HARDY AND MIYACHI THEOREMS FOR HEISENBERG MOTION GROUPS

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Abstract. Let $G = \mathbb{H}^n \rtimes K$ be the Heisenberg motion group, where $K = U(n)$ acts on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ by automorphisms. We formulate and prove two analogues of Hardy's theorem on G. An analogue of Miyachi's theorem for G is also formulated and proved. This allows us to generalize and prove an analogue of the Cowling–Price uncertainty principle and prove the sharpness of the constant 1/4 in all the settings.

§1. Introduction

A classical theorem of Hardy proved in 1933 in the case of the real line, says that an integrable function f and its Fourier transform $\hat{f}$ cannot both have arbitrary Gaussian decay unless f is identically zero. More precisely, if both $f(x)e^{\alpha\|x\|^2}$ and $\hat{f}(\xi)e^{\beta\|\xi\|^2}$ are in $L^\infty(\mathbb{R}^n)$ for some $\alpha, \beta > 0$ then the following conclusions hold:

(1) $f = 0$ whenever $\alpha\beta > 1/4$.
(2) The function f is a constant multiple of $e^{-\alpha\|x\|^2}$ when $\alpha\beta = 1/4$.
(3) When $\alpha\beta < 1/4$, there are infinitely many linearly independent functions satisfying both conditions.

Here the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp(-i\langle x, y\rangle)\, dx$$

and $\|x\| = \langle x, x\rangle^{1/2}$ is the Euclidean norm. Actually, the bound $\frac{1}{4}$ is sharp and the strong Gaussian decay of f and $\widehat{f}$ is only required with respect to a one-dimensional subgroup of $\mathbb{R}^n$. For a detailed proof of Hardy's theorem,

see [7]. Hardy's theorem initially proved for the Euclidean Fourier transform has been extended to several setups including the Heisenberg group and many other various classes of nonabelian connected Lie groups ([3, 4, 6, 12–14, 16, 17] and [19]). Specifically, analogues of Hardy's theorem have been proved for motion groups ([6] and [16]). In case of a Lie group G, the Euclidean norms on $\mathbb{R}^n$ and on $\widehat{\mathbb{R}^n}$ have to be replaced by certain norm functions on G and the Hilbert–Schmidt norm of the operator-valued Fourier transform on the unitary dual $\widehat{G}$ of G, respectively depending upon the parameters involved in the problem in question.

In this context, we mention that in [10], Sarkar and Thangavelu gave an analogue of Hardy's theorem for the Euclidean motion group $M_n := SO(n) \ltimes \mathbb{R}^n$ by means of the heat kernel. Indeed, denoting by Δ_n the standard Laplacian on $\mathbb{R}^n$ and by

$$p_t^n(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{1}{4t}\|x\|^2}$$

the associated heat kernel, they proved the following result.

THEOREM 1.1. *Let $f \in L^1(M_n)$ satisfy the following conditions:*

(1) $|f(k, x)| \leqslant C(1 + \|x\|)^N p_s^n(x), \ (k, x) \in M_n.$
(2) $\|\pi_{r,\sigma}(f)\|_{HS} \leqslant C(1 + r)^N e^{-tr^2} \ (r, \sigma) \in \mathbb{R}_+^* \times \widehat{SO(n-1)}.$

Then $f = 0$ whenever $s < t$. When $s = t$, f can be expressed as a finite linear combination of functions of the form

$$P_{m,j}(x)(-\Delta_{n+2m})^{\frac{j-m}{2}} p_t^{n+2m}(x) g_{m,j}(k)$$

where $P_{m,j}$ are solid harmonics of degree m and $g_{m,j}$ are certain bounded functions in $L^2(SO(n))$.

A weaker version of the last theorem for the case $s < t$ is known from the work of Sundari (cf. [16]). For arbitrary compact extension $M_K := K \ltimes \mathbb{R}^n$ of $\mathbb{R}^n$, the following analogue of Hardy's uncertainty principle was also recently proved (cf. [2]).

THEOREM 1.2. *Let $q \in \mathbb{N}$, α and β be positive real numbers and let f be a measurable function on M_K satisfying the following decay conditions:*

(i) $|f(k, x)| \leqslant \varphi(k)(1 + \|x\|^2)^q e^{-\alpha\|x\|^2}$ *for all $x \in \mathbb{R}^n$ and $k \in K$, where $\varphi \in L^2(K),$*

(ii) $\|\pi_{\ell,\sigma}(f)\|_{HS} \leqslant \psi_\ell(\sigma)(1 + \|\ell\|^2)^q e^{-\beta\|\ell\|^2}$, for all $\ell \in \mathbb{R}^n$ and all $\sigma \in \widehat{K_\ell}$, with $\|\psi_\ell\|_{l^2(\widehat{K_\ell})} \leqslant C$ for some positive constant C independent of ℓ.

Then the following conclusions hold:

(1) $f = 0$ almost everywhere whenever $\alpha\beta > 1/4$.
(2) If $\alpha\beta < 1/4$, then there are infinitely many linearly independent functions satisfying both conditions (i) and (ii).
(3) When $\alpha\beta = 1/4$, the function f is of the form $f(k, x) = \zeta(k, x)e^{-\alpha\|x\|^2}$ where $x \mapsto \zeta(k, x)$ is polynomial on x of degree $\leqslant 2q$ and $k \mapsto \zeta(k, x)$ is an L^2-function on K.

In this paper, our main goal is to prove a suitable analogue of Theorem 1.1 in the context of Heisenberg motion groups. Let $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the group operation defined by

$$(z, t) \cdot (w, s) = (z + w, t + s + \tfrac{1}{2}\text{Im}(z \cdot \overline{w})), z, w \in \mathbb{C}^n; t; \quad s \in \mathbb{R}.$$

Alternatively, we can consider $\mathbb{H}^n$ as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law

$$(x, y, t) \cdot (u, v, s) = (x + u, y + v, t + s + \tfrac{1}{2}(u \cdot y - v \cdot x)),$$
$$x, y, u, v \in \mathbb{R}^n; t, \quad s \in \mathbb{R}.$$

Let $K = U(n)$ act on $\mathbb{H}^n$ by the automorphisms

$$(\sigma, (z, t)) \to (\sigma z, t), \qquad \sigma \in K, (z, t) \in \mathbb{H}^n,$$

and let $G = \mathbb{H}^n \rtimes K$ be the Heisenberg motion group with the group law

$$(z, t, k)\,(w, s, k') = ((z, t) \cdot (kw, s),\ kk').$$

We will prove two analogues of Hardy's uncertainty principle for the Heisenberg motion group. The first one makes use of the heat kernel of the Heisenberg group and that of the associated compact group K and the second utilizes the Euclidean norms. We will thus prove parallel versions of the uncertainty principle as in Euclidean motion groups (cf. Theorems 1.1 and 2.1) without taking into account the polynomial growth in the conditions of decay.

