

# THE CONGRUENCE LATTICE OF A COMBINATORIAL STRICT INVERSE SEMIGROUP

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The congruence lattice of a combinatorial strict inverse semigroup is shown to be isomorphic to a complete subdirect product of congruence lattices of semilattices preserving pseudocomplements.

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## 1. Introduction

Congruence lattices of semilattices admit several remarkable properties. Let  $\mathcal{S}$  denote the class of all semilattices, let  $Y \in \mathcal{S}$  and  $\mathcal{C}(Y)$  be the congruence lattice of  $Y$ . By Papert [11],  $\mathcal{C}(Y)$  is completely (meet-)semidistributive, (weakly) relatively pseudocomplemented and thus also pseudocomplemented, and each non universal congruence  $\rho \in \mathcal{C}(Y)$  is the intersection of coatoms (which are also characterized). Hall [5] proved that  $\mathcal{C}(Y)$  is semimodular (and thus satisfies the Jordan–Dedekind chain condition), and described the atoms of  $\mathcal{C}(Y)$ . Jones [9] has shown that  $\mathcal{C}(Y)$  is  $M$ -symmetric. Freese and Nation [2] proved that the class  $\mathcal{C}(\mathcal{S}) = \{\mathcal{C}(Y) \mid Y \in \mathcal{S}\}$  of all congruence lattices of semilattices does not satisfy any non trivial lattice identity. Finally, Zhitomirskij [13] gave a characterization of all lattices which are isomorphic to congruence lattices of semilattices. Quite recently, Freese and Nation [3] have again considered congruence lattices of finite semilattices from a more lattice theoretic point of view. They have shown, for instance, that the congruence lattice of a finite semilattice (in fact, of a finite combinatorial inverse semigroup) is upper bounded (which is stronger than meet-semidistributivity).

A combinatorial strict inverse semigroup  $S$  is an (inverse) subdirect product of combinatorial Brandt semigroups (and possibly the trivial group) (see Petrich [12]). This class forms a variety of inverse semigroups and will be denoted by  $\mathcal{B}$ . It is well known that  $\mathcal{B}$  is the unique combinatorial cover of  $\mathcal{S}$  in the lattice of inverse semigroup varieties [12]. The purpose of this note is to establish several of the above mentioned properties for congruence lattices of combinatorial strict inverse semigroups. Let  $T \in \mathcal{B}$ . We shall describe atoms and coatoms in  $\mathcal{C}(T)$  and show that each non universal congruence  $\rho \in \mathcal{C}(T)$  is the intersection of coatoms. Further, we shall prove that  $\mathcal{C}(T)$  is a complete subdirect product of congruence lattices of semilattices. We infer that each lattice implication which holds  $\mathcal{C}(\mathcal{S})$  also holds in  $\mathcal{C}(\mathcal{B}) = \{\mathcal{C}(T) \mid T \in \mathcal{B}\}$ . In

particular,  $\mathcal{C}(T)$  is completely semidistributive. In addition,  $\mathcal{C}(T)$  is  $M$ -symmetric (which in fact has been already shown by Jones [9]). Furthermore, it is proved that the mentioned subdirect decomposition of  $\mathcal{C}(T)$  preserves (weak) relative pseudocomplements and hence  $\mathcal{C}(T)$  is pseudocomplemented for any  $T \in \mathcal{B}$ . Finally, an example shows that the classes  $\mathcal{C}(\mathcal{S})$  and  $\mathcal{C}(\mathcal{B})$  do not coincide.

**2. Preliminaries: combinatorial strict inverse semigroups**

For undefined notions, the reader is referred to the books of Howie [7] (general semigroups), Petrich [12] (inverse semigroups) and Grätzer [4] (lattices). For an arbitrary semigroup  $S$  let  $\mathcal{C}(S)$  denote the (complete) lattice of all congruences on  $S$ . Join and meet are denoted by  $\vee$  and  $\wedge$  (or  $\bigvee$  and  $\bigwedge$ ). The lattice of all equivalence relations on a set  $X$  is denoted by  $\mathcal{E}(X)$ . The identical and universal relations on the set  $X$  are denoted by  $\varepsilon = \varepsilon_X$  and  $\omega = \omega_X$ , respectively.

Combinatorial strict inverse semigroups admit the following description (see Petrich [12, XIV] or Nambooripad [10]).

**Theorem 2.1.** *Let  $X$  be a partially ordered set, for each  $\alpha \in X$  let  $I_\alpha^*$  be the non zero part of a combinatorial Brandt semigroup  $I_\alpha$  such that  $I_\alpha^* \cap I_\beta^* = \emptyset$  whenever  $\alpha \neq \beta$ . For  $\alpha \geq \beta$  let  $f_{\alpha,\beta}: I_\alpha^* \rightarrow I_\beta^*$  be a partial homomorphism subject to the following conditions*

- (1)  $f_{\alpha,\alpha} = \text{id}_{I_\alpha^*}$  for all  $\alpha \in X$ ,
- (2)  $f_{\alpha,\beta} f_{\beta,\gamma} = f_{\alpha,\gamma}$  whenever  $\alpha \geq \beta \geq \gamma$ ,
- (3) for any  $x \in I_\alpha^*, y \in I_\beta^*, \alpha, \beta \in X$ , the set

$$D(x, y) = \{ \gamma \leq \alpha, \beta \mid (x f_{\alpha,\gamma})(y f_{\beta,\gamma}) \neq 0 \text{ in } I_\gamma \}$$

has a greatest element, to be denoted by  $\delta(x, y)$ .

Put  $S = \bigcup_{\alpha \in X} I_\alpha^*$  and define a product on  $S$  by

$$xy = (x f_{\alpha,\delta(x,y)})(y f_{\beta,\delta(x,y)})$$

where  $x \in I_\alpha^*, y \in I_\beta^*$ . Then the groupoid  $S$ , to be denoted by  $(X; I_\alpha, f_{\alpha,\beta})$  is a combinatorial strict inverse semigroup. Conversely, every combinatorial strict inverse semigroup can be so constructed.

