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NONUNIFORM DICHOTOMY OF EVOLUTIONARY PROCESSES IN BANACH SPACES

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In this paper we study nonuniform dichotomy concepts of linear evolutionary processes which are defined in a general Banach space and whose norms can increase no faster than an exponential. Connections between the dichotomy concepts and (B, D) admissibility properties are established. These connections have been partially accomplished in an earlier paper by the authors for the case when the process was a semigroup of class C_0 and $(B, D) = \left(L^p, L^q\right)$.

1. Introduction

The dichotomy concepts for linear differential equations and their connections with admissibility properties have been extensively studied among others by Coppel [1] and Massera and Schäffer [3]. The study of the asymptotical behaviour of linear time-varying systems underlines nonuniform stability and nonuniform dichotomy properties (see for example [3], [5], [7] and [8]). Nonuniform dichotomic behaviours for the general case when the evolution of the system is described by a linear evolutionary process $P(\cdot, \cdot)$ on a general Banach space are considered in this paper. Using a fundamental inequality established in [3] we define the concept of (B, D) dichotomic evolutionary process and give a sufficient condition for non-uniform exponential dichotomy of a large class of such processes. We also consider the connections between the dichotomy concepts and the (B, D)

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admissibility property for the case of a linear system described by

$$x(t, t_0, x_0, u) = P(t, t_0)x_0 + \int_{t_0}^{t} P(t, s)u(s)ds .$$

The case when $P(t, t_0) = P(t-t_0, 0)$, that is, $P(\cdot, \cdot)$ is a C_0-semigroup, has been considered in [5]. The obtained results may be regarded as generalizations of well-known results of Coppel, Massera and Schäffer, Palmer and Reghis. They are applicable for a large class of systems described in [2].

Thus this paper is in a sense a sequel to [5].

2. Notation, definitions and terminology

Let $(X, \|\cdot\|)$ be a real or complex Banach space. The space of continuous linear mappings from X into itself is denoted by $L(X)$. The symbol Δ will denote the set defined by

$$\Delta = \{(t, s) : 0 \le s \le t < \infty\} .$$

DEFINITION 2.1. An application $P(\cdot, \cdot) : \Delta \to L(X)$ will be called an *evolutionary process* if and only if

(i) $P(t, s)P(s, t_0) = P(t, t_0)$ for $0 \le t_0 \le s \le t$,

(ii) $P(t, t)x = x$ for every $x \in X$,

(iii) $P(t, s)$ is strongly continuous in s on $[0, t]$ and in t on $[s, \infty)$,

(iv) there exists a nondecreasing function

$$p : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}_+ = (0, \infty)$$

such that

$$\|P(t, s)x\| \le p(t-s)\|x\| \text{ for all } (t, s) \in \Delta \text{ and } x \in X .$$

REMARK 2.1. If $P(\cdot, \cdot)$ is an evolutionary process then its norm can increase no faster than an exponential, that is, there exist $M, \omega > 0$ such that

$$\|P(t, s)\| \le Me^{\omega(t-s)} \text{ for all } (t, s) \in \Delta .$$

Indeed, if $M = p(1)$, $\omega = \ln M$ and n is the positive integer such

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that $n \le t-s < n+1$ then

$$\|P(t, s)\| \le \|P(t, s+n)\| \cdot \|P(s+n, s+n-1)\| \cdot \ldots \cdot \|P(s+1, s)\|$$
$$\le Me^{n\omega}$$
$$\le Me^{\omega(t-s)} .$$

REMARK 2.2. If the evolutionary process $P(\cdot, \cdot)$ satisfies the condition

(v) $P(t, s) = P(t-s, 0)$ for all $(t, s) \in \Delta$, then $P(\cdot, \cdot)$ is a semigroup of class C_0.

The space of X-valued functions f almost defined on $\mathbb{R}_+$ such that f is strongly measurable and locally integrable is denoted by $L^1_{\text{loc}}(X)$. In particular $L^1_{\text{loc}}(\mathbb{R}) = L^1_{\text{loc}}$. If $I = [a, b]$ is a real compact interval, then the characteristic function of I will be denoted by φ_a^b. In the particular cases $a = 0$ and respectively $b = \infty$ we use the notation

$$\varphi_0^b = \varphi^b \text{ and respectively } \varphi_a^\infty = \varphi_a .$$

DEFINITION 2.2. A Banach space $(S, |\cdot|_S)$ is said to be a *Schäffer space* (and we write $S \in \mathcal{S}$) if it has the following properties:

(i) $S \subset L^1_{\text{loc}}$ and there exists $M > 0$ such that

$$\int_0^t |f(s)|ds \le M \cdot |f|_S \text{ for every } S \in S \text{ and } t \ge 0 ;$$

(ii) if $g \in S$ and f is a real measurable function with $|f| \le g$ almost everywhere on $\mathbb{R}_+$ then $f \in S$ and $|f|_S \le |g|_S$;

(iii) for every $f \in S$ and $t \ge 0$ the function

$$g_t(s) = \begin{cases} 0 & , \text{ if } (t, s) \in \Delta , \\ f(s-t) & , \text{ if } (s, t) \in \Delta , \end{cases}$$

is also in S with $|g_t|_S = |f|_S$;

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(iv) for every $t \geq 0$ the function

$$\varphi^t(s) = \begin{cases} 1 , \text{ if } (t, s) \in \Delta , \\ 0 , \text{ if } (t, s) \notin \Delta , \end{cases}$$

is in S .

