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A NEW MEAN WITH INEQUALITIES

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Abstract

We introduce a new mean and compare it to the standard arithmetic, geometric and harmonic means. In fact we identify a generic way of constructing means from existing ones.

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1. Introduction

The notions of arithmetic, geometric and harmonic means (of positive real numbers) can be traced back to Pythagoras himself. Some of the greatest mathematicians in history worked on establishing an order for these means. Augustin Louis Cauchy (1789–1857) established the inequality of arithmetic and geometric means [2, p. 457]. Nowadays, this inequality is simply derived from an application of Jensen's inequality. Many variations of the notion of a mean have since been introduced. The quadratic mean and more generally power means,

$$M^p(x_1, \ldots, x_n) = \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n} x_i^p},$$

are examples of such variations. Indeed, M^{-1}, $M^0 = \lim_{p\to 0} M^p$, M^1 and M^2 are respectively the harmonic, geometric, arithmetic and quadratic means. Using Jensen's inequality, we immediately get that, for $p < q$,

$$\begin{aligned} M^q(x_1, \ldots, x_n)^q &= \frac{1}{n}\sum_{i=1}^{n} x_i^q = \frac{1}{n}\sum_{i=1}^{n} (x_i^p)^{q/p} \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{q/p} \\ &= M^p(x_1, \ldots, x_n)^q, \end{aligned}$$

thus establishing a general inequality for means,

$$M^p(x_1, \ldots, x_n) \le M^q(x_1, \ldots, x_n).$$

The power mean is a continuous and symmetric function of $(x_1, \ldots, x_n)$. Further, for any $p \in \mathbb{R}$ and $\lambda > 0$, $M^p(\lambda x_1, \ldots, \lambda x_n) = \lambda M^p(x_1, \ldots, x_n)$ and $M^p(x, \ldots, x) = x$.

A further generalization is obtained by replacing the power function by an arbitrary continuous strictly monotone function, f,

$$M_f(x_1, \ldots, x_n) = f^{-1}\bigg(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\bigg).$$

Such a mean may not satisfy the homogeneity property, $M_f(\lambda x_1, \ldots, \lambda x_n) = \lambda M_f(x_1, \ldots, x_n)$. However, it retains the fact that it is a continuous and symmetric function of $(x_1, \ldots, x_n)$ with the property that $M_f(x, \ldots, x) = x$. It is these three basic characteristics that Aumann retains in his definition of a mean (see [1]).

DEFINITION 1. A mean is a function $M(x_1, \ldots, x_n)$ (or rather a family of functions indexed by $n \ge 1$) that is continuous, symmetrical—that is, for any permutation τ, $M(x_{\tau(1)}, \ldots, x_{\tau(n)}) = M(x_1, \ldots, x_n)$—and satisfies the property $M(x, \ldots, x) = x$.

REMARK 2. As we do not attempt to relate the means of sequences of different sizes, our definition does not impose any 'projective' property. It does not exclude the possibility of a 'mixture of means'. For example, the family of functions made up of arithmetic means, for n odd, and geometric means, for n even, still qualifies as a mean.

In this paper we introduce a generic way of constructing new means from existing ones. These means are genuinely different in the sense that they do not fall in the general scheme of generalized means or even generalized weighted means (where the equal weighting $1/n$ is replaced by arbitrary weights). The mapping used to construct these means is then studied and shown to be one-to-one. We also look at the particular case of the harmonic $\hbar$-mean and obtain a comparison result with the arithmetic mean.

While the arithmetic mean is defined for any sequence of real numbers, the geometric mean, $G(x_1, \ldots, x_n) = \sqrt[n]{x_1 \cdots x_n} = M_{\ln}(x_1, \ldots, x_n)$, requires all values to be strictly positive. The same restriction removes many unnecessary complications (dealing with $1/0$ and $1/\infty$) in the case of the harmonic mean (see (3)). To avoid having to constantly change the space on which the means are defined, we shall only consider means of strictly positive real numbers.

2. The $\hbar$-mean

First we introduce the following notation. For a given positive integer n, N denotes the set $\{1, \ldots, n\}$ and $\mathbb{K}$ the set of all nonempty subsets of N. For a given sequence $\{x_1, \ldots, x_n\}$ and $K \in \mathbb{K}$, we denote the subsequence $\{x_k, k \in K\}$, $x_{[K]}$. Finally $\hbar(n)$ denotes the harmonic number of n, $\hbar(n) = \sum_{k=1}^{n}(1/k)$.

