GLOBAL NORM-RESIDUE MAP OVER QUASI-FINITE FIELD

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Dedicated to the memory of Professor TADASI NAKAYAMA

A field k is called quasi-finite if it is perfect and if $G_k \approx \hat{Z}$ where G_k is the Galois group of the algebraic closure k_c over k and $\hat{Z}$ is the completion of the additive group of the rational integers. The classical reciprocity law on the local field with finite residue field is well-known to hold on local fields with quasi-finite residue field ([4], [5]). Thus it is natural to ask if the global reciprocity law should hold in the ordinary sense (see § 1 below) on the function-fields of one variable over quasi-finite field. We consider here two basic prototypes of non-finite quasi-finite fields:

(a) field k of non-zero characteristic which is algebraic over the prime subfield k_0 and has a finite p-primary degree for all prime p, i.e., $[k:k_0] = \prod_p p^{\nu_p}$ with $\nu_p < \infty$ for all p.

(b) The formal power-series field of one variable over an algebraically closed field of characteristic zero. In this note we show that the reciprocity law holds in the case (a) whereas it is not so for the case (b). Indeed we show that, for a function-field of one variable of positive genus over the field of type (b), there always exists a non-trivial (abelian) extension in which every prime divisor splits completely, i.e., an extension which can not be distinguised locally.

1. Let k be a quasi-finite field. Given a fixed generator σ of the Galois group G_k, we obtain the identification $\tilde{\sigma} : \chi(G_k) \xrightarrow{\approx} Q/Z$ given by $\chi \to \chi(\sigma)$, where $\chi(G_k)$ denotes the character group of G_k. Therefore if L is a local field with the residue field k, then we obtain Hasse invariant $\mathrm{inv}_L : B(L) \xrightarrow{\approx} Q/Z$ which is the composite of two isomorphisms $B(L) \xrightarrow{\approx} \chi(G_k) \xrightarrow{\approx} Q/Z$ where $B(L)$ is the Brauer group of L. In turn we obtain the norm residue map $(*, L) : L^* \to G_L^a$

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which is characterized by $\chi(a, L) = \mathrm{inv}_L(a, \chi)$ for all $\chi \in \chi(G_L)$, where (a, χ) is the element of $B(L)$ represented by the cup-product, and G_L^a is the Galois group of the maximal abelian extension of L over L. (Thus the local norm residue map depends only on the choice of a generator of the Galois group of the residue field). It follows from the local class field theory that $(^*, L)$ induces the isomorphism $(^*, M/L) : L^*/NM^* \xrightarrow{\approx} G(M/L)$ for all finite abelian extensions $M > L$, where $N = N_{M/L}$ stands for the norm mapping. Now let K be a function-field of one variable over k. For each prime divisor $\mathfrak{p}$ of K (which is trivial on k) the local completion $K_{\mathfrak{p}}$ is a local field whose residue field $\overline{K}_{\mathfrak{p}}$ is finite algebraic over k. Therefore $\overline{K}_{\mathfrak{p}}$ is also quasi-finite and we shall agree to choose $\sigma^{d_{\mathfrak{p}}}$ $(d_{\mathfrak{p}} = [\overline{K}_{\mathfrak{p}} : k])$ as a generator of $G_{\overline{K}_{\mathfrak{p}}}$. Thus we have local norm residue map $(^*, K_{\mathfrak{p}}) : K_{\mathfrak{p}}^* \to G_{K_{\mathfrak{p}}}^a$ for all $\mathfrak{p}$, and in turn we obtain the global norm residue map $(^*, K) : J_K \to G_K^a$ by $(\alpha, K) = \prod_{\mathfrak{p}} (\alpha_{\mathfrak{p}}, K_{\mathfrak{p}})$ which is characterized by $\chi(\alpha, K) = \mathrm{inv}_K(\alpha, \chi) = \sum \mathrm{inv}_{K_{\mathfrak{p}}}(\alpha_{\mathfrak{p}}, \chi)$ for all $\chi \in \chi(G_K)$, where J_K is the group of idèles of K. However, it follows easily from Tzen's theorem and the product formula that the norm residue map is trivial on the principal idèles and hence we have $(^*, K) : C_K \to G_K^a$ where C_K is the idèle class group of K. We shall say that *the reciprocity law holds over k* if, for any function-field of one variable K over k, the norm residue map induces the isomorphism $(^*, L/K) : C_K/NC_L \to G(L/K)$ for all finite abelian extensions $L > K$. Thus the reader should observe that we are concerned with the reciprocity with respect to the norm-residue map on the idèle class group and not an abstract class formation.