REMARK 2.3. It is obvious that the following Banach spaces are Schäffer spaces:

(i) L^p (where $1 \leq p < \infty$), the space of real p-integrable functions f on $\mathbb{R}_+$ with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}_+} |f(s)|^p ds\right)^{1/p} ;$$

(ii) L^∞ , the space of essentially bounded real measurable functions f with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{s\geq 0} |f(s)| ;$$

(iii) M^p (with $1 \leq p < \infty$), the space of real measurable functions f such that

$$|f|_p = \sup_{t\geq 0} \left(\int_t^{t+1} |f(s)|^p ds\right)^{1/p} < \infty ;$$

(iv) $M^\infty = L^\infty$ with the norm $|\cdot|_\infty = \|\cdot\|_\infty$;

(v) C , the space of real bounded and continuous functions f defined on $\mathbb{R}_+$ with the norm

$$|||f||| = \sup_{t\geq 0} |f(t)| .$$

If we denote by $S(X)$ (where $S \in \mathbf{S}$) the space of X-valued functions f almost defined on $\mathbb{R}_+$, such that f is strongly measurable and $\|f\| \in S$ then $S(X)$ is a Banach space with the norm $|||f|||_S$, which we write without ambiguity as $\|f\|_S$.

For every $S \in \mathbf{S}$ we associate the functions $\alpha_S, \beta_S : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

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$$\alpha_S(t) = \inf\left\{M > 0; \int_0^t |f(s)|ds \le M|f|_S, \forall f \in S\right\}$$

and respectively

$$\beta_S(t) = |\varphi^t|_S .$$

REMARK 2.4. In [3, pp. 61-67] it is shown that the function α_S and β_S have the following properties:

(i) $t \le \alpha_S(t)\beta_S(t) \le 2t$ for all $t \ge 0$ and $S \in \mathcal{S}$;

(ii) if $t_1 \le t_2$ then $\alpha_S(t_1) \le \alpha_S(t_2)$ and $\beta_S(t_1) \le \beta_S(t_2)$;

(iii) $\alpha_S(t) = \inf\left\{M > 0 : \int_a^b |f(s)|ds \le M|f|_S, \forall f \in S\right\}$ for all $a, b > 0$ with $b - a = t$.

If $S \in \mathcal{S}$ and $t_0 > 0$ then we denote by $S_{t_0}(X)$ the space of functions $f : [t_0, \infty) \to X$ with the property that $\tilde{f} : \mathbb{R}_+ \to X$ defined by

$$\tilde{f}(t) = \begin{cases} f(t) , & \text{if } (t, t_0) \in \Delta , \\ 0 , & \text{if } (t, t_0) \notin \Delta , \end{cases}$$

is in $S(X)$.

Throughout in this paper we suppose that for all $D \in \mathcal{S}$ and $t_0 \ge 0$ the set

$$X_1^D(t_0) = \{x_0 \in X : P(\cdot, t_0)x_0 \in D_{t_0}(X)\}$$

is a closed complemented subspace.

If $X_2^D(t_0)$ is a subspace such that

$$X = X_1^D(t_0) \oplus X_2^D(t_0)$$

then we denote by $P_1(t_0)$ the projection on to $X_2^D(t_0)$ (that is

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$\operatorname{Ker} P_1(t_0) = X_1^D(t_0)$) and we let $P_2(t_0) = I - P_1(t_0)$ (where I is the identity operator on X), which is the projection on to $X_1^D(t_0)$.

We shall let

$$P_1(t, t_0) = P(t, t_0)P_1(t_0)$$

and

$$P_2(t, t_0) = P(t, t_0)P_2(t_0) .$$

Now let us note two assumptions which will be used at various times.

ASSUMPTION 1. Let $B, D \in S$. We say that the pair (B, D) satisfies Assumption 1 if

$$\lim_{t\to\infty} \alpha_B(t)\beta_D(t) = \infty .$$

ASSUMPTION 2. The evolutionary process $P(\cdot, \cdot)$ satisfies Assumption 2 if there are two applications $m, \delta : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|P_2(t+\delta(t_0), t_0)x\| \geq m(t_0)\|P_2(t, t_0)x\|$$

for all $(t, t_0) \in \Delta$ and $x \in X$.

DEFINITION 2.3. Let $P(\cdot, \cdot)$ be an evolutionary process. Then $P(\cdot, \cdot)$ is said to be

(i) a *(nonuniformly) exponentially dichotomic process* if and only if there are $N, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|P_1(t, t_0)x\| \leq N(t_0)e^{-\nu(t_0)(t-s)}\|P_1(s, t_0)x\|$$

and

$$\|P_2(t, t_0)x\| \geq N(t_0)e^{\nu(t_0)(t-s)}\|P_2(s, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$;

(ii) a *(nonuniformly) dichotomic process* if and only if there exists $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|P_1(t, t_0)x\| \le N(t_0)\|P_1(s, t_0)x\|$$

and

$$\|P_2(t, t_0)x\| \ge N(t_0)\|P_2(s, t_0)x\|$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

REMARK 2.5. It is obvious that

(i) if $P(\cdot, \cdot)$ is an exponentially dichotomic process it is dichotomic,

(ii) if $P(\cdot, \cdot)$ is a dichotomic process then Assumption 2 holds.