The set  $X$  in the construction above is the structure set of  $S = (X; I_\alpha, f_{\alpha,\beta})$ . The partially ordered set  $X$  in fact is the partially ordered set  $S/\mathcal{J}$  of all  $\mathcal{J}$ -classes of  $S = (X; I_\alpha, f_{\alpha,\beta})$ . In the following “structure set” will stand for “structure set of a combinatorial strict inverse semigroup”. Any structure set  $X$  satisfies the following (see [1] or [10]).

**Proposition 2.2.** *For any structure set  $X$  the following assertions hold:*

- (1)  $X$  is downwards directed,
- (2) for any two elements  $\alpha, \beta \in X$  having a common upper bound  $\gamma \geq \alpha, \beta$ , the greatest lower bound  $\inf\{\alpha, \beta\}$  exists in  $X$ .

The following definitions and results are from [1] where congruences on strict regular semigroups (that is, regular subdirect products of completely simple and/or completely 0-simple semigroups) are studied.

**Definition 1.** Let  $S=(X; I_\alpha, f_{\alpha, \beta})$  be a combinatorial strict inverse semigroup and  $\rho \in \mathcal{C}(S)$ . For  $\alpha, \beta \in X$  put

$$\alpha \rho_{\mathcal{J}} \beta \Leftrightarrow (\exists \gamma \leq \alpha, \beta) (\forall x \in I_\alpha^*, y \in I_\beta^*) x \rho x f_{\alpha, \gamma}, y \rho y f_{\beta, \gamma}.$$

An equivalent definition of  $\rho_{\mathcal{J}}$  is:

$$\alpha \rho_{\mathcal{J}} \beta \Leftrightarrow (\exists \gamma \leq \alpha, \beta, x \in I_\alpha^*, y \in I_\beta^*, u, v \in I_\gamma^*) x \rho u, y \rho v.$$

For  $\alpha \in X$  let  $\langle \alpha \rangle = \{\gamma \in X \mid \gamma \leq \alpha\}$  denote the principal order ideal in  $X$  generated by  $\alpha$ . By Proposition 2.3 in [1] we have the following.

**Proposition 2.3.** *The relation  $\rho_{\mathcal{J}}$  is an equivalence relation on  $X$  satisfying:*

- (1) if  $\alpha \rho_{\mathcal{J}} \beta$  then  $\alpha \rho_{\mathcal{J}} \gamma \rho_{\mathcal{J}} \beta$  for some  $\gamma \leq \alpha, \beta$ ,
- (2)  $\rho|_{\langle \alpha \rangle}$  is a semilattice congruence on  $\langle \alpha \rangle$  for each  $\alpha \in X$ .

**Definition 2.** Let  $X$  be a structure set. An equivalence relation  $\xi$  on  $X$  is a congruence on  $X$  if it satisfies the conditions (1) and (2) of Proposition 2.3. The set of all such congruences on  $X$ , ordered by inclusion, will be denoted by  $\mathcal{C}(X)$ .

The following is also from [1] (Corollaries 3.2 and 3.4 and the observation before Corollary 5.14).

**Theorem 2.4.** *Let  $S=(X; I_\alpha, f_{\alpha, \beta})$  be a combinatorial strict inverse semigroup. Then the following assertions hold.*

- (1)  $\mathcal{C}(X)$  is a complete lattice and a complete  $\vee$ -subsemilattice of  $\mathcal{E}(X)$ .
- (2) The mapping  $\mathcal{J}: \rho \mapsto \rho_{\mathcal{J}}$  is an isomorphism between the lattices  $(\mathcal{C}(S), \cap, \vee)$  and  $(\mathcal{C}(X), \wedge, \vee)$ .

Hence, on a combinatorial strict inverse semigroup  $S=(X; I_\alpha, f_{\alpha, \beta})$ , any congruence  $\rho$  is uniquely determined by the relation  $\rho_{\mathcal{J}}$ . This will be essentially used in the following. Notice that  $\mathcal{C}(X)$  in general is not a  $\cap$ -subsemilattice of  $\mathcal{E}(X)$ .

**3. Atoms and coatoms** A congruence  $\rho$  covering the identical relation  $\varepsilon$  in  $\mathcal{C}(S)$  is an

*atom*. Dually, each congruence  $\rho$  covered by the universal relation  $\omega$  is a *coatom*. Since for  $S=(X; I_\alpha, f_{\alpha, \beta})$ ,  $\mathcal{C}(S)$  and  $\mathcal{C}(X)$  are isomorphic, a congruence  $\rho$  is a (co)atom in  $\mathcal{C}(S)$  if and only if  $\rho_{\mathcal{F}}$  is a (co)atom in  $\mathcal{C}(X)$ . We start with coatoms. The following is a reformulation of the analogue for semilattices obtained by Papert [11].

**Definition 3.** Let  $X$  be a structure set. A non empty subset  $F \subseteq X$  is a *c-filter* if:

- (1)  $F$  is an order filter in  $X$ , that is,  $\alpha \in F$  implies  $\beta \in F$  for each  $\beta \geq \alpha$ ,
- (2)  $F$  is downwards directed, that is, for any  $\alpha, \beta \in F$  there is some  $\gamma \in F$  such that  $\gamma \leq \alpha, \beta$ .

**Definition 4.** Let  $X$  be a structure set and  $F \subseteq X$  be a *c-filter*. For  $\alpha, \beta \in X$  let

$$\alpha \rho_F \beta \Leftrightarrow \alpha, \beta \in F \text{ or } \alpha, \beta \notin F.$$

That is,  $\rho_F = F \times F \cup X \setminus F \times X \setminus F$ .

It is easy to prove that  $\rho_F$  is a congruence on  $X$  for each *c-filter*  $F$ .

**Theorem 3.1.** Let  $X$  be a structure set. Then the following assertions hold.

- (1) A congruence  $\rho$  on  $X$  is a *coatom* if and only if  $\rho = \rho_F$  for some *c-filter*  $F \neq X$ .
- (2) Each non universal congruence  $\rho$  on  $X$  is the intersection of some (non empty) set of coatoms. In particular, each non universal congruence  $\rho$  is contained in some *coatom*.