Our second formulation of Hardy's uncertainty principle (cf. Section 3) comes as a direct consequence of Miyachi's uncertainty principle. In 1997 Miyachi [9] proved the following uncertainty principle which includes both Hardy and Cowling–Price theorems as corollaries.

THEOREM 1.3. *Let f be an integrable function on $\mathbb{R}^n$ such that*

$$e^{\alpha\|\cdot\|^2} f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n),$$

for some positive number α. Further assume that

$$\int_{\mathbb{R}^n} \log^+\left(\frac{e^{\beta\|y\|^2}|\widehat{f}(y)|}{c}\right) dy < \infty$$

for some positive numbers β and c. If $\alpha\beta = 1/4$, then f is a constant multiple of the Gaussian $e^{-\alpha\|\cdot\|^2}$.

This result leads to the following corollary.

COROLLARY 1.4. (Cf. [1]) *Under the same conditions of Theorem 1.3, the function f vanishes almost everywhere if $\alpha\beta > 1/4$. When $\alpha\beta < \frac{1}{4}$, there exist infinitely many linearly independent functions satisfying both conditions.*

The last part of the paper is devoted to formulate and prove an analogue of Miyachi's theorem for Heisenberg motion groups. As a direct consequence, we also derive an analogue of Cowling–Price uncertainty principle and prove the sharpness of the constant $1/4$ in both the settings.

§2. A heat kernel version of Hardy's uncertainty principle for Heisenberg motion groups

Let $\pi_\lambda(z,t), \lambda \in \mathbb{R}^*$ be the Schrödinger representation of $\mathbb{H}^n$ realized on $L^2(\mathbb{R}^n)$ and defined as:

$$\pi_\lambda(z,t)f(\xi) = e^{i\lambda t}e^{i\lambda(x\xi+\frac{1}{2}xy)}f(\xi+y), \quad z = (x,y).$$

For each $k \in K \subset U(n), (x,y,t) \mapsto (k \cdot (x,y),t)$ is an automorphism of $\mathbb{H}^n$ because $U(n)$ preserves the symplectic form $xy - yv$ on $\mathbb{R}^{2n}$. If ρ is a representation of $\mathbb{H}^n$ then using this automorphism we can define another representation ρ_k by $\rho_k(x,u,t) = \rho(k \cdot (x,u),t)$ which coincides with ρ at the center. If we take ρ to be the Schrödinger representation π_λ, then by Stone–von Neumann theorem ρ_k is unitarily equivalent to ρ and there exists a unitary operator $\mu_\lambda(k)$ such that:

$$\mu_\lambda(k)\,\pi_\lambda(z,t)\,\mu_\lambda(k)^\star = \pi_\lambda(kz,t).$$

The operator-valued function μ_λ, which can be extended to the double cover of $\mathrm{Sp}(n,\mathbb{R})$, can be chosen so that it becomes a unitary representation

of K on $L^2(\mathbb{R}^n)$ and is called the metaplectic representation. Let $\sigma \in \widehat{K}$ acting on a Hilbert space V_σ. We define for each $\lambda \in \mathbb{R}^*$ and $\sigma \in \widehat{K}$, the representation ρ_σ^λ on $L^2(\mathbb{R}^n) \otimes V_\sigma$ by

$$(2.1) \qquad \rho_\sigma^\lambda(z, t, k) = (\pi_\lambda(z, t)\mu_\lambda(k)) \otimes \sigma(k).$$

Then it is shown that ρ_σ^λ are all irreducible unitary representations of G. They are enough to describe the Plancherel measure of G (cf. [11]). For an integrable function f on G its Fourier transform $\rho_\sigma^\lambda(f)$ is the operator-valued function given by

$$\rho_\sigma^\lambda(f) = \int_G f(g)\rho_\sigma^\lambda(g)\,dg$$

where dg is the Haar measure on G. Observe that $\rho_\sigma^\lambda(f)$ is a bounded linear operator acting on $L^2(\mathbb{R}^n) \otimes V_\sigma$.

Let $\mathscr{L}$ be the sublaplacian on $\mathbb{H}^n$ with the associated heat kernel $p_a(z, t)$, which is given by:

$$p_a^\lambda(z) = \int_{-\infty}^{+\infty} e^{i\lambda t} p_a(z, t)\,dt = c_n \left(\frac{\lambda}{\sinh a\lambda}\right)^n e^{-\frac{\lambda}{4}\coth(a\lambda)|z|^2}.$$

Let Δ_K be the Laplacian on K with the associated heat kernel $q_a(k)$. Let $h_a(z, t, k) = p_a \otimes q_a(z, t, k)$ be the heat kernel on $G = \mathbb{H}^n \rtimes K$ associated to $\mathscr{L} + \Delta_K$. We prove the following two theorems.

THEOREM 2.1. *Let f be a measurable function on G which satisfies:*

(i) $\quad |f(z, t, k)| \leqslant C\, h_a(z, t, k)$.
(ii) *For every $\lambda \in \mathbb{R}^*$ and $\sigma \in \widehat{K}$, $\rho_\sigma^\lambda(f)^\star \rho_\sigma^\lambda(f) \leqslant C\rho_\sigma^\lambda(h_b)^\star \rho_\sigma^\lambda(h_b)$,*

for some positive constant C. Further assume that f is central in the K-variable in the sense that $f(z, t, k) = f(z, t, uku^{-1})$ for any $k, u \in K$ and $(z, t) \in \mathbb{H}^n$. Then $f = 0$ whenever $a < b$.

The following remarks are in order. In the above theorem we are not able to relax the condition that f is central in the last variable. We believe the result is true without this assumption though our proof requires it. Uncertainty principles for general functions with no restrictions on K-types is still an open problem even in the context of semisimple Lie groups. The only group on which a general version of Hardy's uncertainty principle has been proved is $SU(1, 1)$, see [8] and [18].

We also remark that we do not know what happens when $a = b$ in the above theorem. The same comment applies to Hardy's theorem (see [19, Theorem 2.9.2]) on the Heisenberg group also. However, we have the following version which treats the equality case which is the analogue of [19, Theorem 2.9.5].

For an integrable function f on G, define the partial Fourier transform

$$(2.2) \qquad f^\lambda(z, k) = \int_{\mathbb{R}} f(z, t, k) e^{i\lambda t} dt.$$

THEOREM 2.2. *Let f be a measurable function on G which is central in the K-variable and satisfies:*

(i) $|f^\lambda(z, k)| \leqslant C\, p_a^\lambda(z) q_a(k)$ *for all $\lambda \in \mathbb{R}^*$*
(ii) *For every $\lambda \in \mathbb{R}^*$ and $\sigma \in \widehat{K}$, $\rho_\sigma^\lambda(f)^\star \rho_\sigma^\lambda(f) \leqslant C\, \rho_\sigma^\lambda(h_a)^\star \rho_\sigma^\lambda(h_a)$,*

for some positive constant C. Then $f(z, t, k) = \rho_a(z, t)\, \varphi(k)$, for some $\varphi \in L^2(K)$.