DEFINITION 3. Let M be a mean. We call $\hbar$-mean of M the function

$$\mathcal{H}_M(x_1, \ldots, x_n) = \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} M(x_{[K]}),$$

where $|K|$ is the cardinal of K.

PROPOSITION 4. *Let M be a mean. Its $\hbar$-mean is a mean; that is $\mathcal{H}_M$ is continuous, symmetrical and satisfies the property $\mathcal{H}_M(x, \ldots, x) = x$. Furthermore, if M is homogeneous, $M(\lambda x_1, \ldots, \lambda x_n) = \lambda M(x_1, \ldots, x_n)$, then so is $\mathcal{H}_M$.*

PROOF. The continuity, symmetry and homogeneity are obvious. Now, since, for an identical sequence, $x_1 = \cdots = x_n = x$, $M(x_{[K]}) = x$, for any $K \in \mathbb{K}$,

$$\mathcal{H}_M(x, \ldots, x) = \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} x = x \frac{1}{\hbar(n)} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} = x.$$

The last equality follows from a well-known result and is given here for convenience only:

$$\begin{aligned}
\hbar(n) &= \int_0^1 \frac{1 - x^n}{1 - x}\, dx \\
&= \int_0^1 \frac{1 - (1 - x)^n}{x}\, dx = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \int_0^1 x^{k-1}\, dx \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k}.
\end{aligned}$$

□

PROPOSITION 5. *The arithmetic $\hbar$-mean is simply the arithmetic mean. That is, if A is the arithmetic mean, $A(x_1, \ldots, x_n) = (1/n) \sum_{k=1}^{n} x_k$, then $\mathcal{H}_A = A$.*

PROOF. Observe that

$$\sum_{K \in \mathbb{K}, |K| = k} A(x_{[K]}) = \frac{1}{k} \binom{n-1}{k-1} \sum_{i=1}^{n} x_i = \binom{n}{k} A(x_1, \ldots, x_n).$$

The result immediately follows:

$$\begin{aligned}
\mathcal{H}_A(x, \ldots, x) &= \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} A(x_{[K]}) \\
&= A(x_1, \ldots, x_n) \frac{1}{\hbar(n)} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} \\
&= A(x_1, \ldots, x_n).
\end{aligned}$$

□

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PROPOSITION 6. *If $\mathcal{H}_{M_1} = \mathcal{H}_{M_2}$, then $M_1 = M_2$; that is, the mapping $M \to \mathcal{H}_M$ is one-to-one on the space of means. Furthermore, $M' = \mathcal{H}_M$ if and only if*

$$M(x_1, \ldots, x_n) = \sum_{K \in \mathbb{K}} (-1)^{|K|+1} p_{n,|K|} M'(x_{[K]}), \tag{1}$$

where $p_{n,l}$ is obtained recursively as follows:

$$p_{1,1} = 1, \quad p_{n,n} = n\hbar(n) \quad \text{and, for } l < n, \quad p_{n,l} = (-1)^n n \sum_{k=l}^{n-1} \frac{(-1)^k}{k} p_{k,l} \binom{n-l}{n-k}.$$

PROOF. We proceed by induction on the size of the sequence, n. Since $M_1(x) = \mathcal{H}_{M_1}(x) = \mathcal{H}_{M_2}(x) = M_2(x)$, we see that M_1 and M_2 coincide on single value sequences. Assume that M_1 and M_2 coincide for all sequences of size less than or equal to n. Then the requirement that

$$\mathcal{H}_{M_1}(x_1, \ldots, x_n, y) = \mathcal{H}_{M_2}(x_1, \ldots, x_n, y)$$

becomes

$$\begin{aligned} &\frac{1}{\hbar(n+1)}\left(y + \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|}\left(M_1(x_{[K]}) - \frac{|K|}{|K|+1} M_1(x_{[K]}, y)\right)\right) \\ &\quad = \frac{1}{\hbar(n+1)}\left(y + \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|}\left(M_2(x_{[K]}) - \frac{|K|}{|K|+1} M_2(x_{[K]}, y)\right)\right). \end{aligned}$$