REMARK 2.6. Let us denote

$$X_1(t_0) = \{x_0 \in X : P(\cdot, t_0)x_0 \in L^\infty_{t_0}(X)\} .$$

(i) If $P(\cdot, \cdot)$ is a dichotomic process then $X_1^D(t_0) \subset X_1(t_0)$. Indeed, if $x_0 \in X_1^D(t_0)$ then

$$\|P(t, t_0)x_0\| \le N(t_0)\|x_0\|$$

for every $t \ge t_0$ and hence $x_0 \in X_1(t_0)$.

(ii) If $P(\cdot, \cdot)$ is an exponentially dichotomic process then

$$X_1^D(t_0) = X_1(t_0) \quad \text{for all } D \in S \text{ and } t_0 \ge 0 .$$

For this, it is sufficient to observe that if $x_0 \in X_1(t_0)$ and $P_2(t_0)x_0 \ne 0$ then

$$\begin{aligned}\|P(t, t_0)x_0\| &\ge \|P_2(t, t_0)x_0\| - \|P_1(t, t_0)x_0\| \\ &\ge N(t_0)e^{\nu(t_0)(t-t_0)} \cdot \|P_2(t_0)x_0\| - N(t_0)e^{-\nu(t_0)(t-t_0)} \cdot \|P_1(t_0)x_0\| ,\end{aligned}$$

implies that

$$\lim_{t\to\infty} \|P(t, t_0)x_0\| = \infty ,$$

which contradicts the fact that $x_0 \in X_1(t_0)$.

DEFINITION 2.4. Let B, D be two Schäffer spaces. The evolutionary process $P(\cdot, \cdot)$ is (B, D) *dichotomic* if and only if there exists a function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\varphi_{t+\delta}(\cdot)P_1(\cdot, t_0)x\|_D + \left\|\varphi^t(\cdot)P_2(\cdot, t_0)x\right\|_D \le \frac{N(t_0)}{\delta\alpha_B(\delta)} \cdot \int_t^{t+\delta} \|P(s, t_0)x\| ds$$

for all $(t, t_0) \in \Delta$, $\delta > 0$ and $x \in X_k(t_0)$, $k = 1, 2$.

Consider the linear system described by the following integral model

$$\text{(P)} \qquad x(t, t_0, x_0, u) = P(t, t_0)x_0 + \int_{t_0}^t P(t, s)u(s)ds ,$$

where $(t, t_0) \in \Delta$, $x_0 \in X$, $u \in B(X)$.

DEFINITION 2.5. Let B, D be two Schäffer spaces. We say that the pair (B, D) is *admissible* for the system (P) if and only if for all $t_0 \ge 0$ and $u_0 \in B_{t_0}(X)$ there exists $x_0 \in X$ such that $x(\cdot, t_0, x_0, u_0) \in D_{t_0}(X)$.

REMARK 2.7. If the pair (B, L^∞) is admissible for the system (P) and

$$\lim_{t\to\infty} \alpha_B(t) = \infty ,$$

then (see Corollary 4.2) Assumption 2 holds.

3. Preliminary results

We prove the following lemmas which will be used in the sequel.

LEMMA 3.1. *If* $f : \Delta \to \mathbb{R}_+$ *is a function with the property that there is* $\delta : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that* $2f(t+\delta(t_0), t_0) \le f(t, t_0) \le 2f(s, t_0)$ *for all* $s + \delta(t_0) \ge t \ge s \ge t_0 \ge 0$ *then there exists* $\nu : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *such that*

$$f(t, t_0) \le 4e^{-\nu(t_0)(t-s)} f(s, t_0)$$

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for all $t \geq s \geq t_0 \geq 0$.

Proof. If $t \geq s \geq t_0 \geq 0$ and n is the positive integer such that $n\delta \leq t-s < (n+1)\delta$ then

$$\begin{aligned} f(t, t_0) &\leq 2f(s+n\delta(t_0), t_0) \\ &\leq 2(\tfrac{1}{2})^n f(s, t_0) \\ &= 4e^{-(n+1)\nu(t_0)\delta(t\ )} \cdot f(s, t_0) \\ &\leq 4e^{-\nu(t_0)(t-s)} \cdot f(s, t_0) , \end{aligned}$$

where $\nu(t_0) = \ln 2/\delta(t_0)$.

LEMMA 3.2. *Let* $f : \Delta \to \mathbb{R}_+$ *be a function with the property that there is* $\delta : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that* $f(s, t_0) \leq 2f(t, t_0) \leq f(t+\delta(t_0), t_0)$ *for all* $s + \delta(t_0) \geq t \geq s \geq t_0 \geq 0$ *then there is* $\nu : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that*

$$4f(t, t_0) \geq e^{\nu(t_0)(t-s)} \cdot f(s, t_0)$$

for all $t \geq s \geq t_0 \geq 0$.

Proof. This is similar to the proof of the preceding lemma.