**Proof.** (1) For each *c-filter*  $F \neq X$ ,  $\rho_F$  is a congruence on  $X$  and clearly is a *coatom* since there are precisely two  $\rho_F$ -classes. Conversely, let  $\rho$  be a non universal congruence on  $X$ . At most one  $\rho$ -class is an ideal in  $X$ . If  $\alpha\rho$  is an ideal then for any  $\beta \in X$  there is  $\gamma \in X$ ,  $\gamma \leq \alpha, \beta$ . If  $\beta\rho$  is also an ideal then  $\gamma \in \alpha\rho \cap \beta\rho$  and therefore  $\beta\rho\gamma\rho\alpha$ . Let  $\alpha\rho$  be a  $\rho$ -class not being an ideal and let

$$F = F(\alpha\rho) = \{\beta \in X \mid \beta \geq \gamma \text{ for some } \gamma \in \alpha\rho\}.$$

It is easy to show that  $F$  is a *c-filter*. Since  $\alpha\rho$  is not an ideal there are  $\beta \in \alpha\rho, \gamma \in X \setminus \alpha\rho$  such that  $\gamma < \beta$ . Then  $\gamma \notin F$  for if there would be a  $\delta \in \alpha\rho$  such that  $\delta \leq \gamma$  then  $\delta\rho\beta$  and  $\delta \leq \gamma < \beta$  imply  $\alpha\rho\delta\rho\gamma$  since  $\rho|(\beta)$  is a congruence on the semilattice  $(\beta)$ . Consequently,  $F \neq X$  and  $\rho_F$  is a *coatom*. Let  $\beta \in F, \gamma \in X \setminus F$ . If  $\beta\rho\gamma$  then there is  $\delta \leq \beta, \gamma$  such that  $\beta\rho\delta\rho\gamma$ . Now there is  $\varepsilon \in \alpha\rho, \varepsilon \leq \beta$ . Since  $\rho|(\beta)$  is a semilattice congruence we observe  $\varepsilon = \beta \wedge \varepsilon\rho\delta \wedge \varepsilon$  and thus  $\delta \wedge \varepsilon \in \alpha\rho$ . Since  $\gamma \geq \delta \geq \delta \wedge \varepsilon$  we infer that  $\gamma \in F$ , a contradiction. Consequently,  $\rho \subseteq \rho_F$ . Hence, if we assume in addition that  $\rho$  is a *coatom* then  $\rho = \rho_F$ .

(2) Let  $\rho$  be a non universal congruence. Then clearly  $\rho \subseteq \bigcap \{\rho_F \mid \rho \subseteq \rho_F\}$  and therefore also  $\rho \subseteq \bigwedge \{\rho_F \mid \rho \subseteq \rho_F\}$ . Note that by the proof of (1), the set  $\{\rho_F \mid \rho \subseteq \rho_F\}$  is not empty.

We only have to show that for any non  $\rho$ -related elements  $\alpha, \beta \in X$  there is some  $\rho_F$  containing  $\rho$  and separating  $\alpha$  and  $\beta$ . Let  $\alpha, \beta \in X$  be such that  $(\alpha, \beta) \notin \rho$ . Suppose there are  $\gamma \in \alpha\rho, v \in \beta\rho$  such that  $\gamma \geq v$ . Let  $\gamma' \in \alpha\rho$ ; then there is  $\delta \leq \gamma, \gamma'$  such that  $\delta \in \alpha\rho$ . Now  $\rho|(\gamma)$  is a semilattice congruence so that  $\gamma\rho\delta$  implies  $v = \gamma \wedge v\rho\delta \wedge v \leq \gamma'$ . Therefore, either  $\alpha\rho \subseteq F(\beta\rho)$  or  $\alpha\rho \cap F(\beta\rho) = \emptyset$ . If there are  $\gamma, \delta \in \alpha\rho, \mu, v \in \beta\rho$  such that  $\gamma \geq \mu$  and  $v \geq \delta$  then by [1, Lemma 2.6.(2)] we observe that  $\alpha\rho = \beta\rho$ . Hence if  $\alpha\rho \neq \beta\rho$  then  $\alpha\rho \cap F(\beta\rho) = \emptyset$  or  $\beta\rho \cap F(\alpha\rho) = \emptyset$  (or both) hold. Consequently, the elements  $\alpha$  and  $\beta$  are separated either by  $\rho_{F(\alpha\rho)}$  or by  $\rho_{F(\beta\rho)}$  and thus  $\rho = \bigcap \{ \rho_{F(\alpha\rho)} \mid \alpha\rho \in X/\rho \} = \bigwedge \{ \rho_{F(\alpha\rho)} \mid \alpha\rho \in X/\rho \}$ . □

From the description of the coatom  $\rho_F$  in  $\mathcal{C}(X)$  one can easily reconstruct the corresponding coatom  $\theta \in \mathcal{C}(S)$  for which  $\theta_{\mathcal{F}} = \rho_F$ . A congruence  $\rho$  on  $X$  is a coatom if and only if there are precisely two  $\rho$ -classes. This is not true for the coatoms in  $\mathcal{C}(S)$ . A trivial example is any combinatorial Brandt semigroup  $S$  having more than two elements. The corresponding structure set is the two element chain. The only coatom in  $\mathcal{C}(S)$  is the identical relation  $\varepsilon_S$  which has exactly  $|S|$  classes. Less trivial examples can be provided easily.

Next we give a description of the atoms in  $\mathcal{C}(X)$ . It is similar to Hall's description [5] of the atoms in the congruence lattice of a semilattice.