The first result is obtained by observing that, under the induction assumption, for any subset K, $M_1(x_{[K]}) = M_2(x_{[K]})$ and, for any proper subset K, $M_1(x_{[K]}, y) = M_2(x_{[K]}, y)$. Indeed, after simplification, one ends up with

$$M_1(x_1, \ldots, x_n, y) = M_2(x_1, \ldots, x_n, y).$$

To prove the second statement, we show that applying the mapping $\mathcal{H}$ to the right-hand side of (1), we recover M'. To begin with, let $M''(x_1, \ldots, x_n)$ denote the right-hand side of (1). Then

$$\begin{aligned} \mathcal{H}_{M''}(x_1, \ldots, x_n) &= \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} \sum_{L \subset K, L \neq \emptyset} (-1)^{|L|+1} p_{|K|,|L|} M'(x_{[L]}) \\ &= \frac{1}{\hbar(n)} \sum_{L \in \mathbb{K}} (-1)^{|L|+1} \left(\sum_{L \subset K \subset N} \frac{(-1)^{|K|+1}}{|K|} p_{|K|,|L|}\right) M'(x_{[L]}) \\ &= \frac{1}{\hbar(n)} \sum_{L \in \mathbb{K}} (-1)^{|L|+1} \left(\sum_{k=|L|}^{n} \frac{(-1)^{k+1}}{k} p_{k,|L|} \binom{n-|L|}{n-k}\right) M'(x_{[L]}). \end{aligned} \tag{2}$$

Next, let $q_{n,l} = \sum_{k=l}^{n} ((-1)^{k+1}/k) p_{k,l} \binom{n-l}{n-k}$. We show that $q_{n,l} = 0$ for $l < n$ and $q_{n,n} = (-1)^{n+1}\hbar(n)$. Again, we proceed by induction on n. First, $q_{1,1} = p_{1,1} = 1$.

Then, assuming that the property is true up to $n-1$, for $l \leq n-1$,

$$\begin{aligned}
q_{n,l} &= \sum_{k=l}^{n} \frac{(-1)^{k+1}}{k} p_{k,l} \binom{n-l}{n-k} \\
&= \frac{(-1)^{n+1}}{n} p_{n,l} + \sum_{k=l}^{n-1} \frac{(-1)^{k+1}}{k} p_{k,l} \binom{n-l}{n-k} \\
&= \frac{(-1)^{n+1}}{n} (-1)^n n \sum_{k=l}^{n-1} \frac{(-1)^{k}}{k} p_{k,l} \binom{n-l}{n-k} + \sum_{k=l}^{n-1} \frac{(-1)^{k+1}}{k} p_{k,l} \binom{n-l}{n-k} \\
&= 0.
\end{aligned}$$

Finally,

$$q_{n,n} = \frac{(-1)^{n+1}}{n} p_{n,n} = \frac{(-1)^{n+1}}{n} n\hbar(n) = (-1)^{n+1}\hbar(n).$$

It follows that (2) reduces to

$$\begin{aligned}
\mathcal{H}_{M''}(x_1, \ldots, x_n) &= \frac{1}{\hbar(n)} (-1)^{n+1} q_{n,n} M'(x_{[N]}) \\
&= \frac{1}{\hbar(n)} (-1)^{n+1} (-1)^{n+1} \hbar(n) M'(x_{[N]}) \\
&= M'(x_1, \ldots, x_n). \qquad \square
\end{aligned}$$

3. The harmonic $\hbar$-mean

In this section, we look at the specific case of $\mathcal{H}_H$ where H is the harmonic mean

$$H(x_1, \ldots, x_n) = \frac{n}{\sum_{k=1}^{n} (1/x_k)}. \tag{3}$$

For $n = 2$,

$$\mathcal{H}_H(x_1, x_2) = \frac{2}{3}\left(x_1 + x_2 - \frac{x_1 x_2}{x_1 + x_2}\right) = \frac{2(x_1^2 + x_1 x_2 + x_2^2)}{3(x_1 + x_2)}.$$

This mean is shown in [4] to exceed the arithmetic, geometric and harmonic means. In fact the author unifies all of these means, and many others, under one family

$$F_\alpha(x_1, x_2) = \frac{\int_{x_1}^{x_2} t^\alpha \, dt}{\int_{x_1}^{x_2} t^{\alpha-1} \, dt},$$

observes that F_{-2} is the harmonic mean, $F_{-1/2}$ is the geometric mean, $F_{1/2}$ is the Heron mean $(x_1 + \sqrt{x_1 x_2} + x_2)/3$, F_1 is the arithmetic mean, $F_2(x_1, x_2) = \mathcal{H}_H(x_1, x_2)$, and establishes that F_α increases with α.