LEMMA 3.3. *Let* (B, D) *be a pair of Schäffer spaces which satisfy Assumption* 1. *If the evolutionary process* $P(\cdot, \cdot)$ *is* (B, D) *dichotomic then there exists an application* $\eta : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that*

(i) $2\int_t^{t+\delta} \|P_1(\tau, t_0)x\| d\tau \leq \int_s^{s+\delta} \|P_1(\tau, t_0)x\| d\tau$ *for all* $s \geq t_0 \geq 0$, $\delta \geq \eta(t_0)$, $t \geq s + \eta(t_0)$ *and* $x \in X$, *and*

(ii) $2\int_s^{s+\delta} \|P_2(\tau, t_0)x\| d\tau \leq \int_t^{t+\delta} \|P_2(\tau, t_0)x\| d\tau$ *for all* $s \geq t_0 \geq 0$, $\delta > \eta(t_0)$, $t \geq s + 2\eta(t_0)$ *and* $x \in X$.

Proof. If $P(\cdot, \cdot)$ is a (B, D) dichotomic process then there exists an application $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality from Definition 2.4 holds.

For every $t_0 \geq 0$ let $\delta_0 = \eta(t_0) > 0$ be sufficiently large to satisfy

$$\frac{8N(t_0)}{\alpha_B(\delta_0)\beta_D(\delta_0)} < 1 .$$

Let now $s \geq t_0 \geq 0$, $\delta \geq \delta_0$ and let n be the positive integer such that $n\delta_0 \leq \delta < (n+1)\delta_0$. If we denote by $\delta_1 = \delta/n$ then for $t \geq s + \delta_0$ and $\tau = s + k\delta_1$, $k = 0, 1, \ldots, n-1$ we obtain

$$\begin{aligned}\int_{\tau+t-s}^{\tau+t-s+\delta_1} \|P_1(u, t_0)x\|du &\leq \alpha_D(\delta_1)\|\varphi_{\tau+\delta_0}(\cdot)P_1(\cdot, t_0)x\|_D \\ &\leq \frac{N(t_0)\alpha_D(\delta_1)}{\delta_0\alpha_B(\delta_0)} \cdot \int_\tau^{\tau+\delta_0} \|P_1(u, t_0)x\|du \\ &\leq \frac{2N(t_0)\delta_1}{\delta_0\alpha_B(\delta_0)\beta_D(\delta_1)} \cdot \int_\tau^{\tau+\delta_0} \|P_1(u, t_0)x\|du \\ &\leq \frac{4N(t_0)}{\alpha_B(\delta_0)\beta_D(\delta_0)} \cdot \int_\tau^{\tau+\delta_0} \|P_1(u, t_0)x\|du \\ &\leq \tfrac{1}{2} \int_\tau^{\tau+\delta_0} \|P_1(u, t_0)x\|du .\end{aligned}$$

Taking $\tau = s + k\delta_1$, $k = 0, 1, 2, \ldots, n-1$ and adding we obtain

$$\int_t^{t+\delta} \|P_1(u, t_0)x\|du \leq \tfrac{1}{2} \int_s^{s+\delta} \|P_1(u, t_0)x\|du ,$$

which proves the inequality *(i)*.

For the proof of *(ii)* let $s \geq t_0$, $t \geq s + 2\delta_0$ and $\tau = s + k\delta_1$ with $k = 0, 1, \ldots, n-1$. Then as before we obtain

$$\begin{aligned}\int_\tau^{\tau+\delta_1} \|P_2(u, t_0)x\|du &\leq \alpha_D(\delta_1)\left\|\varphi^{\tau+s}(\cdot)P_2(\cdot, t_0)x\right\|_D \\ &\leq \tfrac{1}{2} \int_{\tau+t-s}^{\tau+t-s+\delta_1} \|P_2(u, t_0)x\|du\end{aligned}$$

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and adding, we obtain the inequality *(ii)*.

LEMMA 3.4. *Let* (B, D) *be a pair of Schäffer spaces which satisfy Assumption* 1. *If the evolutionary process* $P(\cdot, \cdot)$ *is* (B, D) *dichotomic then there are* $\nu : \mathbb{R}_+ \to \mathbb{R}_+^*$, $M : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ *such that*

$$(i)\quad \int_t^{t+\delta} \|P_1(u, t_0)x\| du \le M(t_0, \delta) e^{-\nu(t_0)(t-s)} \|P_1(s, t_0)x\| \quad \text{and}$$

$$(ii)\quad \int_s^{s+\delta} \|P_2(u, t_0)x\| du \le M(t_0, \delta) e^{-\nu(t_0)(t-s)} \|P_2(t, t_0)x\|$$

for all $t \ge s \ge t_0 \ge 0$, $\delta > 0$ *and* $x \in X$.

Proof. Let $\eta(\cdot)$ be the function defined in the preceding theorem. For $\delta > 0$ and $x \in X$ consider the function

$$f : \Delta \to \mathbb{R}_+ , \quad f(t, t_0) = \int_t^{t+n\delta} \|P_1(u, t_0)x\| du ,$$

where $n = n(t_0, \delta)$ is sufficiently large such that $n\delta > 4\eta(t_0)$.