The above theorems are proved by reducing them to the following result for Laguerre expansions. We let

$$\varphi_{k,\lambda}^{n-1}(z) = L_k^{n-1}(\tfrac{1}{2}|\lambda||z|^2) e^{-\frac{1}{4}|\lambda||z|^2}$$

stand for Laguerre functions of type $(n - 1)$ on $\mathbb{C}^n$. Any radial function f on $\mathbb{C}^n$ can be expanded in terms of $\varphi_{k,\lambda}^{n-1}(z)$. Let $f(z, k)$ be a radial function on $\mathbb{C}^n$ taking values in $L^2(K)$. Define

$$R_m^\lambda(f, k) = \frac{\Gamma(m+1)}{\Gamma(m+n)} \int_{\mathbb{C}^n} f(z, k) \varphi_{k,\lambda}^{n-1}(z)\, dz$$

as the Laguerre coefficients of f. We have the following theorem.

THEOREM 2.3. *Let $\lambda \in \mathbb{R}^*$ be fixed. Assume that $f \in L^2(\mathbb{C}^n, L^2(K))$ is a radial function which satisfies:*

(i) $\left(\int_K |f(z, k)|^2\, dk \right)^{\frac{1}{2}} \leqslant C p_a^\lambda(z)$,
(ii) *For all m,* $\left(\int_K |R_m^\lambda(f, k)|^2\, dk \right)^{\frac{1}{2}} \leqslant C \left((2m+n)|\lambda| \right)^{\frac{n-1}{2}} e^{-(2m+n)|\lambda|b}$

for some positive constant C. Then $f = 0$ whenever $a < b$ and $f(z, k) = c\, p_a^\lambda(z)\varphi(k)$ for some $\varphi \in L^2(K)$ when $a = b$.

We do not prove the above theorem but remark that the proof is similar to that of Hardy's theorem for the Heisenberg group [19, Theorem 2.9.2]. In fact, Hardy's theorem is proved by reducing it to the above. See the proofs of [19, Theorems 2.9.4 and 2.9.5].

In proving Theorems 2.1 and 2.2, we make use of certain results such as Peter–Weyl theorem from the representation theory of compact Lie groups. For the convenience of the reader we collect the relevant results referring the reader to [15] for details. Given a compact Lie group K let $\widehat{K}$ stand for its unitary dual consisting of equivalence classes of irreducible unitary representations. If σ is one such representation realized on a d_σ dimensional Hilbert space H_σ with an orthonormal basis v_j, $j = 1, 2, \ldots, d_\sigma$ we denote the matrix coefficients $(\sigma(k)v_j, v_i)$ by $\sigma_{ij}(k)$. Then Peter–Weyl theorem asserts that the functions $\{\sqrt{d_\sigma}\sigma_{ij} : 1 \leqslant i, j \leqslant d_\sigma, \sigma \in \widehat{K}\}$ is an orthonormal basis for $L^2(K)$. Thus one has the expansion

$$f = \sum_{\sigma \in \widehat{K}} \sum_{i,j=1}^{d_\sigma} d_\sigma(f, \sigma_{ij})\sigma_{ij}$$

and the Plancherel formula holds:

$$\int_K |f(k)|^2 \, dk = \sum_{\sigma \in \widehat{K}} \sum_{i,j=1}^{d_\sigma} d_\sigma |(f, \sigma_{ij})|^2.$$

By introducing the character $\chi_\sigma(k)$ as the trace of $\sigma(k)$ the Peter–Weyl expansion can also be written as

$$f(k) \sum_{\sigma \in \widehat{K}} d_\sigma f * \chi_\sigma(k).$$

We will make use of these results in the case of $K = U(n)$.

Proof of Theorem 2.1. The proof is long and is given in several steps. We start the proof with the following calculation of $\rho_\sigma^\lambda(h_a)$. Let $\{v_1, v_2, \ldots, v_{d(\sigma)}\}$ be an orthonormal basis for V_σ. Let $(\Phi_\alpha^\lambda)_{\alpha \in \mathbb{N}^n}$ stand for the Hermite basis for $L^2(\mathbb{R}^n)$. We have the following:

$$\begin{aligned}
\rho_\sigma^\lambda(h_a)(\Phi_\alpha^\lambda \otimes v_j) &= \int_{\mathbb{H}^n \times K} p_a(z, t)q_a(k) \\
&\quad \times (\pi_\lambda(z, t)\mu_\lambda(k) \otimes \sigma(k))(\Phi_\alpha^\lambda \otimes v_j) \, dz \, dt \, dk \\
(2.3) \qquad &= \int_K (\widehat{p_a}(\lambda)\mu_\lambda(k)\Phi_\alpha^\lambda) \otimes (\sigma(k)v_j)q_a(k) \, dk.
\end{aligned}$$

Here $\widehat{p_a}(\lambda) = \pi_\lambda(p_a)$ is the Fourier transform of the heat kernel p_a and it is well known that $\widehat{p_a}(\lambda) = e^{-aH(\lambda)}$ is the Hermite semigroup generated by the Hermite operator $H(\lambda) = (-\Delta + \lambda^2 |x|^2)$, see [19, Section 2.8]. If $|\alpha| = N$, then the space E_N^λ spanned by Φ_β^λ, $|\beta| = N$ is invariant under $\mu_\lambda(k)$. Thus there exist some functions $c_{\alpha\beta}^\lambda(k)$ on K such that

$$(2.4) \qquad \mu_\lambda(k)\, \Phi_\alpha^\lambda = \sum_{|\beta|=N} c_{\alpha\beta}^\lambda(k)\, \Phi_\beta^\lambda.$$

As $\mu_\lambda(k)$ is unitary, we have

$$(2.5) \qquad \sum_{|\beta|=N} |c_{\alpha\beta}^\lambda(k)|^2 = 1.$$

We also know that the matrix coefficients $(\sigma(k)v_j, v_i)$ form an orthogonal system in $L^2(K)$. We can write

$$(2.6) \qquad \sigma(k)\, v_j = \sum_{i=1}^{d(\sigma)} (\sigma(k)v_j, v_i)\, v_i.$$