THEOREM 1. *Let K be a function-field of one variable over a quasi-finite field k, and let $\widetilde{K} = k_c K$ where k_c is the algebraic closure of k. Then we have the exact sequence*

$$0 \to H^1(G_k, E(\widetilde{K})) \to B(K) \to \textstyle\sum_{\mathfrak{p}} B(K_{\mathfrak{p}}) \xrightarrow{\mathrm{inv}_K} Q/Z \to 0$$

where $E(\widetilde{K})$ is the divisor class group of $\widetilde{K}$. If $H^1(G_k, E(\widetilde{K})) = 0$ for all function-fields K of one variable over k, then the reciprocity law holds over k.

Proof. Consider the exact sequences

$$0 \to P(\widetilde{K}) \to D(\widetilde{K}) \to E(\widetilde{K}) \to 0$$
$$0 \to k_c^* \to \widetilde{K}^* \to P(\widetilde{K}) \to 0$$

where $D(\tilde{K})$, $P(\tilde{K})$ are the group of divisors, principal divisors resp. Since $G_k \cong \hat{Z}$, we have scd $G_k = 2$ ([3]) and hence we obtain the exact sequences

$$(*)\qquad H^1(G_k, D(\tilde{K})) \to H^1(G_k, E(\tilde{K})) \to H^2(G_k, P(\tilde{K})) \to H^2(G_k, D(\tilde{K})) \to H^2(G_k, E(\tilde{K})) \to 0$$

and

$$H^2(G_k, k_c^*) \to H^2(G_k, \tilde{K}^*) \to H^2(G_k, P(\tilde{K})) \to 0.$$

Since k is a quasi-finite field, we have $H^2(G_k, k_c^*) = 0$ and hence $B(K) \simeq H^2(G_k, \tilde{K}^*) \simeq H^2(G_k, P(\tilde{K}))$. Furthermore $D(\tilde{K}) = \sum_p \mathrm{Hom}_{Z(D_p)}(Z(G_k), Z)$, where D_p is the decomposition subgroup of the prime divisor p, entails $H^1(G_k, D(\tilde{K})) = \sum_p H^1(D_p, Z) = 0$. Therefore the exact sequence $(*)$ becomes

$$(**)\qquad 0 \to H^1(G_k, E(\tilde{K})) \to B(K) \to H^2(G_k, D(\tilde{K})) \to H^2(G_k, E(\tilde{K})) \to 0.$$

Now let us consider the exact sequence $0 \to E_0(\tilde{K}) \to E(\tilde{K}) \xrightarrow{d} Z \to 0$ where d is the degree map. Since $E_0(\tilde{K})$ is a divisible group, we get $H^2(G_k, E_0(\tilde{K})) = 0$ ([3]) and hence we obtain the isomorphism $\tilde{d} : H^2(G_k, E(\tilde{K})) \to H^2(G_k, Z)$. Now the composite map $H^2(D_p, Z) \to \sum_p H^2(D_p, Z) = H^2(G_k, D(\tilde{K})) \to H^2(G_k, E(\tilde{K})) \xrightarrow{\tilde{d}} H^2(G_k, Z)$ is nothing but the corestriction map ([3]), and hence the composite map $H^2(G_k, D(\tilde{K})) \to H^2(G_k, E(\tilde{K})) \xrightarrow{\tilde{d}} H^2(G_k, Z) \xrightarrow{\tilde{\sigma}} Q/Z$ is given by $\sum_p \chi_p \to \sum_p \chi_p\,(\sigma^{d_p})$ where $d_p = [G_k;\ D_p]$. Consequently the diagram