From the preceding lemma we have

$$2f(t+\eta(t_0), t_0) \le f(t, t_0)$$

and for $s + \eta(t_0) \ge t \ge s \ge t_0 \ge 0$ the following inequalities hold:

$$\begin{aligned} f(t, t_0) &\le \int_s^{s+n\delta+\eta(t_0)} \|P_1(u, t_0)x\| du \\ &\le \int_s^{s+\eta(t_0)} \|P_1(u, t_0)x\| du + \int_s^{s+n\delta} \|P_1(u, t_0)x\| du \\ &\le \frac{3}{2} \cdot f(s, t_0) \\ &\le 2f(s, t_0) . \end{aligned}$$

By Lemma 3.1 it follows that there is $\nu : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

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$$\int_t^{t+\delta} \|P_1(u, t_0)x\| du \le f(t, t_0)$$
$$\le 4e^{-\nu(t_0)(t-s)} f(s, t_0)$$
$$\le 4e^{-\nu(t_0)(t-s)} \cdot \int_s^{s+n\delta} \|P(u, s)\| du \cdot \|P_1(s, t_0)x\|$$
$$\le M(t_0, \delta)^{-\nu(t_0)(t-s)} \cdot \|P_1(s, t_0)x\| ,$$

where $M : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}_+$ is defined by

$$M(t_0, \delta) = 4n\delta p(n\delta) .$$

Thus the inequality *(i)* is proved.

For the proof of *(ii)* we consider the function

$$g : \Delta \to \mathbb{R}_+ , \quad g(t, t_0) = \int_t^{t+n\delta} \|P_2(u, t_0)x\| du ,$$

where n is a natural number such that $n\delta > 4\eta(t_0)$. It is easy to see that

$$g(t+2\eta(t_0), t_0) \ge 2g(t, t_0) \ge g(s, t_0)$$

for all $s + \eta(t_0) \ge t \ge s \ge t_0 \ge 0$.

By Lemma 3.2 we obtain

$$\int_s^{s+\delta} \|P_2(u, t_0)x\| du \le \int_s^{s+n\delta} \|P_2(u, t_0)x\| du$$
$$= g(s, t_0)$$
$$\le 4e^{-\nu(t_0)(t-s)} \cdot g(t, t_0)$$
$$\le 4e^{-\nu(t_0)(t-s)} \cdot \|P_2(t, t_0)x\| \cdot \int_t^{t+n\delta} \|P(u, t)\| du$$
$$\le M(t_0, \delta)e^{-\nu(t_0)(t-s)} \|P_2(t, t_0)x\|$$

and *(ii)* is proved.

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4. The main results

We prove the following

THEOREM 4.1. *If the pair* (B, D) *is admissible for the system* (P) *then there exists an application* $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that for all* $t_0 \geq 0$ *and* $u \in B_{t_0}(X)$ *there is an unique* $x_2(u) \in X_2^D(t_0)$ *with the properties*

(i) $x(\cdot, t_0, x_2(u), u) \in D_{t_0}(X)$ *, and*

(ii) $\|x(\cdot, t_0, x_2(u), u)\|_D \leq N \cdot \|u\|_B$.

Proof. By admissiblity of (B, D) for the system (P) it follows that for all $t_0 \geq 0$ and $u \in B_{t_0}(X)$ there is $x_0 \in X$ such that $x(\cdot, t_0, x_0, u) \in D_{t_0}(X)$.

If we let $x_k = P_k(t_0)x_0$ $(k = 1, 2)$ then from

$$x(\cdot, t_0, x_0, u) = P(\cdot, t_0)x_1 + x(\cdot, t_0, x_2, u) \in D_{t_0}(X)$$

and $P(\cdot, t_0)x_1 \in D_{t_0}(X)$ it follows that $x(\cdot, t_0, x_2, u) \in D_{t_0}(X)$ and hence for all $t_0 \geq 0$ and $u \in B_{t_0}(X)$ there exists $x_2 \in X_2^D(t_0)$ such that $x(\cdot, t_0, x_2, u) \in D_{t_0}(X)$.

If we suppose that for $u \in B_{t_0}(X)$ there exist $x_2', x_2'' \in X_2^D(t_0)$ such that

$$x(\cdot, t_0, x_2', u) \in D_{t_0}(X) \quad \text{and} \quad x(\cdot, t_0, x_2'', u) \in D_{t_0}(X)$$

then

$$\begin{aligned} x(\cdot, t_0, x_2', u) - x(\cdot, t_0, x_2'', u) &= x(\cdot, t_0, x_2'-x_2'', 0) \\ &= P(t, t_0)\left(x_2'-x_2''\right) \in D_{t_0}(X) , \end{aligned}$$

which implies

$$x_2' - x_2'' \in X_1^D(t_0) \cap X_2^D(t_0) = \{0\} ,$$

and hence $x_2' = x_2''$.

Consider the Banach space $Y_{t_0}(X) = X_2^D(t_0) \times D_{t_0}(X)$ with the norm

$$\|(x_2, d)\|_1 = \|x_2\| + \|d\|_D .$$

Let $A : B_{t_0}(X) \to Y_{t_0}(X)$ be the linear operator defined by

$$Au = (x_2(u), x(\cdot, t_0, x_2(u), u)) .$$

To establish property *(ii)* it is enough by the closed graph theorem to show that A is a closed operator.

Let $u_n \to u$ in $B_{t_0}(X)$ and $Au_n \to (x_2, y)$ in $Y_{t_0}(X)$. Then $x_2(u_n) \to x_2$ in $X_2^D(t_0)$ and $x(\cdot, t_0, x_2(u_n), u_n) \to y$ in $D_{t_0}(X)$. From $D(X) \longrightarrow L^1_{\text{loc}}(X)$ it follows that there exists a subsequence (u_{n_k}) of (u_n) such that

$$y(t) = \lim_{k\to\infty} x(t, t_0, x_2(u_{n_k}), u_{n_k})$$

almost everywhere.

