ON CONNECTIONS OF CARTAN

SHÔSHICHI KOBAYASHI

Introduction. Consider a differentiable manifold M and the tangent bundle $T(M)$ over M, the structure group of which is usually the general linear group G'. Let P' be the principal fibre bundle associated with $T(M)$. Consider the fibre F of $T(M)$ as an affine space, then we have acting on F the affine transformation group G, which contains G' as the isotropic subgroup. Following the idea of Klein, it is more natural to take G as the structure group of the bundle $T(M)$. Let P be the principal fibre bundle associated to $T(M)$ with group G.

In the classical theory of affine connections, there are two points of view. The one is due to Levi-Civita, who considered each tangent space of M as a vector space and explained a connection as a law of parallel displacement of vectors along curves. From the point of view of the theory of connections in fibre bundles, a connection in the sense of Levi-Civita is a connection in the principal fibre bundle P' with group G'. The other point of view is due to E. Cartan. Following him, each tangent space of M is an affine space on which the affine transformation group G acts transitively, and an affine connection is a law of development of tangent spaces along curves; it is a connection in P.

The idea of Cartan was rigorously established by Ehresmann **(3)** as follows. Consider a fibre bundle B satisfying the conditions of *soudure* (see §2); the fibre F is homeomorphic to a homogeneous space G/G' and the structure group G of B can be reduced to G'. As in the case of tangent bundle, we obtain two principal fibre bundles P and P' with group G and G' respectively and P' is contained in P. A connection in P is called a connection of Cartan, if it satisfies the following condition: the differential form ω defining the connection gives an absolute parallelism on P'. The importance of this condition was shown in previous papers **(4; 5)**.

It is known that there is a correspondence between affine connections in the sense of Cartan and those in the sense of Levi-Civita; there is a canonical one-to-one correspondence between the set of connections in P and the set of connections in P' **(7)**.

The purpose of the present paper is to show that there exists a one-to-one correspondence between the set of Cartan connections in P and the set of infinitesimal connections in P', if the homogeneous space $F = G/G'$ is *weakly reductive* (see §2). We shall show also that in such a case the torsion forms can be defined. The last section will be devoted to the application to invariant connections.

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1. Tangent vectors. The manifolds and the mappings considered in this paper are all of class C^∞. For the definition of tangent vector and the differential of a mapping, the reader is referred to Chevalley's book **(2)**.

Let M be a manifold. We denote by $T(M)$ the set of all tangent vectors to M. For any two manifolds M and M', we have a natural isomorphism

$$T(M \times M') = T(M) \times T(M').$$

Let G be a Lie group and $\phi: G \times G \rightarrow G$ be the mapping defining group operation:

$$\phi(s, s') = s \cdot s', \qquad s,s' \in G.$$

Consider the differential mapping[1] $\delta\phi: T(G \times G) \rightarrow T(G)$. $T(G \times G)$ being identified with $T(G) \times T(G)$, $\delta\phi$ can be considered as a mapping of $T(G) \times T(G)$ onto $T(G)$ and defines a group operation in $T(G)$. The Lie group $T(G)$, obtained in this way, is called the *tangent group to* G. We have a natural imbedding of G into $T(G)$ and G is considered as a subgroup of $T(G)$. The set of all tangent vectors to G at the unit, which we shall denote by $T_e(G)$, is a normal subgroup of $T(G)$ and will be identified with the Lie algebra of G.

Suppose G acts, as a transformation group, on a manifold P on the right and let $\psi: P \times G \rightarrow P$ be the mapping defining the transformation law. Then, the differential mapping

$$\delta\psi: T(P) \times T(G) \rightarrow T(P)$$

defines $T(G)$ as a transformation group on $T(P)$ acting on the right. If P is a principal fibre bundle over M with group G and with projection π, then $T(P)$ is a principal fibre bundle over $T(M)$ with group $T(G)$ and with projection $\delta\pi$.

2. Soudure. Let B be a fibre bundle over base manifold M, with fibre F and with Lie structure group G. B is *soudé* **(3)** to M, if the following conditions are satisfied:

(s.1) G acts on F transitively: then F can be identified with the homogeneous space G/G', where G' is the isotropic group at a point o of F.