Using (2.4) and (2.6) we have

$$\rho_\sigma^\lambda(h_a)\, (\Phi_\alpha^\lambda \otimes v_j) = \sum_{|\beta|=N} \sum_{i=1}^{d(\sigma)} \left(\int_K c_{\alpha\beta}^\lambda(k)(\sigma(k)v_j, v_i)q_a(k)\, dk \right)$$
$$\times\, (\Phi_\beta^\lambda \otimes v_i)e^{-(2N+n)a|\lambda|},$$

where we have used the relation

$$e^{-aH(\lambda)}\, \Phi_\beta^\lambda = e^{-(2N+n)|\lambda|a}\, \Phi_\beta^\lambda$$

which follows from the fact that Φ_β^λ are eigenfunctions of $H(\lambda)$ with eigenvalue $(2|\beta| + n)|\lambda|$, see [19, Section 2.3]. Thus,

$$\|\rho_\sigma^\lambda(h_a)(\Phi_\alpha^\lambda \otimes v_j)\|^2 = \sum_{|\beta|=N} \sum_{i=1}^{d(\sigma)} \left| \int_K c_{\alpha\beta}^\lambda(k)(\sigma(k)v_j, v_i)q_a(k)\, dk \right|^2 e^{-2a(2N+n)|\lambda|}.$$

Summing over all $j = 1, 2, \ldots, d(\sigma)$ and then over all $\sigma \in \widehat{K}$ and using Peter–Weyl theorem we get

$$\sum_{\sigma \in \widehat{K}} \sum_{j=1}^{d(\sigma)} \left\| \rho_\sigma^\lambda(h_a)(\Phi_\alpha^\lambda \otimes v_j) \right\|^2 d(\sigma) = e^{-2a(2N+n)|\lambda|} \sum_{|\beta|=N} \int_K \left| c_{\alpha\beta}^\lambda(k) \right|^2 |q_a(k)|^2 \, dk$$

$$= e^{-2a(2N+n)|\lambda|} \int_K |q_a(k)|^2 \, dk$$

in view of (2.5). Hence we note that

$$\| \rho_\sigma^\lambda(h_a)(\Phi_\alpha^\lambda \otimes v_j) \| \leqslant C \, e^{-a(2N+n)|\lambda|}.$$

The hypothesis $\rho_\sigma^\lambda(f)^\star \, \rho_\sigma^\lambda(f) \leqslant C \, \rho_\sigma^\lambda(h_a)^\star \, \rho_\sigma^\lambda(h_a)$ leads to:

$$(2.7) \qquad \sum_{\sigma \in \widehat{K}} \sum_{j=1}^{d(\sigma)} \| \rho_\sigma^\lambda(f) \, (\Phi_\alpha^\lambda \otimes v_j) \|^2 \, d(\sigma) \leqslant C \, e^{-2a(2N+n)|\lambda|}, \quad |\alpha| = N.$$

Let $M = U(n-1)$ considered as a subgroup of $U(n) = K$ and let $\widehat{K_M}$ stand for the set of all class one representations in $\widehat{K}$. By Peter–Weyl theorem, we have

$$f(u'z, t, k) = \sum_{\delta \in \widehat{K}} d(\delta) \int_K f(u'u^{-1}z, t, k) \, \chi_\delta(u) \, du.$$

It can be shown that the integral is nonzero only when $\delta \in \widehat{K_M}$. Hence evaluating at $u' = e$, we get:

$$f(z, t, k) = \sum_{\delta \in \widehat{K_M}} d(\delta) \int_K f(u^{-1}z, t, k) \chi_\delta(u) \, du.$$

Define $f_\delta(z, t, k) = \int_K f(u^{-1}z, t, k) \chi_\delta(u) \, du$, so that

$$f(z, t, k) = \sum_{\delta \in \widehat{K_M}} d(\delta) \, f_\delta(z, t, k).$$

Note that f_δ satisfies hypothesis (i) of Theorems 2.1 and 2.2. We will show that they also satisfy hypothesis (ii) whenever f satisfies the same. Hence Theorems 2.1 and 2.2 will be proved once we prove them for f_δ for each $\delta \in \widehat{K_M}$.

PROPOSITION 2.4.   *Let $f(z, t, k)$ be central in the K-variable. Then the condition*

$$\rho_\sigma^\lambda(f_\delta)^\star \rho_\sigma^\lambda(f_\delta) \leqslant C \rho_\sigma^\lambda(h_b)^\star \rho_\sigma^\lambda(h_b)$$

holds whenever

$$\rho_\sigma^\lambda(f)^\star \rho_\sigma^\lambda(f) \leqslant C \rho_\sigma^\lambda(h_b)^\star \rho_\sigma^\lambda(h_b)$$

holds.

Proof.   Recall that for $\delta \in \widehat{K_M}$,

$$
\begin{aligned}
f_\delta(z, t, k) &= \int_K f(u^{-1}z, t, k)\, \chi_\delta(u)\, du \\
&= \int_K f(u^{-1}z, t, u^{-1}ku)\chi_\delta(u)\, du,
\end{aligned}
$$

and

$$\rho_\sigma^\lambda(f_\delta) = \int_K \chi_\delta(u)\, d(u) \int_{\mathbb{H}^n \times K} f(u^{-1}z, t, u^{-1}ku)\, \rho_\sigma^\lambda(z, t, k)\, dz\, dt\, dk.$$

We now compute:

$$
\begin{aligned}
&\int_{\mathbb{H}^n \times K} f(u^{-1}z, t, u^{-1}ku)\pi_\lambda(z, t)\mu_\lambda(k) \otimes \sigma(k)\, dz\, dt\, dk \\
&= \int_{\mathbb{H}^n \times K} f(z, t, k)\pi_\lambda(uz, t)\mu_\lambda(uku^{-1}) \otimes \sigma(uku^{-1})\, dz\, dt\, dk \\
&= \int_{\mathbb{H}^n \times K} f(z, t, k)\mu_\lambda(u)\pi_\lambda(z, t)\mu_\lambda(k)\mu_\lambda(u)^\star \otimes \sigma(u)\sigma(k)\sigma(u)^\star\, dz\, dt\, dk \\
&= \big(\mu_\lambda(u) \otimes \sigma(u)\big)\rho_\sigma^\lambda(f)\big(\mu_\lambda(u)^\star \otimes \sigma(u)^\star\big).
\end{aligned}
$$

Thus:

$$\rho_\sigma^\lambda(f_\delta) = \int_K \big(\mu_\lambda(u) \otimes \sigma(u)\big)\rho_\sigma^\lambda(f)\big(\mu_\lambda(u)^\star \otimes \sigma(u)^\star\big)\chi_\delta(u)\, du.$$