From

$$\begin{aligned}
&\|y(t)-x(t, t_0, x_2, u)\| \\
&\le \|y(t)-x(t, t_0, x_2(u_{n_k}), u_{n_k})\| + \|x(t, t_0, x_2(u_{n_k}), u_{n_k})-x(t, t_0, x_2, u)\| \\
&\le \|y(t)-x(t, t_0, x_2(u_{n_k}), u_{n_k})\| + \|P(t, t_0)\| \cdot \|x_2(u_{n_k})-x_2\| \\
&\qquad + \int_{t_0}^{t} \|P(t, s)\| \|u_{n_k}(s)-u(s)\| ds \\
&\le \|y(t)-x(t, t_0, x_2(u_{n_k}), u_{n_k})\| + p(t-t_0)\left(\|x_2(u_{n_k})-x_2\|+\alpha_B(t)\cdot\|u_{n_k}-u\|_B\right) ,
\end{aligned}$$

for $k \to \infty$ we obtain that

$$y(t) = x(t, t_0, x_2, u)$$

almost everywhere. Hence $x(\cdot, t_0, x_2, u) \in D_{t_0}(X)$, which implies that $x_2 = x_2(u)$ and

$$Au = (x_2(u), x(t, t_0, x_2(u), u)) = (x_2, y) .$$

COROLLARY 4.2. *If the pair* (B, L^∞) *is admissible for the system* (P) *and*

$$\lim_{t\to\infty} \alpha_B(t) = \infty$$

then there is $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ *such that*

$$\|P_2(t, t_0)x\| \le M(t_0)\|P(t+1, t_0)x\|$$

for all $(t, t_0) \in \Delta$ *and* $x \in X$.

Proof. Let $(t, t_0) \in \Delta$, $x \in X$ and the input function

$$u : \mathbb{R}_+ \to X , \quad u(s) = \varphi_{t+1}^{t+2}(x)P(s, t_0)x .$$

From

$$\|u(s)\| \le p(1)\varphi_{t+1}^{t+2}(s) \cdot \|P(t+1, t_0)x\|$$

it follows that $u \in B_{t_0}(X)$ and

$$\|u\|_B \le p(1)\beta_B(1) \cdot \|P(t+1, t_0)x\| .$$

From

$$x(s, t_0, -P_2(t_0)\hat{x}, u) = -P_2(s, t_0)x + \int_{t_0}^{s} \varphi_{t+1}^{t+2}(u)P(s, t_0)x\,du$$

$$= \begin{cases} P_1(s, t_0)x , & \text{if } s \ge t+2 , \\ -P_2(s, t_0)x , & \text{if } s \le t+1 , \end{cases}$$

it follows that $x(\cdot, t_0, -P_2(t_0)x, u) \in L_{t_0}^\infty(X)$.

By Theorem 4.1 there is an application $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

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$$\begin{aligned}\|P_2(t, t_0)x\| &\le N(t_0) \cdot \|u\|_B \\ &\le N(t_0)p(1)\beta_B(1)\|P(t+1, t_0)x\| \\ &= M(t_0)\|P(t+1, t_0)x\|\end{aligned}$$

for all $(t, t_0) \in \Delta$ and $x \in X$.

THEOREM 4.3. *If the pair* (L^1, L^∞) *is admissible for the system then* $P(\cdot, \cdot)$ *is a dichotomic process.*

Proof. Let $(s, t_0) \in \Delta$, $x \in X$ and

$$u(t) = \varphi_s^{s+1}(t)P(t, t_0)x .$$

Clearly $u \in L^1_{t_0}(X)$ and from

$$x(t, t_0, -P_2(t_0)x, u) = \begin{cases} -P_2(t, t_0)x , & \text{if } t \le s , \\ P_1(t, t_0)x , & \text{if } t \ge s+1 , \end{cases}$$

it follows that $x(\cdot, t_0, -P_2(t_0)x, u) \in L^\infty_{t_0}(X)$.

By Theorem 4.1 there is $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|P_1(t, t_0)x\| \le N(t_0)p(1) \cdot \|P(s, t_0)x\| \quad \text{for all} \quad t \ge s+1$$

and

$$\|P_2(t, t_0)x\| \le N(t_0)p(1) \cdot \|P(s, t_0)x\| \quad \text{for} \quad t \le s .$$

If $t \in [s, s+1]$ then

$$\begin{aligned}\|P_1(t, t_0)x\| &\le p(1) \cdot \|P_1(s, t_0)x\| \\ &\le p(1)\left(\|P(s, t_0)x\|+\|P_2(s, t_0)x\|\right) \\ &\le M(t_0)\|P(s, t_0)x\| ,\end{aligned}$$

where

$$M(t_0) = \max\left\{N(t_0)p(1), p(1)+N(t_0)p(1)^2\right\} .$$

Finally we obtain

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$$\|P_1(t, t_0)x\| \le M(t_0)\|P(s, t_0)x\| \quad \text{for all} \quad t \ge s \ge t_0 \ge 0$$

and

$$\|P_2(t, t_0)x\| \le M(t_0)\|P(s, t_0)x\| \quad \text{for} \quad 0 \le t_0 \le t \le s \ .$$

These inequalities show that $P(\cdot, \cdot)$ is a dichotomic process.