(s.2) $\dim F = \dim M$.

(s.3) The structure group G of the bundle B can be reduced to G': in other words, B admits a cross-section, which we shall denote by σ. When B is considered as the fibre bundle with structure group G', it will be denoted by B'.

(s.4) Two fibre bundles $T(M)$ and $T_F(B)$ over M, with group $GL(n, R)$ (where $n = \dim M$), are equivalent, where $T(M)$ is the space of all tangent vectors to M and $T_F(B)$ the space of all tangent vectors to F_x at $\sigma(x)$, x running through M.

Let P (resp. P') be the principal fibre bundle associated to B (resp. B').

[1]Chevalley denotes the differential of ϕ by $d\phi$.

The structure group and the fibre of P (resp. P') are G (resp. G'). P' can be considered as a submanifold of P.

Let $\mathfrak{g}$, $\mathfrak{g}'$ be the Lie algebras of G and G' respectively. Take a vector subspace $\mathfrak{f}$ of $\mathfrak{g}$ such that

(2.1) $$\mathfrak{g} = \mathfrak{g}' + \mathfrak{f}, \qquad \mathfrak{g}' \cap \mathfrak{f} = \{o\}.$$

The tangent space $T_o(F)$ to F at o can be identified with $\mathfrak{f}$; let p be the natural projection of G onto $F = G/G'$, then δp maps $T_e(G)$ onto $T_o(F)$, and since $T_e(G)$ and $\mathfrak{g}$ are identified, δp maps $\mathfrak{f}$ onto $T_o(F)$ isomorphically.

Each element s of G' induces a linear transformation of $T_o(F)$, which we shall denote by L_s. If $\mathfrak{f}$ satisfies

(2.2) $$ad(s)\cdot\mathfrak{f} \subseteq \mathfrak{f} \qquad s \in G',$$

then L_s corresponds to $ad(s)$, when we identify $T_o(F)$ with $\mathfrak{f}$.

Now we shall consruct a $T_o(F)$-valued linear differential form θ on P' satisfying the following conditions:

(θ.1) If $\bar{u} \in T(P')$ and $\theta(\bar{u}) = 0$, then $\delta\pi(\bar{u})$is the zero vector, where π is the projection of P' onto M.

(θ.2) $$\theta(\bar{u}s) = L_s^{-1}\theta(\bar{u}) \qquad \bar{u} \in T(P'), \quad s \in G'.$$

(θ.3) $$\theta(u\bar{s}) = 0 \qquad u \in P', \quad \bar{s} \in T(G').$$

Let $\bar{u}$ be a tangent vector to P' at u. The projection $\pi: P' \to M$ induces the projection $\delta\pi: T(P') \to T(M)$, and $\delta\pi(\bar{u})$ is a vector tangent to M at $\pi(u)$. As the bundle B is *soudé* to M, the vector $\delta\pi(u)$ can be identified with a vector tangent to F_x at $\sigma(x)$, where $x = \pi(u)$. We shall denote by $\bar{u}^*$ this vector tangent to F_x at $\sigma(x)$. The element $u \in P'$ is considered as a mapping of the standard fibre F onto F_x such that $u(o) = \sigma(x)$, where o is the point of F which defined the isotropic group G'. The map u induces the differential map δu of $T(F)$ onto $T(F_x)$. The inverse image $\delta u^{-1}(\bar{u}^*)$ of $\bar{u}^* \in T(F_x)$ by δu is a vector tangent to F at o, which we denote by $\theta(\bar{u})$. Clearly θ is a linear differential form on P'. If $\theta(\bar{u}) = 0$, then $\bar{u}^*$ is the zero vector; consequently $\delta\pi(\bar{u})$ is also the zero vector, which proves the property (θ.1).

Now we shall verify (θ.2).

We see that $\bar{u}s$ is a tangent vector to P' at us. As $\delta\pi(\bar{u}) = \delta\pi(\bar{u}s)$, we have $\bar{u}^* = (\bar{u}s)^*$.

