^2) = \int d\Gamma(\mathbf{k}_1, \mathbf{k}_2) f_{a,n} \delta\big(\epsilon(\mathbf{k}_1) + \epsilon(\mathbf{k}_2) - \omega\big) f^*_{b,n} \,.$$

It resembles the two-reggeon unitarity condition (11.37) if we identify

$$j - 1 \;=\; \omega, \quad \alpha(t) - 1 \;=\; \epsilon(\mathbf{k}).$$

The pomeron case corresponds to *massless* excitations (like phonons in a solid state) with the dispersion law without a 'mass-gap': $\epsilon(\mathbf{0}) = 0$. Such multi-phonon thresholds accumulate to $\omega = 0$ as the pomeron branchings do.

There is, however, one but essential difference between the two problems, namely the signature factor $\bar{\xi}_j$ in (11.37). To have a complete analogy with a quantum-mechanical system, we would like the amplitude to be real below its singularities in energy. To try to get rid of $\bar{\xi}_j$ by simply absorbing $\sqrt{\bar{\xi}_j}$ into the reggeon production factors N is dangerous: $\bar{\xi}_j$ may be negative and this would introduce an unwanted singularity into the vertex function.

In non-relativistic quantum mechanics the discontinuity of the forward amplitude $(a = b)$ is given by the product $f \times f^*$ and is positive. What about our problem? Near the most interesting point $\alpha_1 \approx \alpha_2 \approx j \approx 1$ we have $\bar{\xi}_j = \xi_j \, \xi_{\alpha_1} \, \xi_{\alpha_2} \simeq (i)^3$, producing in the unitarity condition (11.37)

$$\delta_j f_j^{(2)} \propto \frac{1}{i}\bar{\xi}_j \simeq -1.$$

This means that, contrary to common particles, the contribution of a two-reggeon branching is *negative*. (We shall see shortly that in fact every additional reggeon introduces (-1), and thus the signs are alternating.)

Still, except for the signs, the reggeon branching is similar to usual branching in a system of particles, only in an unusual space, with t-channel angular momentum j in the rôle of 'energy' ω.

We already remarked more than once that when $t < 0$ the branching of two pomerons is positioned *on the right* from the pole. Now that we have established an analogy with the non-relativistic theory, this observation becomes quite dramatic, since in NQM a pole cannot sit on top of a cut.

A multi-reggeon state in the t-channel, looks from s-channel as a *repetition* of the one-reggeon exchange, and repetitions is the domain of the unitarity. In the next lecture we start to construct the field theory of interacting reggeons using unitarity in the s-channel. This will allow us, in particular, to bypass the problem of non-uniqueness of the analytic continuation (11.28). We will also fix the signature factors γ_j in the reggeon unitarity conditions (11.40) which, as we have mentioned before, must vanish for physical integer values of j of the proper signature.