**Theorem 3.2.** *Let  $X$  be a structure set, let  $\alpha, \beta \in X$  be such that  $\beta < \alpha$  and if  $\gamma < \alpha$  for some  $\gamma \in X$  then  $\gamma \leq \beta$ . Define an equivalence relation  $\theta(\alpha, \beta)$  on  $X$  by*

$$\gamma \theta(\alpha, \beta) \delta \Leftrightarrow \{\gamma, \delta\} \subseteq \{\alpha, \beta\} \text{ or } \gamma = \delta.$$

*Then  $\theta(\alpha, \beta)$  is an atom in  $\mathcal{C}(X)$ . Conversely, every atom in  $\mathcal{C}(X)$  can be so constructed.*

**Proof.** Let  $\alpha, \beta \in X$  be as in the theorem. Then  $\theta(\alpha, \beta)$  is obviously an atom in  $\mathcal{C}(X)$ . Let  $\gamma \in X$ . Then  $\theta(\alpha, \beta)|(\gamma) = \varepsilon_{(\gamma)}$  or  $\theta(\alpha, \beta)|(\gamma)$  is an atom in  $\mathcal{C}((\gamma))$  by Hall [5] and thus  $\theta(\alpha, \beta)|(\gamma)$  is a congruence on  $(\gamma)$ . Hence  $\theta(\alpha, \beta)$  is a congruence on  $X$  and thus is an atom in  $\mathcal{C}(X)$ . Conversely, let  $\rho$  be an atom in  $\mathcal{C}(X)$ . Since  $\rho \neq \varepsilon_X$  there are  $\alpha, \beta \in X, \beta < \alpha$  such that  $\alpha\rho\beta$ . Let  $\chi_{(\alpha)}$  denote the "Rees congruence" with respect to the principal ideal  $(\alpha)$ . That is,

$$\gamma \chi_{(\alpha)} \delta \Leftrightarrow \gamma = \delta \text{ or } \gamma, \delta \in (\alpha).$$

It is easily verified that  $\chi_{(\alpha)}$  and  $\rho \cap \chi_{(\alpha)}$  are congruences on  $X$ . In particular,  $\rho \cap \chi_{(\alpha)} = \rho \wedge \chi_{(\alpha)}$ . Since  $\varepsilon \neq \rho \cap \chi_{(\alpha)}$  and  $\rho$  is an atom we observe that  $\rho \subseteq \chi_{(\alpha)}$ . Hence  $\rho = \rho|(\alpha) \cup \varepsilon_X$ . Now  $\rho|(\alpha)$  is an atom in  $\mathcal{C}((\alpha))$  for if there is some  $\tau \in \mathcal{C}((\alpha))$  such that  $\tau \neq \varepsilon_{(\alpha)}$  and  $\tau$  is strictly contained in  $\rho|(\alpha)$  then  $\tau \cup \varepsilon_X$  is a non identical congruence on  $X$ , strictly contained in  $\rho$ . By Hall's characterization of the atoms in  $\mathcal{C}((\alpha))$ ,  $\rho|(\alpha)$  admits the description as  $\rho|(\alpha) = \theta^{(\alpha)}(\alpha, \beta)$  (the upper index denoting the domain of the relation under consideration). Since  $\rho = \rho|(\alpha) \cup \varepsilon_X = \theta^{(\alpha)}(\alpha, \beta) \cup \varepsilon_X = \theta^X(\alpha, \beta)$  the assertion follows. □

Reconstructing the corresponding atom in  $\mathcal{C}(S)$  we obtain the following description.

**Corollary 3.3.** *Let  $S=(X; I_{\alpha}, f_{\alpha, \beta})$  be a combinatorial strict inverse semigroup. Then  $\rho \in \mathcal{C}(S)$  is an atom if and only if there are  $\alpha, \beta \in X, \beta < \alpha$  such that  $\gamma < \alpha$  implies  $\gamma \leq \beta$  and*

$$\rho = (f_{\alpha, \beta} \cup \varepsilon_S)(f_{\alpha, \beta}^{-1} \cup \varepsilon_S)$$

(the product denoting the usual composition of binary relations on  $S$ ). That is, for  $x \in I_{\gamma}^*, y \in I_{\delta}^*$  we have  $x \rho y$  if and only if one of the following holds

- (1)  $x = y$
- (2)  $\gamma = \alpha, \delta = \beta$  and  $y = x f_{\alpha, \beta}$
- (3)  $\delta = \alpha, \gamma = \beta$  and  $x = y f_{\alpha, \beta}$
- (4)  $\gamma = \delta = \alpha$  and  $x f_{\alpha, \beta} = y f_{\alpha, \beta}$ .

A congruence  $\rho$  on the structure set  $X$  is an atom if and only if  $\rho$  has precisely one class containing two elements, the other  $\rho$ -classes being singletons. Similarly as for the case of coatoms the analogue does not hold for the corresponding atoms in  $\mathcal{C}(S)$ . A trivial example is again any combinatorial Brandt semigroup containing more than two elements, and less trivial examples can be provided easily.

**4. A subdirect decomposition of  $\mathcal{C}(S)$**

For the combinatorial strict inverse semigroup  $S$  we provide a decomposition of  $\mathcal{C}(S)$  into a subdirect product of congruence lattices of semilattices. As in the previous section, we consider the structure set  $X$  of  $S$  as a partial semilattice by setting  $\alpha \wedge \beta = \inf\{\alpha, \beta\}$ , provided this greatest lower bound exists.

**Definition 4.** A subset  $Z \subseteq X$  of a structure set  $X$  is a *subsemilattice* of  $X$  if for any  $\alpha, \beta \in Z$ , their infimum in  $X$   $\inf_X\{\alpha, \beta\}$  exists and is contained in  $Z$ . That is,  $Z$  is a subsemilattice of  $X$  if the partial operation  $\wedge$ , when restricted to  $Z$ , provides a total operation on  $Z$ .

**Remark.** A subset  $Z \subseteq X$ , endowed with the induced partial order may be a semilattice but not a subsemilattice of  $X$ .

Let  $X$  be a structure set. By Proposition 2.3, each principal ideal  $(\alpha]$  in  $X$  is a subsemilattice of  $X$ . A subsemilattice  $Y$  of  $X$  is a *maximal* subsemilattice if there is no subsemilattice  $A$  of  $X$  strictly containing  $Y$ .

**Lemma 4.1.** *Each maximal subsemilattice  $Y$  of  $X$  is an (order) ideal of  $X$ .*

**Proof.** Let  $Z$  be a subsemilattice of  $X$  and let  $\alpha \in Z$ . Let  $\beta \in Z$  and  $\gamma \in (\alpha]$ . Since  $Z$  is a subsemilattice,  $\inf\{\alpha, \beta\}$  exists and is contained in  $(\alpha]$ . Since  $(\alpha]$  is a subsemilattice,

$\inf\{\gamma, \inf\{\alpha, \beta\}\} = \inf\{\alpha, \beta, \gamma\} = \inf\{\beta, \gamma\}$  exists and is contained in  $(\alpha]$ . Consequently,  $Z \cup (\alpha]$  is a subsemilattice of  $X$  for any  $\alpha \in Z$ . We infer that for any maximal subsemilattice  $Y$  of  $X$ ,  $Y = \bigcup_{\alpha \in Y} (\alpha]$ . □

The existence of maximal subsemilattices of  $X$  follows by a usual Zorn's Lemma argument. Denote by  $\mathcal{Y}(X) = \{Y_i | i \in I\}$  the collection of all maximal subsemilattices of  $X$ .