$$\begin{array}{ccc} H^2(G_k, D(\tilde{K})) & \to & H^2(G_k, E(\tilde{K})) \\ \| & & \downarrow \tilde{\sigma}\circ\tilde{d} \\ \sum_p B(K_p) & \xrightarrow[\mathrm{inv}_K]{} & Q/Z \end{array}$$

is commutative. It follows that

$$0 \to H^1(G_k, E(\tilde{K})) \to B(K) \to \sum_p B(K_p) \xrightarrow{\mathrm{inv}_K} Q/Z \to 0$$

is exact. Now assume that $H^1(G_k, E(\tilde{K})) = 0$ for every function-field K of one variable over k, and let $L > K$ be any Galois extension. Then the exact sequence $0 \to B(L) \to \sum_p B(L_p) \xrightarrow{\mathrm{inv}_L} Q/Z \to 0$ of $G = G(L/K)$-modules gives us the exact sequence $0 \to B(L)^G \to (\sum_p B(L_p))^G \to Q/Z \to H^1(G, B(L)) \cdots$. Now the exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & B(K) & \to & \sum_p B(K_p) & \xrightarrow{\mathrm{inv}_K} & Q/Z & \to 0 & & \\ & \downarrow & & \downarrow & & \downarrow [L:K] & & & \\ 0 \to & B(L)^G & \to & (\sum_p B(L_p))^G & \to & Q/Z & \to & H^1(G, B(L)) & \cdots \end{array}$$

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gives us ([1] p. 40) the exact sequence

(1) $$0 \to H^2(G(L/K), L^*) \to H^2(G(L/K), J_L) \xrightarrow{\text{inv}_K} Z/nZ \to B(L)^G/\text{Im}(B(K) \to B(L)) \cdots \text{where } n = [L : K].$$

On the other hand, the exact sequence $0 \to L^* \to J_L \to C_L \to 0$ gives us the exact sequence

(2) $$0 \to H^1(G(L/K), C_L) \to H^2(G(L/K), L^*) \to H^2(G(L/K), J_K) \to H^2(G(L/K), C_L) \to H^3(G(L/K), L^*).$$ Now suppose that

$L > K$ is a cyclic extension. Then the exact sequence $0 \to H^2(G(L/K), L^*) \to B(K) \to B(L)^{G(L/K)} \longrightarrow H^3(G(L/K), L^*) \to \cdots$ entails that $B(K) \to B(L)^G$ is an epimorphism, and hence (1) becomes

(1)′ $$0 \to H^2(G(L/K), L^*) \longrightarrow H^2(G(L/K), J_L) \xrightarrow{\text{inv}_K} Z/nZ \to 0$$

Comparing (1)′ and (2), we conclude that for all cyclic extensions $L > K$ we have $H^1(G(L/K), C_L) = 0$ and $|H^2(G(L/K), C_L)| = [L : K]$. In particular it follows that for any Galois extension $L > K$ there exists at least one prime divisor in K which does not split completely in L. Consequently $(*, L)$: $C_K \to G_K^a$ induces the epimorphism $(*, L/K) : C_K/NC_L \to G(L/K)$ for all cyclic extensions $L > K$ of prime degree, and hence $(*, L/K)$ is the epimorphism for all finite abelian extensions $L > K$. On the other hand, $|H^2(G(L/K), C_L)| = [L : K]$ for all cyclic extensions $L > K$ entails that $|H^0(G(L/K), C_L)| \leq [L : K]$ for all solvable extensions, and hence $(*, L/K) : C_K/NC_L \to G(L/K)$ is an isomorphism for all finite abelian extension $L > K$. This completes our proof.