Then

$$\theta(\bar{u}s) = \delta(us)^{-1}\cdot(\bar{u}s)^* = \delta(us)^{-1}\cdot\bar{u}^* = \delta s^{-1}\cdot\delta u^{-1}(\bar{u}^*) = \delta s^{-1}\theta(\bar{u}) = L_s^{-1}\theta(\bar{u}).$$

Finally we shall prove (θ.3). For any $u \in P'$ and $\bar{s} \in T(G')$, $\delta\pi(u\bar{s})$ is the zero vector. From the definition of θ, it is clear that $\theta(u\bar{s}) = 0$.

Suppose our fibre bundle satisfies only the conditions (s.1)–(s.3). We shall prove that, if there exists a $T_o(F)$-valued linear differential form θ on P', which possesses the properties (θ.1)–(θ.3), then the bundle B satisfies also the condition (s.4).

Let $\bar{x}$ be a tangent vector to M at x and $\bar{u}$ be a tangent vector to P' at u such that

$$\delta\pi(\bar{u}) = \bar{x}.$$

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Then $\pi(u) = x$. As $\theta(\bar{u})$ is an element of $T_o(F)$ and u is a map of F onto F_x such that $u(o) = \sigma(x)$, the image $\delta u(\theta(\bar{u}))$ of $\theta(\bar{u})$ by the differential of u is a tangent vector to F_x at $\sigma(x)$. Now we shall show that $\delta u(\theta(\bar{u}))$ depends only on $\bar{x}$ and is independent of the choice of $\bar{u}$ such that $\delta\pi(\bar{u}) = \bar{x}$. If $\bar{u}'$ is a tangent vector to P at the same point u such that $\delta\pi(\bar{u}') = \bar{x}$, from the property ($\theta$.3), $\theta(\bar{u}' - \bar{u}) = 0$; hence

$$\theta(\bar{u}') = \theta(\bar{u}), \qquad \delta u(\theta(\bar{u}')) = \delta u(\theta(\bar{u})).$$

If $\bar{u}' = \bar{u}s$ for some $s \in G'$, then $\bar{u}'$ is tangent to P' at us and

$$\theta(\bar{u}') = L_s^{-1}\theta(\bar{u}).$$

Hence

$$\delta(us)\,\theta(\bar{u}') = \delta u\cdot\delta s\cdot L_s^{-1}\theta(\bar{u}) = \delta u\cdot\theta(\bar{u}).$$

This completes the proof, because, for any $\bar{u}' \in T(P')$ such that

$$\delta\pi(\bar{u}') = \delta\pi(\bar{u}),$$

there is an element $s \in G'$ such that $\bar{u}'s$ is tangent at the same point as u and

$$\delta\pi(\bar{u}'s) = \delta\pi(\bar{u}).$$

If the vector subspace $\mathfrak{f}$ of $\mathfrak{g}$ satisfies (2.1) and (2.2), it can be identified with $T_o(F)$. Therefore θ is considered as an $\mathfrak{f}$-valued linear differential form and the property (θ.2) is replaced by

$$(\theta.2') \qquad \theta(\bar{u}s) = s^{-1}\theta(\bar{u})s \qquad \bar{u} \in T(P'), \quad s \in G'.$$

A homogeneous space $F = G/G'$ is called *weakly reductive* **(8)**, if there is a vector subspace $\mathfrak{f}$ of $\mathfrak{g}$ satisfying (2.1) and (2.2).

THEOREM 1. *A fibre bundle B satisfying the condition* (s.1)–(s.3) *is soudé to M, if and only if there exists a $T_o(F)$-valued linear differential form θ on P' possessing the properties* (θ.1)–(θ.3). *If the homogeneous space $F = G/G'$ is weakly reductive then θ is considered as an $\mathfrak{f}$-valued linear differential form and the property* (θ.2) *is replaced by* (θ.2′).

Remarks on weakly reductive homogeneous spaces. In either of the following cases, the homogeneous space F is weakly reductive:

(1) G' is compact,

(2) G' is semi-simple and connected,

(3) G' is discrete.