**Lemma 4.2.** *Let  $\rho$  be a congruence on the structure set  $X$  and  $Y$  be a subsemilattice of  $X$ . Then  $\rho|Y$  is a semilattice congruence on  $Y$ .*

**Proof.** Let  $\alpha, \beta, \gamma \in Y$  and  $\alpha \rho \beta$ . By hypothesis,  $\alpha \wedge \beta$  exists. There is some  $\delta \leq \alpha, \beta$  such that  $\alpha \rho \delta \rho \beta$ . Since  $\rho|(\alpha]$  is a semilattice congruence on  $(\alpha]$  we have  $\alpha \wedge \beta = \alpha \wedge (\alpha \wedge \beta) \rho \delta \wedge (\alpha \wedge \beta) = \delta$ . We infer that  $\alpha \rho \alpha \wedge \beta \rho \beta$ . The meet  $\alpha \wedge \gamma$  exists since  $Y$  is a subsemilattice of  $X$ . Hence  $\alpha \wedge \gamma = \alpha \wedge (\alpha \wedge \gamma) \rho (\alpha \wedge \beta) \wedge (\alpha \wedge \gamma) = \inf\{\alpha, \beta, \gamma\}$  and dually also  $\beta \wedge \gamma \rho \inf\{\alpha, \beta, \gamma\}$  so that  $\alpha \wedge \gamma \rho \beta \wedge \gamma$ . □

**Corollary 4.3.** *If  $X$  is a semilattice then the congruences on  $X$  are precisely the semilattice congruences.*

We now are able to prove the main result. The idea is similar to Hamilton's idea to decompose the congruence lattice of a tree into a subdirect product of the congruence lattices of the maximal subchains (see [6]).

**Theorem 4.4.** *Let  $X$  be the structure set of a combinatorial strict inverse semigroup and let  $\{Y_i | i \in I\}$  be the collection of all maximal subsemilattices of  $X$ . Then the mapping*

$$\mathcal{C}(X) \rightarrow \prod_{i \in I} \mathcal{C}(Y_i), \quad \rho \mapsto (\rho|Y_i)_{i \in I}$$

*is a lattice isomorphism between  $\mathcal{C}(X)$  and the complete subdirect product*

$$\Phi = \left\{ (\rho_i)_{i \in I} \in \prod_{i \in I} \mathcal{C}(Y_i) \mid \rho_i|Y_i \cap Y_j = \rho_j|Y_i \cap Y_j, i, j \in I \right\}.$$

**Proof.** Let  $i \in I$  be a fixed element and let  $\{\rho_k | k \in K\}$  be a collection of congruences on  $X$ . By Theorem 2.4 it follows that  $\bigwedge_{k \in K} \rho_k = (\bigcap_{k \in K} \rho_k)^0$ , that is, the greatest congruence on  $X$  contained in  $\bigcap_{k \in K} \rho_k$ . In the following we shall omit the subscript " $k \in K$ ". First we observe that  $(\bigwedge \rho_k)|Y_i \subseteq (\bigcap \rho_k)|Y_i = \bigcap \rho_k|Y_i$ . For each  $k$  put  $\rho'_k = \rho_k|Y_i \cup \varepsilon_X$ . Then  $\rho'_k$  is a congruence on  $X$  and  $\rho'_k \subseteq \rho_k$ . Now  $\bigcap \rho'_k = \bigcap (\rho_k|Y_i \cup \varepsilon_X) = (\bigcap \rho_k|Y_i) \cup \varepsilon_X$  which is a congruence on  $X$ . Hence  $\bigwedge \rho'_k = \bigcap \rho'_k \subseteq \bigwedge \rho_k$  since  $\rho'_k \subseteq \rho_k$  for all  $k$ . Restricting to  $Y_i$  we get  $(\bigwedge \rho'_k)|Y_i = (\bigcap \rho'_k)|Y_i = (\bigcap \rho_k|Y_i \cup \varepsilon_X)|Y_i = \bigcap \rho_k|Y_i$  and hence  $\bigcap \rho_k|Y_i \subseteq (\bigwedge \rho_k)|Y_i$ . Consequently,  $(\bigwedge \rho_k)|Y_i = \bigcap \rho_k|Y_i$  and thus the mapping

$$\mathcal{C}(X) \rightarrow \mathcal{C}(Y_i), \quad \rho \mapsto \rho|Y_i$$

is a complete  $\bigwedge$ -homomorphism. For the join we observe that the inclusion  $\bigvee \rho_k|Y_i \subseteq (\bigvee \rho_k)|Y_i$  is trivial. To show the converse, let  $\alpha, \beta \in Y_i$  be such that  $\alpha \bigvee \rho_k \beta$ . There exist  $\alpha_0, \dots, \alpha_n \in X$  and  $\rho_1, \dots, \rho_n \in \{\rho_k | k \in K\}$  such that  $\alpha = \alpha_0 \rho_1 \alpha_1 \dots \alpha_{n-1} \rho_n \alpha_n = \beta$ . For each  $j$ ,  $1 \leq j \leq n$  there is  $\beta_j \leq \alpha_{j-1}, \alpha_j$  such that  $\alpha_{j-1} \rho_j \beta_j \rho_j \alpha_j$ . Since  $\beta_1, \beta_2 \leq \alpha_1$ , the meet  $\gamma_2 = \inf\{\beta_1, \beta_2\}$  exists. By induction,  $\gamma_j = \gamma_{j-1} \wedge \beta_j = \inf\{\beta_1, \beta_2, \dots, \beta_j\}$  exists and  $\gamma_j \leq \alpha_0, \dots, \alpha_j$ . The relation  $\alpha_j \rho_{j+1} \beta_{j+1}$  yields  $\gamma_j = \alpha_j \wedge \gamma_j \rho_{j+1} \beta_{j+1} \wedge \gamma_j = \gamma_{j+1}$ . Since the maximal subsemilattice  $Y_i$  is an ideal (Lemma 4.1),  $\gamma_j \in Y_i$  for all  $j$ . Put  $\gamma_n = \gamma$  then  $\gamma \leq \alpha, \beta$  and  $\alpha \bigvee \rho_k|Y_i \gamma$ . By the same argument there is some  $\delta \leq \alpha, \beta$  such that  $\beta \bigvee \rho_k|Y_i \delta$ . Now  $\alpha \geq \alpha \wedge \beta \geq \gamma$  and  $\beta \geq \alpha \wedge \beta \geq \delta$  imply  $\alpha \bigvee \rho_k|Y_i \alpha \wedge \beta \bigvee \rho_k|Y_i \beta$ . In particular,  $\bigvee \rho_k|Y_i = (\bigvee \rho_k)|Y_i$  and thus the mapping