Therefore, we get

$$
\begin{aligned}
\|\rho_\sigma^\lambda(f_\delta)(\varphi \otimes v)\| \leqslant \int_K &|\chi_\delta(u)|\|(\mu_\lambda(u) \otimes \sigma(u)) \\
&\times \rho_\sigma^\lambda(f)(\mu_\lambda(u)^\star \otimes \sigma(u)^\star)(\varphi \otimes v)\|\, du
\end{aligned}
$$

$$\leqslant C \int_K |\chi_\delta(u)| \|\rho_\sigma^\lambda(f)(\mu_\lambda(u)^\star \otimes \sigma(u)^\star)(\varphi \otimes v)\| \, du.$$

$$\leqslant C \int_K |\chi_\delta(u)| \|\rho_\sigma^\lambda(h_a)(\mu_\lambda(u)^\star \otimes \sigma(u)^\star)(\varphi \otimes v)\| \, du.$$

Since $h_a(z, t, k)$ is central in K and radial in z, we see that:

$$(\mu_\lambda(u) \otimes \sigma(u))\rho_\sigma^\lambda(h_a)(\mu_\lambda(u)^\star \otimes \sigma(u)^\star) = \rho_\sigma^\lambda(h_a).$$

Consequently,

$$\|\rho_\sigma^\lambda(f_\delta)(\varphi \otimes v)\| \leqslant \|\rho_\sigma^\lambda(h_a)(\varphi \otimes v)\| \left(\int_K |\chi_\delta(u)| \, du \right).$$

This completes the proof of Proposition 2.4. □

From the above proposition it is clear that f_δ satisfies the same hypotheses as f. Our strategy is to show that under the hypothesis of Theorem 2.1, $f_\delta = 0$ for each δ which will immediately imply that $f = 0$.

COROLLARY 2.5. *Under the hypothesis (ii) of Theorems 2.1 and 2.2 we have:*

$$\sum_{\sigma \in \widehat{K_M}} \sum_{j=1}^{d(\sigma)} d(\sigma) \|\rho_\sigma^\lambda(f_\delta)(\Phi_\alpha^\lambda \otimes v_j)\|^2 \leqslant C \, e^{-2(2|\alpha|+n)b|\lambda|},$$

for some positive constant C provided that f is central in the K-variable.

Proof. The corollary is proved by repeating the argument leading to (2.7) by replacing f with f_δ. □

For each p and $q \in \mathbb{N}$, there exists an irreducible unitary class-1 representation $\delta_{p,q}$ of K realized on $\mathscr{H}_{p,q}$, the space of all bigraded spherical harmonics of bidegree (p, q). Moreover, each $\delta \in \widehat{K_M}$ is unitarily equivalent to one and only one of $\delta_{p,q}$. When $\delta = \delta_{p,q}$, we can expand f_δ in terms of solid harmonics:

$$f_\delta(z, t, k) = \sum_{l=1}^{d(p,q)} f_l(z, t, k) = \sum_{l=1}^{d(p,q)} g_l(z, t, k) \, P_l(z),$$

where $d(p, q) = \dim \mathscr{H}_{p,q}$, $g_l(z, t, k)$ are radial in z and P_l are solid harmonics whose restrictions to S^{2n-1} form an orthonormal basis for $\mathscr{H}_{p,q}$.

In what follows we make use of several results from the theory of Weyl transform and Weyl correspondence. For all the results and unexplained terminologies used we refer the reader to Sections 2.6 and 2.7 of the book [19]. First we make use of the Hecke–Bochner formula [19, Theorem 2.6.2] for the Weyl transform:

$$\widehat{f_l}(\lambda, k) = \pi_\lambda(g_l \, P_l) = G_\lambda(P_l) \, T_l^\lambda(k),$$

where $G(P_l)$ is the Weyl correspondence associated to P_l and (assuming $\lambda > 0$)

$$T_l^\lambda(k) = \sum_{m=p}^\infty R_{m-p,\lambda}^N \, (g_l^\lambda, k) \, P_m(\lambda),$$

where $N = n + p + q$ and

$$R_{m,\lambda}^N \, (g_l^\lambda, k) = \frac{\Gamma(m+1)}{\Gamma(m+N)} \int_{\mathbb{C}^N} g_l^\lambda(z, k) \, \varphi_{m,\lambda}^{N-1}(z) \, dz.$$

Also, $P_m(\lambda)$ stands for the spectral projections associated to the Hermite operator $H(\lambda) = (-\Delta + \lambda^2 |x|^2)$. Thus we have

$$\rho_\sigma^\lambda(f_\delta) = \sum_{l=1}^{d(p,q)} \int_K G_\lambda(P_l) \, T_l^\lambda(k)\mu_\lambda(k) \otimes \sigma(k) \, dk.$$

Fix α with $|\alpha| = m + p$ and choose an orthonormal basis $\{v_i : i = 1, 2, \ldots, d(\sigma)\}$ for V_σ. Then

$$\begin{aligned}
\rho_\sigma^\lambda(f_\delta) \, (\Phi_\alpha^\lambda \otimes v_j) &= \sum_{|\beta|=m+p} \sum_{i=1}^{d(\sigma)} \int_K c_{\alpha\beta}^\lambda(k) R_{m,\lambda}^N(g_l^\lambda, k)(\sigma(k)v_j, v_i) \\
&\quad \times G_\lambda(P_l)(\Phi_\beta^\lambda \otimes v_i) \, dk \\
&= \sum_{|\beta|=m+p} \sum_{i=1}^{d(\sigma)} A_j(\beta, l, i) \, G_\lambda(P_l)(\Phi_\beta^\lambda \otimes v_i),
\end{aligned}$$

where

$$A_j(\beta, l, i) = \int_K c_{\alpha\beta}^\lambda(k) \, R_{m,\lambda}^N \, (g_l^\lambda, k) \, (\sigma(k)v_j, v_i) \, dk.$$

We will calculate:

$$\|\rho_\sigma^\lambda(f_\delta)(\Phi_\alpha^\lambda \otimes v_j)\|^2 = \sum_i \sum_{\beta,\gamma} \sum_{l,r} A_j(\beta, l, i)\overline{A_j(\gamma, r, i)}(G_\lambda(P_l) \, \Phi_\beta^\lambda, G_\lambda(P_r)\Phi_\gamma^\lambda).$$

We make the observation that:

$$(G_\lambda(P_l)\,\Phi_\beta^\lambda,\; G_\lambda(P_r)\,\Phi_\gamma^\lambda) = 0, \quad \text{for } \beta \neq \gamma.$$

This can be easily verified using results (see [19, equation (2.6.22)]):

$$G_\lambda(P_l)\,\Phi_\beta^\lambda = G_\lambda(P_l)\,P_{m+p}(\lambda)\,\Phi_\beta^\lambda = W_\lambda(P_l\,\varphi_{m,\lambda}^{N-1})\,\Phi_\beta^\lambda$$

and

$$G_\lambda(P_r)\,\Phi_\gamma^\lambda = G_\lambda(P_r)\,P_{m+p}(\lambda)\,\Phi_\gamma^\lambda = W_\lambda(P_r\,\varphi_{m,\lambda}^{N-1})\,\Phi_\gamma^\lambda.$$