THEOREM 4.4. *Let* (B, D) *be a pair of Schäffer spaces satisfying Assumption* 1 *and suppose that Assumption* 2 *holds. If* $P(\cdot, \cdot)$ *is* (B, D) *dichotomic then it is exponentially dichotomic.*

Proof. Let $t \ge s \ge t_0 \ge 0$ and $x \in X$. If $t \ge s+1$ and $u \in [t-1, t]$ then

$$\|P_1(t, t_0)x\| \le \|P(t, u)\| \cdot \|P_1(u, t_0)x\| \le p(t-u)\|P_1(u, t_0)x\| \ .$$

By Lemma 3.4 there are $\nu : \mathbb{R}_+ \to \mathbb{R}_+^*$, $M : \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}_+$ such that

$$\begin{aligned}\|P_1(t, t_0)x\| &\le \int_{t-1}^{t} p(t-u)\|P_1(u, t_0)x\|du \\ &\le p(1) \int_{t-1}^{t} \|P_1(u, t_0)x\| \cdot du \\ &\le p(1)M(t_0, 1)e^{-\nu(t_0)(t-s)} \cdot \|P_1(s, t_0)x\| \ ,\end{aligned}$$

for $t \ge s+1$ and $s \ge t_0 \ge 0$.

If $t \in [s, s+1]$ then

$$\|P_1(t, t_0)x\| \le p(t-s)\|P_1(s, t_0)x\| \le p(1)e^{\nu(t_0)}e^{-\nu(t_0)(t-s)} \cdot \|P_1(s, t_0)x\| \ .$$

Finally we obtain

$$\|P_1(t, t_0)x\| \le N_1(t_0)e^{-\nu(t_0)(t-s)}\|P_1(s, t_0)x\|$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$, where

$$N_1(t_0) = \max\left\{p(1)M\ t_0,\ 1\ ,\ p(1)e^{\nu(t_0)}\right\} \ .$$

Let $m, \delta : \mathbb{R}_+ \to \mathbb{R}_+^*$ given by Assumption 2 and let $\delta_0 = \delta(t_0)$ and

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$m_0 = m(t_0)$.

Then

$$m_0 \cdot \|P_2(s, t_0)x\| \le \|P_2(s+\delta_0, t_0)x\| \le p(s+\delta_0-u) \cdot \|P_2(u, t_0)x\|$$

for all $(s, t_0) \in \Delta$, $u \in [s, s+\delta_0]$ and $x \in X$.

By integration in raport with u on $[s, s+\delta_0]$ from Lemma 3.4 we have

$$m_0\delta_0\|P_2(s, t_0)x\| \le p(\delta_0) \cdot \int_s^{s+\delta_0} \|P_2(u, t_0)x\|du$$

$$\le p(\delta_0)M(t_0, \delta_0) \cdot e^{-\nu(t_0)(t-s)} \cdot \|P_2(t, t_0)x\| ,$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

THEOREM 4.5. *If the pair* (B, D) *is admissible for the system then the evolutionary process* $P(\cdot, \cdot)$ *is* (B, D) *dichotomic.*

Proof. Let $t_0 \ge 0$, $\delta > 0$ and $x \in X$ such that

$$P_1(t, t_0)x \ne 0 \text{ for all } t \ge t_0 .$$

Then $P(t, t_0) \ne 0$ for every $t \ge t_0$.

For each $(t, t_0) \in \Delta$ we consider the function

$$u_0(s) = \begin{cases} \dfrac{P(s,t_0)x}{\|P(s,t_0)x\|} , & \text{if } s \in [t, t+\delta] , \\ 0 , & \text{if } s \notin [t, t+\delta] , \end{cases}$$

$$= \frac{\varphi_t^{t+\delta}(s)P(s,t_0)x}{\|P(s,t_0)x\|} .$$

From

$$\|u_0(x)\| \le \beta_B(\delta)$$

it results that $u_0 \in B_{t_0}(X)$ and $\|u_0\|_B \le \beta_B(\delta)$. If

$$f(t, t_0) = \int_t^{t+\delta} \frac{ds}{\|P(s, t_0)x\|} ,$$

and $x_0 = -f(t, t_0)P_2(t_0)x$ then

$$x(s, t_0, x_0, u_0) = \begin{cases} f(t, t_0)P_1(s, t_0)x & , \text{ if } s \geq t+\delta , \\ -f(t, t_0)P_2(s, t_0)x & , \text{ if } t_0 \leq s \leq t , \end{cases}$$

and hence $x(\cdot, t_0, x_0, u_0) \in D_{t_0}(X)$.

Since $x_0 \in X_2^D(t_0)$, by Theorem 4.1 we have that there is $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|x(\cdot, t_0, x_0, u_0)\|_D \leq N(t_0) \cdot \|u_0\|_B \leq N(t_0)\beta_B(\delta) .$$

From here, Remark 2.4 and by Schwartz's inequality we obtain

$$\begin{aligned} \|\varphi_{t+\delta}(\cdot)P_1(\cdot, t_0)x\|_D + \left\|\varphi_{t_0}^t(\cdot)P_2(\cdot, t_0)x\right\|_D & \\ &\leq \frac{2N(t_0)\beta_B(\delta)}{f(t, t_0)} \\ &\leq \frac{2N(t_0)\beta_B(\delta)}{\delta^2} \cdot \int_t^{t+\delta} \|P(s, t_0)x\|ds \\ &\leq \frac{4N(t_0)}{\delta\alpha_B(\delta)} \cdot \int_t^{t+\delta} \|P(\tau, t_0)x\|d\tau . \end{aligned}$$

If for $x \in X$ there is $t_1 \geq t_0$ such that $P(t_1, t_0)x = 0$ then $P(t, t_0)x = P(t, t_1)P(t_1, t_0)x = 0$ for all $t \geq t_1$ and thus $x \in X_1^D(t_0)$, which implies $x = P_1(t_0)x$.