$$\mathcal{C}(X) \rightarrow \mathcal{C}(Y_i), \quad \rho \mapsto \rho|Y_i$$

is a complete  $\bigvee$ -homomorphism. For  $\rho_i \in \mathcal{C}(Y_i)$  the relation  $\rho = \rho_i \cup \varepsilon_X$  is a congruence on  $X$  such that  $\rho|Y_i = \rho_i$ . Hence the homomorphism  $\rho \mapsto \rho|Y_i$  is surjective. Now let  $\rho, \eta \in \mathcal{C}(X)$  be such that  $\rho \neq \eta$ . We may assume that there are  $\alpha, \beta \in X$  such that  $\alpha \rho \beta$  and  $(\alpha, \beta) \notin \eta$ . By a Zorn's Lemma argument there are maximal subsemilattices  $Y_i, Y_j$  of  $X$  such that  $(\alpha) \subseteq Y_i$  and  $(\beta) \subseteq Y_j$ . Now there is  $\gamma \leq \alpha, \beta$  such that  $\alpha \rho \gamma \rho \beta$ . Then either  $(\alpha, \gamma) \notin \eta$  or  $(\beta, \gamma) \notin \eta$  and hence either  $\rho|Y_i \neq \eta|Y_i$  or  $\rho|Y_j \neq \eta|Y_j$ . It follows that the mapping  $\mathcal{C}(X) \rightarrow \prod \mathcal{C}(Y_i), \rho \mapsto (\rho|Y_i)_{i \in I}$  is injective. Finally, let  $(\rho_i) \in \prod \mathcal{C}(Y_i)$  be such that for all  $i, j \in I, \rho_i|Y_i \cap Y_j = \rho_j|Y_i \cap Y_j$ . Put  $\rho'_i = \rho_i \cup \varepsilon_X$ . Then  $\rho'_i \in \mathcal{C}(X)$  for each  $i \in I$ . Let  $\rho = \bigvee \rho'_i$ . We obviously have  $\rho_i \subseteq \rho|Y_i$ . Choose a fixed  $j \in I$  and let  $\alpha, \beta \in Y_j$  be such  $\alpha \rho \beta$ . There are  $\alpha_0, \dots, \alpha_n \in X$  and  $\rho'_1, \dots, \rho'_n \in \{\rho'_i | i \in I\}$  such that  $\alpha = \alpha_0 \rho'_1 \alpha_1 \dots \alpha_{n-1} \rho'_n \alpha_n = \beta$ . Similarly as above there are  $\beta_1, \dots, \beta_n \in X$  such that  $\alpha \geq \beta_1 \geq \dots \geq \beta_n \leq \beta$  and  $\beta_{k-1} \rho'_k \beta_k$ . Since  $(\alpha) \subseteq Y_j$  we have  $\beta_k \in Y_j$  for all  $k$ . Since  $\rho'_k|Y_j \cap Y_k = \rho'_j|Y_j \cap Y_k = \rho_j|Y_j \cap Y_k$  we obtain  $\beta_{k-1} \rho_j \beta_k$  for all  $k$  and thus  $\alpha \rho_j \beta$ , that is,  $\alpha \rho_j \gamma$  for some  $\gamma \leq \alpha, \beta$ . Dually also  $\beta \rho_j \delta$  for some  $\delta \leq \alpha, \beta$ , and therefore, as above,  $\alpha \rho_j \alpha \wedge \beta \rho_j \beta$ . Consequently,  $\rho|Y_j \subseteq \rho_j$  and thus  $\rho|Y_j = \rho_j$ . This holds for each  $j \in I$  and thus the proof is complete.  $\square$

We have thus shown that the congruence lattice of a combinatorial strict inverse semigroup is a subdirect product of congruence lattices of semilattices.

**Definition 5.** A (complete) lattice  $L$  is

- (1) *completely semidistributive* if for any  $\sigma, \rho_i \in L, i \in I$  such that  $\rho_i \wedge \sigma = \rho_j \wedge \sigma$  for all  $i, j \in I$ , then  $(\bigvee \rho_i) \wedge \sigma = \bigvee (\rho_i \wedge \sigma)$ .
- (2) *M-symmetric* if the modularity relation  $M$  on  $L$ , defined by  $\rho M \sigma$  if and only if  $\tau = (\tau \vee \rho) \wedge \sigma$  for all  $\tau \in [\rho \wedge \sigma, \sigma]$ , is a symmetric relation.

It has been shown by Papert [11] that the congruence lattice of any semilattice is completely semidistributive, and by Jones [9, Theorem 3.3] that the congruence lattice



of a semilattice is  $M$ -symmetric. By Jones [8, Proposition 5.1],  $M$ -symmetry is preserved by subdirect products. Theorem 4.4 therefore yields the following:

**Corollary 4.5.** *Let  $T \in \mathcal{B}$ . Then the following assertions hold:*

- (1)  $\mathcal{C}(T)$  is completely semidistributive,
- (2)  $\mathcal{C}(T)$  is  $M$ -symmetric.

Furthermore, the classes  $\mathcal{C}(\mathcal{B})$  and  $\mathcal{C}(\mathcal{S})$  generate the same quasivariety of lattices.