Therefore,

$$(G_\lambda(P_l)\,\Phi_\beta^\lambda,\; G_\lambda(P_r)\,\Phi_\gamma^\lambda) = (W_\lambda(P_r\,\varphi_{m,\lambda}^{N-1})^\star W_\lambda(P_l\,\varphi_{m,\lambda}^{N-1})\,\Phi_\beta^\lambda\,,\; \Phi_\gamma^\lambda)$$

$$= c \int_{\mathbb{C}^N} F_{l,r}(z)\,(\pi_\lambda(z,0)\,\Phi_\beta^\lambda\,,\; \Phi_\gamma^\lambda)\,dz,$$

where

$$F_{l,r} = \int_{\mathbb{C}^N} \overline{P_r(-z+w)}\,P_l(w)\,\varphi_{m,\lambda}^{N-1}(z-w)\,\varphi_{m,\lambda}^{N-1}(w)\,e^{\frac{i}{2}Im(z\cdot\overline{w})}\,dw.$$

Using the homogeneity of P_l and P_r, we see that $F_{l,r}(z)$ is polyradial, that is, radial in each z_j separately. But $(\pi_\lambda(z,0)\,\Phi_\beta^\lambda\,,\; \Phi_\gamma^\lambda)$ is homogeneous of type $(\gamma - \beta)$ and hence the integral is zero if $\beta \neq \gamma$.

In view of the above observation, we get:

$$\|\rho_\sigma^\lambda(f_\delta)(\Phi_\alpha^\lambda \otimes v_j)\|^2 = \sum_i \sum_\beta \sum_{l,r} A_j(\beta,l,i)\overline{A_j(\beta,r,i)}(G_\lambda(P_l)\Phi_\beta^\lambda,\; G_\lambda(P_r)\Phi_\beta^\lambda).$$

Consider now the sum $S = \sum_\sigma d(\sigma)\sum_{i,j=1}^{d(\sigma)} A_j(\beta,l,i)\,\overline{A_j(\beta,r,i)}$. By Peter–Weyl theorem, we see that

$$S = \sum_\sigma d(\sigma)\sum_{i,j=1}^{d(\sigma)}\left(\int_K c_{\alpha,\beta}^\lambda(k)\,R_{m,\lambda}^N(g_l^\lambda,k)\,(\sigma(k)v_j,\;v_i)\,dk\right)$$

$$\times \overline{\left(\int_K c_{\alpha,\beta}^\lambda(k)\,R_{m,\lambda}^N(g_r^\lambda,k)\,(\sigma(k)v_j,\;v_i)\,dk\right)}$$

$$= \int_K |c_{\alpha\beta}^\lambda(k)|^2\,R_{m,\lambda}^N(g_l^\lambda,k)\,\overline{R_{m,\lambda}^N(g_r^\lambda,k)}\,dk.$$

Therefore,

$$
S' = \sum_{\sigma \in \widehat{K}} d(\sigma) \sum_{|\alpha|=m+p} \sum_{j=1}^{d(\sigma)} \|\rho_\sigma^\lambda(f_\delta)(\Phi_\alpha^\lambda \otimes v_j)\|^2 = \sum_{l,r} \sum_{|\alpha|=|\beta|=m+p} \int_K |c_{\alpha\beta}^\lambda(k)|^2
$$
$$
\times R_{m,\lambda}^N(g_l^\lambda, k)\overline{R_{n,\lambda}^N(g_r^\lambda, k)}\, dk (G_\lambda(P_l)\Phi_\beta^\lambda, G_\lambda(P_r)\Phi_\beta^\lambda).
$$

Since $\sum_{|\alpha|=m+p}|c_{\alpha\beta}^\lambda(k)|^2 = 1$, we get

$$
S' = \sum_{l,r} \int_K R_{m,\lambda}^N(g_l^\lambda, k)\, \overline{R_{m,\lambda}^N(g_r^\lambda, k)}\, dk \sum_{|\beta|=m+p} (G_\lambda(\rho_l)\, \Phi_\beta^\lambda, \, G_\lambda(\rho_r)\, \Phi_\beta^\lambda).
$$

But the later sum is $(P_l, P_r)_{L^2(S^{n-1})}$ which equals zero for $l \neq r$. Hence:

$$
\sum_{\sigma \in \widehat{K}} d(\sigma) \sum_{|\alpha|=m+p} \sum_{i=1}^{d(\sigma)} \|\rho_\sigma^\lambda(f_\delta)(\Phi_\alpha^\lambda \otimes v_j)\|^2 = C \sum_{l} \int_K |R_{m,\lambda}^N(g_l^\lambda, k)|^2\, dk.
$$

In view of Corollary 2.5, we see that $\int_K |R_{m,\lambda}^N(g_l^\lambda, k)|^2\, dk \leqslant Ce^{-(2m+n)2b|\lambda|}$ for any l.

By appealing to Theorem 2.3, we can prove Theorems 2.1 and 2.2.

§3. Miyachi Theorem on Heisenberg motion groups

Our attention is focused in this section on an analogue of Miyachi's Theorem for the Heisenberg motion group $G = \mathbb{H}^n \rtimes K$. Given $f \in L^2(G)$, consider the group Fourier transform:

$$
\hat{f}(\lambda, \sigma) = \int_K \int_{\mathbb{R}} \int_{\mathbb{C}} f(z, t, k)\rho_\sigma^\lambda(z, t, k)\, dz\, dt\, dk
$$
$$
= \int_K \int_{\mathbb{C}^n} f^\lambda(z, k)(\pi_\lambda(z, 0)\mu_\lambda(k)) \otimes \sigma(k)\, dz\, dk,
$$

where the partial Fourier transform $f^\lambda(z, k)$ is as in formula (2.2). Then for $f \in (L^1 \cap L^2)(G)$, one gets as in [11, page 30] that

$$
(3.1) \qquad \int_{\mathbb{C}^n \times K} |f^\lambda(z, k)|^2\, dz\, dk = \frac{|\lambda|^n}{(2\pi)^n} \sum_{\sigma \in \widehat{K}} d(\sigma)\|\widehat{f}(\lambda, \sigma)\|_{HS}^2,
$$

and the Plancherel formula for the group G reads:

$$
\int_K \int_{\mathbb{H}^n} |f(z, t, k)|^2\, dt\, dz\, dk = (2\pi)^n \sum_{\sigma \in \widehat{K}} d(\sigma) \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda, \sigma)\|_{HS}^2 |\lambda|^n d\lambda,
$$

where $d(\sigma)$ stands for the dimension of the space V_σ of the representation σ as in the second section. We also introduce the Euclidean norm on $\mathbb{H}^n$ as

$$\|(z,t)\| := \sqrt{t^2 + \|x\|^2 + \|u\|^2}, \quad z = (x,u), x, u \in \mathbb{R}^n \quad \text{and} \quad t \in \mathbb{R}.$$

Our first result in this section is the following theorem.