COROLLARY. *The reciprocity law holds over a quasifinite field of type a).*

Proof. Let k be a quasi-finite field of type a), and let K be a function-field of one variable over k. Then we can find a finite subfield k_0 and a function-field K_0 over k_0 such that $K = kK_0$. It follows from Hasse's norm theorem that $H^1(G_k, E(\tilde{K})) = 0$ for all finite extensions $k_1 > k_0$. Since $K = kK_0 = \bigcup_j k_jK_0$ where $k > k_j > k_0$ runs through intermediate finite fields, we obtain that $H^1(G_k, E(\tilde{K})) = \varinjlim H^1(G_{kj}, E(\tilde{K})) = 0$.

2. The reciprocity law fails over the quasi-finite field of type b). Indeed, there exists an abelian extension $L \gneqq K$ of function fields of one variable over

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the quasi-finite field of type (b), in which every prime of K splits completely in L, i.e., L, K cannot be distinguished locally. One way to see this is to construct an elliptic curve whose index is bigger than one; if ind $K>1$ with the constant field k, then $Kk'>K$ provides such an example, if we set $[k':k]=\text{ind } K$. It is not difficult to construct such a field following the homological interpretation of principal homogeneous spaces (see [2]). Our theorem below, which is of interest in itself, provides a different kind of example, namely an extension $L\gneqq K$ which is linearly disjoint from the constant field extension and in which every prime in K splits completely in L.

Let k be a quasi-finite field of type (b), or more generally let $k=k_0((t))$ where k_0 is an algebraically closed field of any characteristic. If K_0 is a function-field of one variable over k_0, then $K=kK_0$ is a function-field of one variable over k. Let us denote by Σ_K, Σ_{K_0} the set of prime divisors of K, K_0 over k, k_0 resp. We define a map $\pi:\Sigma_K\to\Sigma_{K_0}$ as follows: let v be a non-trivial discrete valuation on K over k. If its restriction on K_0 is non-trivial, we set $\pi(v)=$ the restriction of v on K_0. If the restriction of v on K_0 is trivial, then v induces the inclusion map $i_v:K_0\to O_v$ where O_v denotes the residue field of the valuation ring of v. Since O_v is a finite extension of $k=k_0((t))$ which is complete under t-valuation, O_v is provided with the t-valuation which is the unique extension of the t-valuation on k, and its restriction on K_0 through the embedding i_v is non-trivial. (Indeed, let $O_v=k_0((T))$ where T is a prime element with respect to t-valuation, and set, for each $x\in K_0$, $i_v(x)=\Sigma a_i(x)T^i$ with $a_i(x)\in k_0$. If $i_v(x)$ is a t-unit then $i_v(x-a_0(x))$ is not a t-unit since i_v is k_0-linear map.) We then define $\pi(v)$ to be the restriction of the t-valuation through the embedding $i_v:K_0\to O_v$. We observe here that the embedding i_v: $K_0\to O_v$ induces the unique extension $i_v:\hat{K}_0\to O_v$ where $\hat{K}_0$ denotes the completion of K_0 under the valuation $\pi(v)$. We shall denote the degree $[O_v:\hat{K}_0]$ by $\delta(v)$. We also observe that there is a canonical map $\varepsilon:\Sigma_{K_0}\to\Sigma_K$ such that $\pi\cdot\varepsilon=\text{identity}$, where $\varepsilon(v)$ is the unique extension of v to K which is trivial on k.

Now let $L_0>K_0$ be a finite extension. This gives rise to a finite extension $L>K$ under the constant field extension from k_0 to k, and we obtain the commutative diagram

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$$\begin{array}{ccc} \Sigma_L & \xrightarrow{\pi} & \Sigma_{L_0} \\ \gamma\downarrow & & \downarrow\gamma \\ \Sigma_K & \xrightarrow[\pi]{} & \Sigma_{K_0} \end{array}$$

LEMMA. *Let v be in Σ_K.*

(1) *If $v = \varepsilon\pi(v)$, then v splits completely in $L > K$ if and only if $\pi(v)$ splits completely in $L_0 > K_0$.*