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DEFINITION 7. The harmonic $\hbar$-mean is the function

$$\mathcal{H}_H(x_1, \ldots, x_n) = \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} H(x_{[K]}) = \frac{1}{\hbar(n)} \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{\sum_{k \in K}(1/x_k)}.$$

THEOREM 8. *The harmonic $\hbar$-mean is greater than or equal to the arithmetic mean (and therefore greater than or equal to any of the geometric or harmonic means); that is, for any sequence $\{x_1, \ldots, x_n\}$,*

$$H(x_1, \ldots, x_n) \le G(x_1, \ldots, x_n) \le A(x_1, \ldots, x_n) \le \mathcal{H}_H(x_1, \ldots, x_n).$$

The proof of this result immediately follows from an application of [3, Proposition 5] to the case of exponential random variables. For completeness, we include all steps of the proof.

PROOF. The double inequality $H \le G \le A$ is well known. We only prove the last inequality.

Because the arithmetic mean is greater than the geometric mean, for any sequence $\{u_1, \ldots, u_n\} \in [0, 1]$,

$$1 - u_1 u_2 \cdots u_n \ge \frac{1}{n} \sum_{i=1}^{n} (1 - u_i^n).$$

This implies that, for any sequence $\{F_1, \ldots, F_n\}$ of distribution functions on $[0, +\infty)$,

$$\frac{1}{n} \sum_{i=1}^{n} \int_0^\infty [1 - F_i(u)^n]\, du \le \int_0^\infty [1 - F_1(u) F_2(u) \cdots F_n(u)]\, du. \tag{4}$$

In particular, let $F_i(u) = 1 - e^{-u/x_i}$, $u \ge 0$ (exponential distribution with mean x_i). Then

$$\int_0^\infty [1 - F_i^n(u)]\, du = \int_0^\infty [1 - (1 - e^{-u/x_i})^n]\, du = x_i \int_0^1 \frac{1 - v^n}{1 - v}\, dv = x_i \hbar(n)$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \int_0^\infty [1 - F_i(u)^n]\, du = \hbar(n) A(x_1, \ldots, x_n).$$

Furthermore, the left-hand side of (4) yields

$$\begin{aligned}
&\int_0^\infty [1 - F_1(u) F_2(u) \cdots F_n(u)]\, du \\
&\quad = \int_0^\infty \left[1 - \prod_{i=1}^{n} (1 - e^{-u/x_i})\right] du \\
&\quad = \int_0^\infty \left[1 - \left(1 + \sum_{K \in \mathbb{K}} (-1)^{|K|} e^{-\sum_{k \in K} u/x_k}\right)\right] du \\
&\quad = -\sum_{K \in \mathbb{K}} (-1)^{|K|} \int_0^\infty e^{-\sum_{k \in K} u/x_k}\, du = -\sum_{K \in \mathbb{K}} (-1)^{|K|} \frac{1}{\sum_{k \in K}(1/x_k)}
\end{aligned}$$

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$$= \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} \frac{|K|}{\sum_{k \in K}(1/x_k)} = \sum_{K \in \mathbb{K}} \frac{(-1)^{|K|+1}}{|K|} H(x_{[K]})$$
$$= \hbar(n)\mathcal{H}_H(x_1, \ldots, x_n).$$

The result immediately follows by application of (4). □

4. A monotonicity conjecture

What can be said about other $\hbar$-means? In particular, how do the arithmetic, geometric and harmonic $\hbar$-means compare? And how do they compare to the arithmetic, geometric and harmonic means? Unfortunately, Theorem 8 aside, we do not have complete answers to these questions. Tedious calculations can provide a hint in the case of sequences of two and three values where

$$H \leq G \leq A = \mathcal{H}_A \leq \mathcal{H}_G \leq \mathcal{H}_H.$$

CONJECTURE 9. If $M \leq M'$ then $\mathcal{H}_M \geq \mathcal{H}_{M'}$.

As usual, by $M \leq M'$ we mean, for any sequence $\{x_1, \ldots, x_n\}$, $M(x_1, \ldots, x_n) \leq M'(x_1, \ldots, x_n)$.

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