THEOREM 3.1. *Let f be a square integrable function on G which satisfies:*

(i) $e^{a\|x\|^2} f(x,k) \in L^1(G) + L^\infty(G)$.
(ii) $\int_{\mathbb{R}} \log^+((\sum_{\delta \in \widehat{K}} d(\delta)|\lambda|^n e^{2b\lambda^2} \|\widehat{f}(\lambda,\delta)\|_{HS}^2)/(c))\, d\lambda < \infty$ *for some positive constant c. Then*

    (1)   *If $ab > \frac{1}{4}$, $f = 0$ almost everywhere on G.*
    (2)   *If $ab = \frac{1}{4}$, $f(z,t,k) = e^{-at^2} f(z,0,k)$ for any $(z,t,k) \in \mathbb{C}^n \times \mathbb{R} \times K$.*
    (3)   *If $ab < \frac{1}{4}$, there exist an infinite number of linearly independent functions meeting hypotheses (i) and (ii).*

Proof. Let φ be a Schwartz function on $\mathbb{C}^n \times K$ and F the function defined on $\mathbb{R}$ by

$$F(t) = \int_{\mathbb{C}^n \times K} f(z,t,k)\varphi(z,k)\, dz\, dk.$$

Then for $\lambda \in \mathbb{R}$,

$$\widehat{F}(\lambda) = \int_{\mathbb{C}^n \times K} f^\lambda(z,k)\varphi(z,k)\, dz\, dk$$

and therefore

$$|\widehat{F}(\lambda)|^2 \leqslant C \int_{\mathbb{C}^n \times K} |f^\lambda(z,k)|^2\, dz\, dk$$

for some positive constant C. On the other hand, we get by virtue of formula (3.1) that:

$$|\widehat{F}(\lambda)|^2 \leqslant \frac{C}{(2\pi)^n} |\lambda|^n \sum_{\delta \in \widehat{K}} d(\delta) \|\widehat{f}(\lambda,\delta)\|_{HS}^2.$$

As such, for some constant $c > 0$, we get:

$$\int_{\mathbb{R}} \log^+\left(\frac{e^{b\lambda^2}|\widehat{F}(\lambda)|}{c} \right) d\lambda \leqslant \int_{\mathbb{R}} \log^+\left(\frac{\sum_{\delta \in \widehat{K}} d(\delta)|\lambda|^n e^{2b\lambda^2} \|\widehat{f}(\lambda,\delta)\|_{HS}^2}{c} \right) d\lambda$$
$$< +\infty.$$

On the other hand, clearly $F \in L^\infty(\mathbb{R}) + L^1(\mathbb{R})$. So by Theorem 1.3, if $ab > \frac{1}{4}$, $F = 0$ almost everywhere and so is f as φ is taken arbitrary on $\mathbb{C}^n \times K$.

When $ab = \frac{1}{4}$, taking a well chosen Hilbert basis of $L^2(\mathbb{C}^n \times K)$ consisting of Schwartz functions, we get the second assertion.

We now look at the third case where $ab < \frac{1}{4}$. Let f be a function of $L^2(G)$ of the from $f = f_1 \otimes f_2$, where f_1 is defined on $\mathbb{H}^n$ and f_2 on K. Let as earlier $(\Phi_\alpha^\lambda)_{\alpha \in \mathbb{N}^n}$ be the Hermite basis for $L^2(\mathbb{R}^n)$ and let $c_{\alpha\beta}^\lambda(k)$ be the functions defined on K by the relation

$$\mu_\lambda(k)\, \Phi_\alpha^\lambda = \sum_{|\beta|=|\alpha|} c_{\alpha\beta}^\lambda(k)\, \Phi_\beta^\lambda.$$

Then for $(\Phi_\alpha^\lambda \otimes v_i)_{\alpha \in \mathbb{N}^n,\ i=1,\ldots,d(\sigma)}$ as before, we get

$$\begin{aligned}
\rho_\delta^\lambda(f_1 \otimes f_2)(\Phi_\alpha^\lambda \otimes v_i) &= \int_K \pi_\lambda(f_1)\mu_\lambda(k)\Phi_\alpha^\lambda \otimes \delta(k)v_i f_2(k)\, dk \\
&= \sum_{|\beta|=|\alpha|} \pi_\lambda(f_1)\Phi_\beta^\lambda \otimes \int_K f_2(k)c_{\alpha,\beta}^\lambda(k)\delta(k)v_i\, dk \\
&= \sum_{|\beta|=|\alpha|} \pi_\lambda(f_1)\Phi_\beta^\lambda \otimes \delta(c_{\alpha,\beta}^\lambda \cdot f_2)v_i.
\end{aligned}$$

By the Cauchy–Schwartz inequality

$$\begin{aligned}
\|\rho_\delta^\lambda(f_1 \otimes f_2)(\Phi_\alpha^\lambda \otimes v_i)\|^2 &\leqslant \left(\sum_{|\beta|=|\alpha|} \|\pi_\lambda(f_1)\Phi_\beta^\lambda\|\, \|\delta(c_{\alpha,\beta}^\lambda f_2)v_i\| \right)^2 \\
&\leqslant C(2|\alpha|+n)^{n-1} \sum_{|\beta|=|\alpha|} \|\pi_\lambda(f_1)\Phi_\beta^\lambda\|^2 \|\delta(c_{\alpha,\beta}^\lambda f_2)v_i\|^2.
\end{aligned}$$

On the other hand, for $A = \sum_{\delta \in \widehat{K}}\ d(\delta) \sum_{i=1}^{d(\delta)} \|\rho_\delta^\lambda(f_1 \otimes f_2)(\Phi_\alpha^\lambda \otimes v_i)\|^2$, we have

$$\begin{aligned}
A &\leqslant C(2|\alpha|+n)^{n-1} \sum_{|\beta|=|\alpha|} \|\pi_\lambda(f_1)\Phi_\beta^\lambda\|^2 \sum_{\delta \in \widehat{K}} d(\delta) \sum_{i=1}^{d(\delta)} \|\delta(c_{\alpha,\beta}^\lambda f_2)v_i\|^2 \\
&= C(2|\alpha|+n)^{n-1} \sum_{|\beta|=|\alpha|} \|\pi_\lambda(f_1)\Phi_\beta^\lambda\|^2 \int_K |c_{\alpha,\beta}^\lambda(k)|^2 |f_2(k)|^2\, dk.
\end{aligned}$$