(2) *If $v \neq \varepsilon\pi(v)$, then v splits completely in $L > K$ if and only if $\delta(v)$ is divisible by the ramification indices of $\pi(v)$ in $L_0 > K_0$.*

Proof. (1) $\sharp\gamma^{-1}(v) = \sharp\gamma^{-1}\varepsilon\pi(v) = \sharp\gamma^{-1}\pi(v)$

(2) Let O_v be the valuation ring of v. $L_0 \otimes_{K_0} O_v$ is an integral domain since L_0, K are linearly disjoint over K_0, and furthermore the Jacobson-radical of $L_0 \otimes_{K_0} O_v$ is a principal ideal. It follows that $L_0 \otimes_{K_0} O_v$ is the integral closure of O_v in L. Now $L_0 \otimes_{K_0} O_v = L_0 \otimes_{K_0} O_v \otimes_{\hat{K}_0} O_v = (\hat{L}_0^{(1)} \oplus \hat{L}_0^{(2)} \oplus \cdots \oplus \hat{L}_0^{(q)}) \otimes_{\hat{K}_0} O_v$ where $\hat{K}_0$ is the completion of K_0 under the valuation $\pi(v)$ and $\hat{L}_0^{(1)}, \ldots, \hat{L}_0^{(q)}$ are the completions of L under the extensions of $\pi(v)$. Consequently v splits completely in L, i.e., $L_0 \otimes_{K_0} O_v$ is $[L : K]$-copies of O_v if and only if there exist K_0-imbeddings $L_0 < O_v$ for all j, i.e., if and only if $\delta(v) = [O_v : \hat{K}_0]$ is divisible by $[\hat{L}_0^{(j)} : \hat{K}_0]$ for all j. However, $[\hat{L}_0^{(j)} : \hat{K}_0]$ is nothing but the ramification index of the corresponding valuation.

THEOREM 2. *Let v_0 be in Σ_{K_0}.*

(1) *If v_0 is unramified in L_0/K_0, then $\pi^{-1}(v_0)$ splits completely in L/K.*

(2) *If v_0 is ramified in L_0/K_0 then $\pi^{-1}(v_0)$ contains infinitely many primes which do not split completely in L/K as well as infinitely many primes which do split completely in L/K.*

Proof. (1) This follows immediately from the above lemma. (2) Let us simply denote by $\hat{K}_0 = k_0((T))$ the completion of K_0 under the valuation v_0, where T is a fixed prime element taken inside K_0. For each element λ in k with $v_t(\lambda) > 0$ (where v_t is the t-valuation on k), we obtain the k-algebra map $\hat{K}_0 \otimes_{k_0} k = k_0((T)) \otimes_{k^0} k \to k$ determined by $T \to \lambda$ and the continuity. Restricting this map on $K_0 \otimes_{k_0} k$ we obtain the k-algebra map $\varphi_\lambda : K_0 \otimes_{k0} k \to k$ such that $\varphi_\lambda(T) = \lambda$, and φ_λ determines the valuation v_λ on K such that $\pi(v_\lambda) = v_0$. We observe that $\delta(v_\lambda) = [k : \hat{K}_0] = v_t(\lambda)$, and that v_λ, v_μ are inequivalent valuations of K if

$\lambda \neq \mu$ Therefore, for any given natural integer m, there exist infinitely many primes v in Σ_K such that $\pi(v) = v_0$ and $\delta(v) = m$, and consequently our statement follows from the above lemma.

COROLLARY. *Every prime splits completely in $L > K$ if and only if it is so in $L_0 > K_0$.*

Now let K_0 be a function field of one variable over k_0 with positive genus. Then K_0 admits an unramified abelian extension $L_0 \gneqq K_0$, and every prime splits completely since the constant field k_0 is algebraically closed. It follows from the above corollary that every prime splits completely in the abelian extension $L \gneqq K$, and thus the reciprocity law fails over the quasi-finite field of type (b).

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