Let τ be a positive number such that $P(\tau, t_0)x = 0$ and $P(t, t_0)x \neq 0$ for every $t \in [t_0, \tau]$.

Let $t \geq t_0$, $\delta > 0$, $t_2 \in (t, t+\delta]$ and $\delta_2 > 0$ such that $t + \delta_2 < t_2$. If $t + \delta < \tau$ then we consider the input function defined by

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$$u(s) = \begin{cases} \dfrac{P(s,t_0)x}{\|P(x,t_0)x\|} & , \text{ if } s \in [t, t_2-\delta_2] , \\ \dfrac{P(s,t_0)x}{\|P(t_2-\delta_2,t_0)\|} & , \text{ if } s \in [t_2-\delta_2, t_2] , \\ 0 & , \text{ if } s \notin [t, t_2] . \end{cases}$$

From Remarks 2.1 and 2.4 it follows that there exists $\omega > 0$ such that

$$\|u\|_B \le e^{\omega\delta_2} \cdot \beta_B(\delta) .$$

Hence $u \in B_{t_0}(X)$ and

$$x(s, t_0, 0, u) = \left(\int_t^{t_2-\delta_2} \frac{du}{\|P_1(u,t_0)x\|} + \frac{\delta_2}{\|P_1(t_2-\delta_2,t_0)x\|}\right) \cdot P_1(s, t_0)x$$

for all $s \ge t_2$.

Because $0 \in X_2^D(t_0)$ and $x(\cdot, t_0, 0, u) \in D_{t_0}(X)$ from Theorem 4.1 we obtain that there is $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|x(\cdot, t_0, 0, u)\|_D \le N(t_0)\beta_B(\delta)e^{\omega\delta_2} .$$

Hence

$$\|\varphi_{t_2}(\cdot)P_1(\cdot, t_0)x\|_D \cdot \int_t^{t_2-\delta_2} \frac{du}{\|P_1(u,t_0)x\|} \le N(t_0)\beta_B(\delta)e^{\omega\delta_2}$$

for every $\delta_2 > 0$ with $t + \delta_2 < t_2$.

If $t_2 = t + \delta$ and $\delta_2 \to 0$ then

$$\|\varphi_{t+\delta}(\cdot)P_1(\cdot, t_0)x\|_D \cdot \int_t^{t+\delta} \frac{du}{\|P_1(u,t_0)x\|} \le N(t_0)\beta_B(\delta) .$$

As in the preceding case (using Remark 2.4 and Schwartz's inequality) we obtain

$$\|\varphi_{t+\delta}(\cdot)P_1(\cdot,\ t_0)x\| \le \frac{2N(t_0)}{\delta\alpha_B(\delta)} \cdot \int_t^{t+\delta} \|P(u,\ t_0)x\|du .$$

If $t + \delta \ge \tau$ this inequality remains obviously true. Thus

$$\|\varphi_{t+\delta}(\cdot)P_1(\cdot,\ t_0)x\|_D = \|\varphi_{t+\delta}(\cdot)P_1(\cdot,\ t_0)x\|_D + \left\|\varphi^t(\cdot) \cdot P_2(\cdot,\ t_0)x\right\|_D$$

$$\le \frac{2N(t_0)}{\delta\alpha_B(\delta)} \cdot \int_t^{t+\delta} \|P(u,\ t_0)x\|du ,$$

for all $(t,\ t_0) \in \Delta$, $\delta > 0$ and $x \in X$. This shows that $P(\cdot,\ \cdot)$ is $(B,\ D)$ dichotomic.

COROLLARY 4.6. *Let* $(B,\ D)$ *be a pair of Schäffer spaces satisfying Assumption* 1 *and suppose that Assumption* 2 *holds.*

If the pair $(B,\ D)$ *is admissible for the system* (P) *then the process* $P(\cdot,\ \cdot)$ *is exponentially dichotomic.*

Proof. This is a consequence of Theorems 4.4 and 4.5.

COROLLARY 4.7. *Let* $P(\cdot,\ \cdot)$ *be an evolutionary process satisfying Assumption* 2. *If*

(i) $B = L^p$, $p > 1$ *or* $B = M^p$, $p \ge 1$ *or* $B = C$ *and* $D \in S$, *or*

(ii) $B \in S$ *and* $D = L^q$, $q < \infty$ *or* $D = C$

and $(B,\ D)$ *is admissible for the system* (P) *then* $P(\cdot,\ \cdot)$ *is an exponential dichotomic process.*

Proof. In the above hypotheses we have (see [3], pp. 61-67 that the pair $(B,\ D)$ verifies Assumption 1. Then the result is obvious from the preceding corollary.

COROLLARY 4.8. *If* $\lim_{t\to\infty} \alpha_B(t) = \infty$ *and the pair* $(B,\ L^\infty)$ *is admissible for the system* (P) *then the evolutionary process* $P(\cdot,\ \cdot)$ *is exponentially dichotomic.*

Proof. This results from Remark 2.7 and Corollary 4.6.

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