Hence for $B = \sum_{\delta \in \widehat{K}} d(\delta) \|\widehat{f}(\lambda, \delta)\|^2_{HS}$, one gets:

$$B \leqslant C \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{n-1} \sum_{|\beta|=|\alpha|} \|\pi_\lambda(f_1)\Phi^\lambda_\beta\|^2 \int_K |c^\lambda_{\alpha,\beta}(k)|^2 |f_2(k)|^2 \, dk$$

$$= C \sum_{j=0}^{\infty} (2j+n)^{n-1} \sum_{|\beta|=j} \|\pi_\lambda(f_1)\Phi^\lambda_\beta\|^2 \int_K \sum_{|\alpha|=j=|\beta|} |c^\lambda_{\alpha,\beta}(k)|^2 |f_2(k)|^2 \, dk$$

$$= C \sum_{\beta \in \mathbb{N}^n} (2|\beta| + n)^{n-1} \|\pi_\lambda(f_1)\Phi^\lambda_\beta\|^2 \int_K |f_2(k)|^2 \, dk$$

in view of equality (2.5). Since $\pi_\lambda(\mathscr{L} f_1) = \pi_\lambda(f_1)H(\lambda)$ and $H(\lambda)\Phi^\lambda_\beta = (2|\beta| + n)|\lambda|\Phi^\lambda_\beta$ as explained earlier we see that

$$\sum_{\delta \in \widehat{K}} d(\delta) \|\widehat{f}(\lambda, \delta)\|^2_{HS} \leqslant C|\lambda|^{-n+1} \sum_{\beta \in \mathbb{N}^n} \|\pi_\lambda(\mathscr{L}^{\frac{n-1}{2}} f_1)\Phi^\lambda_\beta\|^2_2 \|f_2\|^2_2$$

$$= C|\lambda|^{-n+1} \|\pi_\lambda(\mathscr{L}^{\frac{n-1}{2}} f_1)\|^2_{HS} \|f_2\|^2_2.$$

Consequently,

$$(3.2) \quad |\lambda|^n e^{2b\lambda^2} \sum_{\delta \in \widehat{K}} d(\delta) \|\widehat{f}(\lambda, \delta)\|^2_{HS} \leqslant C|\lambda| e^{2b\lambda^2} \|\pi_\lambda(\mathscr{L}^{\frac{n-1}{2}} f_1)\|^2_{HS} \|f_2\|^2_2.$$

It is possible to choose functions $f_1 \in L^1 \cap L^2(\mathbb{H}^n)$ such that

$$|\lambda| \|\pi_\lambda(\mathscr{L}^{\frac{n-1}{2}} f_1)\|^2_{HS} \leqslant e^{-2b\lambda^2}.$$

So, for well chosen functions $f \in L^2(K)$ we can arrange

$$\log^+ \left(\frac{\sum_{\delta \in \widehat{K}} d(\delta)|\lambda|^n e^{2b\lambda^2} \|\widehat{f}(\lambda, \delta)\|^2_{HS}}{c} \right) = 0.$$

This achieves the proof of (iii).   □

As a direct consequence of Theorem 3.1, we get the following theorem.

THEOREM 3.2. (Hardy's theorem for Heisenberg motion groups.) *Let f be a measurable function on G such that:*

(i) $|f(z, t, k)| \leqslant Ce^{-a\|(z,t)\|^2}$ *for all* $(z, t, k) \in G$.
(ii) $|\lambda|^{\frac{n}{2}} \|\rho^\lambda_\delta(f)\|_{HS} \leqslant C_\delta e^{-b\lambda}$

for some $C > 0$ and $C_\delta > 0$ such that $\sum_{\delta \in \widehat{K}} d(\delta) C_\delta^2 < +\infty$. Then we have:

(1) *If $ab > \frac{1}{4}$, $f = 0$ almost everywhere on G.*
(2) *If $ab = \frac{1}{4}$, $f(z, t, k) = e^{-at^2} f(z, 0, k)$ for any $(z, t, k) \in \mathbb{C}^n \times \mathbb{R} \times K$.*
(3) *If $ab < \frac{1}{4}$, there exist an infinite number of linearly independent functions meeting hypotheses (i) and (ii).*

As a further generalization, our next objective in this paper is to prove the following analogue of Cowling–Price theorem for Heisenberg motion groups. We have the following theorem.

THEOREM 3.3. (An analogue of Cowling–Price theorem for Heisenberg motion groups.) *Let α and β be positive real numbers, $1 \leqslant p, q \leqslant \infty$ such that $\min(p, q)$ is finite, and let f be a square integrable function on G satisfying the following decay conditions:*

(i) $e^{\alpha \|\cdot\|^2} f \in L^p(G)$.
(ii) $e^{2\beta|\lambda|^2} |\lambda|^n \sum_{\sigma \in \widehat{K}} d(\sigma) \|\rho_\delta^\lambda(f)\|_{HS}^2 \in L^q(\mathbb{R})$.

Then the following conclusions hold:

(1) *$f = 0$ almost everywhere whenever $\alpha\beta \geqslant 1/4$.*
(2) *If $\alpha\beta < 1/4$, then there are infinitely many linearly independent functions satisfying both conditions (i) and (ii).*

Proof. Let f be a function meeting the hypotheses of the theorem. Then, first we obtain that f verifies the condition (i) of Theorem 3.1 since $L^p \subset L^1 + L^\infty$. Furthermore, based on the inequality, $Log^+(x) \leqslant x$ for $x \in \mathbb{R}_+$, we can see that f verifies the second condition of Theorem 3.1. Hence $f = 0$ almost everywhere whenever $\alpha\beta > 1/4$. Second, if $\alpha\beta = 1/4$ then $f(z, t, k) = \lambda(k, z) e^{-\alpha t^2}$ for some function $\lambda \in L^2(K \times \mathbb{C}^n)$. But then $e^{\alpha \|\cdot\|^2} f$ is independent of the variable t and hence the condition $e^{\alpha \|\cdot\|^2} f \in L^p(G)$ forces $\lambda(k, z)$ to be zero for almost every $k \in K$ and $z \in \mathbb{C}^n$. Finally for $\alpha\beta < 1/4$, restricting to the setting of Heisenberg groups, we know that there exists an infinite family of linearly independent functions $f_\varepsilon, \alpha < \varepsilon < 1/\beta$ with the property:

$$|\lambda|^n \|\pi_\lambda(\mathscr{L}^{\frac{n-1}{2}} f_\varepsilon)\|_{HS}^2 \leqslant C_\varepsilon e^{-2\varepsilon\lambda^2}$$

some positive constant C_ε. Hence inequality (3.2) enables us to achieve the proof.   □

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