Notice that  $M$ -symmetry of  $\mathcal{C}(T)$  also follows from the result of Jones [9, Theorem 3.3] by taking into account that for any  $\rho, \eta \in \mathcal{C}(T)$ ,  $\rho = \eta$  if and only  $\rho|E = \eta|E$ ,  $E$  denoting the set of idempotents of  $T$ .

**5. Weak relative pseudocomplements and pseudocomplements**

Let  $L$  be a lattice,  $\rho, \sigma \in L$ ,  $\rho \leq \sigma$ . If there is a greatest element  $\tau \in L$  such that  $\tau \wedge \sigma = \rho$  then  $\tau$  is a *weak relative pseudocomplement* of  $\rho$  and  $\sigma$ , to be denoted by  $\sigma * \rho$ . If  $\sigma * \rho$  exists for all pairs  $\rho \leq \sigma$  in  $L$  then  $L$  is *weakly relatively pseudocomplemented*. If  $L$  has a least element  $0$  then for any  $\sigma \in L$  the weak relative pseudocomplement  $\sigma * 0$  is called the *pseudocomplement* of  $\sigma$  (provided it exists). An important result of Papert [11] asserts that the congruence lattice  $\mathcal{C}(Y)$  of any semilattice  $Y$  is weakly relatively pseudocomplemented and hence also is pseudocomplemented. We shall extend this result to congruence lattices of combinatorial strict inverse semigroups. As in the previous section, for a combinatorial strict inverse semigroup  $S$ , the congruence lattice  $\mathcal{C}(S)$  will be realized as the lattice  $\mathcal{C}(X)$  of all congruences on the structure set  $X$  of  $S$  and thus as a subdirect product of the congruence lattices of the maximal subsemilattices  $Y_i$  of  $X$  as described in Theorem 4.4.

Let  $X$  be a structure set,  $\mathcal{Y}(X) = \{Y_i | i \in I\}$  be the collection of all maximal subsemilattices of  $X$ . For each  $\rho \in \mathcal{C}(X)$ ,  $i \in I$  let  $\rho_i = \rho|Y_i \in \mathcal{C}(Y_i)$ . Let  $\rho, \sigma \in \mathcal{C}(X)$ ,  $\rho \leq \sigma$ . Then  $\rho_i \leq \sigma_i$  for all  $i \in I$ . Further, if  $\tau \cap \sigma = \rho$  for some  $\tau \in \mathcal{C}(X)$  then  $\tau_i \cap \sigma_i = \rho_i$  for each  $i$  and hence  $\tau_i \leq \sigma_i * \rho_i$ ,  $*_i$  denoting the weak relative pseudocomplement in  $\mathcal{C}(Y_i)$ .

**Lemma 5.1.** *Let  $Y$  be a semilattice,  $Z$  be an ideal (and thus a subsemilattice) of  $Y$ ,  $\rho, \sigma \in \mathcal{C}(Y)$ ,  $\rho \leq \sigma$ . Then  $\sigma|Z *_Z \rho|Z = \sigma *_Y \rho|Z$  ( $*_Z$  respectively  $*_Y$  denoting weak relative pseudocomplementation in  $\mathcal{C}(Z)$  respectively  $\mathcal{C}(Y)$ ).*

**Proof.** Put  $\rho' = \rho|Z$  and  $\sigma' = \sigma|Z$ . Then  $\rho' \leq \sigma'$  and thus  $\sigma' *_Z \rho'$  exists. Now  $\sigma \cap (\sigma *_Y \rho)|Z = \sigma' \cap (\sigma *_Y \rho|Z) = \varepsilon_Z$  so that  $\sigma *_Y \rho|Z \subseteq \sigma' *_Z \rho'$ . By [11, Theorem 2], for  $\alpha, \beta \in Z$  we have

$$\alpha \sigma' *_Z \rho' \beta \Leftrightarrow ((\forall \delta, \gamma \in Z) \delta \sigma' \gamma \Rightarrow (\alpha \wedge \delta \rho' \alpha \wedge \gamma \Leftrightarrow \beta \wedge \delta \rho' \beta \wedge \gamma)).$$

We intend to prove that  $\alpha \sigma *_Y \rho \beta$  whenever  $\alpha \sigma' *_Z \rho' \beta$ . Let  $\alpha, \beta \in Z$  and  $\alpha \sigma' *_Z \rho' \beta$ . We

have to prove that  $\alpha \sigma *_Y \rho \beta$ , that is,  $\alpha \wedge \delta \rho \alpha \wedge \gamma \Leftrightarrow \beta \wedge \delta \rho \beta \wedge \gamma$  for any  $\gamma, \delta \in Y$  such that  $\delta \sigma \gamma$ . Let  $\gamma, \delta \in Y$  be such that  $\delta \sigma \gamma$ . Since  $Z$  is an ideal,  $\alpha \wedge \delta, \alpha \wedge \gamma, \beta \wedge \delta, \beta \wedge \gamma \in Z$ , and  $\alpha \wedge \delta \sigma' \alpha \wedge \gamma, \beta \wedge \delta \sigma' \beta \wedge \gamma$ . By  $\alpha \sigma' *_Z \rho' \beta$  we have

$$\alpha \wedge \delta = \alpha \wedge (\alpha \wedge \delta) \rho' \alpha \wedge (\alpha \wedge \gamma) = \alpha \wedge \gamma \Leftrightarrow \beta \wedge (\alpha \wedge \delta) \rho' \beta \wedge (\alpha \wedge \gamma)$$

and

$$\beta \wedge \delta = \beta \wedge (\beta \wedge \delta) \rho' \beta \wedge (\beta \wedge \gamma) = \beta \wedge \gamma \Leftrightarrow \alpha \wedge (\beta \wedge \delta) \rho' \alpha \wedge (\beta \wedge \gamma).$$

Since  $\alpha \wedge (\beta \wedge \delta) = \beta \wedge (\alpha \wedge \delta)$  and  $\alpha \wedge (\beta \wedge \gamma) = \beta \wedge (\alpha \wedge \gamma)$  we infer that  $\alpha \wedge \delta \rho' \alpha \wedge \gamma \Leftrightarrow \beta \wedge \delta \rho' \beta \wedge \gamma$ , that is,  $\alpha \wedge \delta \rho \alpha \wedge \gamma \Leftrightarrow \beta \wedge \delta \rho \beta \wedge \gamma$ . Consequently,  $\alpha \sigma *_Y \rho \beta$  and the assertion is proved. □

Using Theorem 4.4 it is now easy to obtain the mentioned result.

**Theorem 5.2.** *The congruence lattice of a combinatorial strict inverse semigroup  $S$  is isomorphic to a subdirect product of congruence lattices of semilattices closed under componentwise formation of weak relative pseudocomplements and pseudocomplements. In particular,  $\mathcal{C}(S)$  is pseudocomplemented.*

**Proof.** Let  $X$  be the structure set of  $S$  and  $\{Y_i \mid i \in I\}$  be the collection of all maximal subsemilattices of  $S$ . By Theorem 4.4,  $\mathcal{C}(X)$  is isomorphic to

$$\Phi = \left\{ (\rho_i)_{i \in I} \in \prod_{i \in I} \mathcal{C}(Y_i) \mid \rho_i \mid Y_i \cap Y_j = \rho_j \mid Y_i \cap Y_j, i, j \in I \right\}.$$

Let  $\rho, \sigma \in \mathcal{C}(X)$ ,  $\rho \subseteq \sigma$ . Then  $\rho_i \subseteq \sigma_i$  and hence  $\sigma_i *_i \rho_i$  exists for each  $i$ ,  $*_i$  denoting the weak relative pseudocomplements in  $\mathcal{C}(Y_i)$ . By Lemma 5.1,

$$\begin{aligned} \sigma_i *_i \rho_i \mid Y_i \cap Y_j &= (\sigma_i \mid Y_i \cap Y_j) * (\rho_i \mid Y_i \cap Y_j) \\ &= (\sigma_j \mid Y_i \cap Y_j) * (\rho_j \mid Y_i \cap Y_j) \\ &= \sigma_j *_j \rho_j \mid Y_i \cap Y_j, \end{aligned}$$

$*$  denoting the weak relative pseudocomplement in  $\mathcal{C}(Y_i \cap Y_j)$ . Hence  $(\sigma_i *_i \rho_i)_{i \in I} \in \Phi$  and thus  $\sigma *_\rho = \bigvee_{i \in I} \sigma_i *_i \rho_i \cup \varepsilon_X$  is the weak relative pseudocomplement of  $\rho$  and  $\sigma$  in  $\mathcal{C}(X)$ . □

**6. An example**

In [1], it is shown that there is a (finite) combinatorial strict inverse semigroup  $S$  whose structure set  $X$  consists of five elements and is depicted by Fig. 1.

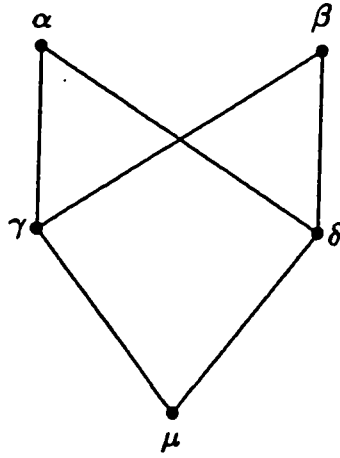


FIGURE 1

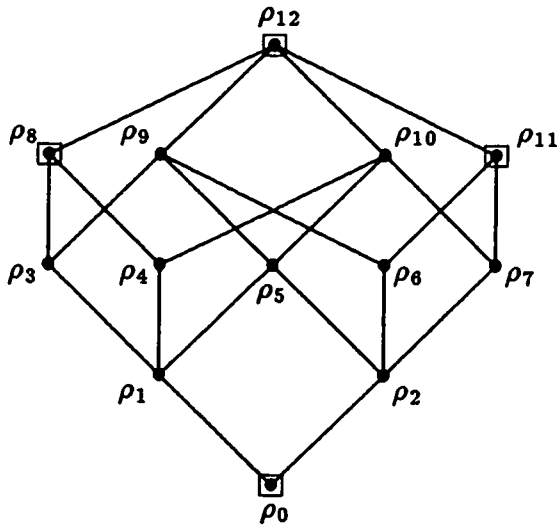


FIGURE 2

In particular,  $X$  is not a semilattice. All congruences on  $X$  are listed below (by denoting the corresponding partitions, and only the non singleton classes are mentioned):

$$\begin{aligned} \rho_0 &: \varepsilon_X \\ \rho_1 &: \{\mu, \gamma\} \\ \rho_2 &: \{\mu, \delta\} \\ \rho_3 &: \{\mu, \gamma\}, \{\delta, \alpha\} \\ \rho_4 &: \{\mu, \gamma\}, \{\delta, \beta\} \\ \rho_5 &: \{\mu, \gamma, \delta\} \\ \rho_6 &: \{\mu, \delta\}, \{\gamma, \alpha\} \\ \rho_7 &: \{\mu, \delta\}, \{\gamma, \beta\} \\ \rho_8 &: \{\mu, \gamma\}, \{\alpha, \beta, \delta\} \\ \rho_9 &: \{\mu, \gamma, \alpha, \delta\} \\ \rho_{10} &: \{\mu, \gamma, \beta, \delta\} \\ \rho_{11} &: \{\mu, \delta\}, \{\alpha, \beta, \gamma\} \\ \rho_{12} &: \omega_X. \end{aligned}$$

The lattice  $\mathcal{C}(X)$  is depicted by Fig. 2. The Boolean elements in  $\mathcal{C}(X)$ , that is, those elements  $\rho$  in  $\mathcal{C}(X)$  which are pseudocomplements of elements of  $\mathcal{C}(X)$  are precisely the elements  $\rho_0, \rho_8, \rho_{11}, \rho_{12}$ . If  $\mathcal{C}(X)$  was isomorphic to the congruence lattice of some semilattice  $Y$  then by [11, Theorem 5],  $Y$  could be embedded into the (lattice of the) Boolean elements of  $\mathcal{C}(Y) \cong \mathcal{C}(X)$ . But no subsemilattice of the four element diamond lattice has a congruence lattice isomorphic to  $\mathcal{C}(X)$ . Hence  $\mathcal{C}(X)$  and thus also  $\mathcal{C}(S)$  cannot be isomorphic to the congruence lattice of any semilattice.

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