If F is an affine space (resp. Euclidean space) and G is the affine transformation group (resp. the group of motion) of F, then F is weakly reductive.

If $F = G/G'$ is weakly reductive, then there exists an affine connection on F invariant by G **(8)**. Therefore the linear isotropic group G' is isomorphic to the isotropic group G. If F is a real projective space and G is the projective transformation group of F, then $F = G/G'$ is not weakly reductive.

3. Connections of Cartan. We shall use the same notations as in §2. An infinitesimal connection in P is defined by a $\mathfrak{g}$-valued linear differential form $\tilde{\omega}$ on P with

$$(\tilde{c}.1) \qquad \tilde{\omega}(u\bar{s}) = s^{-1}\bar{s} \qquad u \in P, \quad \bar{s} \in T_s(G),$$

$$(\tilde{c}.2) \qquad \tilde{\omega}(\bar{u}s) = s^{-1}\tilde{\omega}(\bar{u})s \qquad u \in T(P), \quad s \in G.$$

The meaning of $s^{-1}\bar{s}$ and $s^{-1}\tilde{\omega}(\bar{u})s$ is explained in §1.

Let ω be the restriction of the form $\tilde{\omega}$ on P'. Then ω is also a $\mathfrak{g}$-valued linear differential form such that

$$(c.1) \qquad \omega(u\bar{s}) = s^{-1}\bar{s} \qquad u \in P', \quad s \in T_s(G'),$$

$$(c.2) \qquad \omega(\bar{u}s) = s^{-1}\omega(\bar{u})s \qquad u \in T(P'), \quad s \in G'.$$

The form ω does not give a connection in P', because it is not $\mathfrak{g}'$-valued. It is clear that, if ω is a $\mathfrak{g}$-valued linear differential form on P' satisfying the conditions (c.1) and (c.2), then it is the restriction of a unique differential form $\tilde{\omega}$ on P satisfying the conditions (c.1) and (c.2).

An infinitesimal connection in P defined by $\tilde{\omega}$ is called a *connection of Cartan* **(3)**, if the restricted form ω satisfies the following condition:

(c.3) If $\bar{u} \in T(P')$ and $\omega(\bar{u}) = 0$, then $\bar{u}$ is the zero vector. This implies that ω defines an absolute parallelism on P'.

Suppose the homogeneous space $F = G/G'$ is weakly reductive, and let ω' be a $\mathfrak{g}'$-valued linear differential form on P', which defines an infinitesimal connection in P'. The form ω' satisfies the same conditions (c.1) and (c.2) as the form ω; the difference is that the one is $\mathfrak{g}'$-valued and the other is $\mathfrak{g}$-valued. Let θ be the $\mathfrak{k}$-valued linear differential form on P' in Theorem 1. We shall show that the sum $\theta + \omega'$ satisfies the conditions (c.1)–(c.3). Put

$$(3.1) \qquad \omega = \theta + \omega'.$$

Then

$$(3.2) \qquad \omega(u\bar{s}) = \theta(u\bar{s}) + \omega'(u\bar{s}) \qquad u \in P', \quad \bar{s} \in T_s(G').$$

From (θ.3), we obtain

$$(3.3) \qquad \omega(u\bar{s}) = \omega'(u\bar{s}) \qquad u \in P', \quad \bar{s} \in T_s(G').$$

As ω' is a form of connection in P', we have

$$(3.4) \qquad \omega'(u\bar{s}) = s^{-1}\bar{s},$$

which proves that ω satisfies (c.1).

We have

$$(3.5) \qquad \omega(\bar{u}s) = \theta(\bar{u}s) + \omega'(\bar{u}s) \qquad \bar{u} \in T(P'), \quad s \in G'.$$

Since ω' is a form of connection in P', we have

$$(3.6) \qquad \omega'(\bar{u}s) = s^{-1}\omega'(\bar{u})s \qquad \bar{u} \in T(P'), \quad s \in G'.$$

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From $(\theta.2')$ and (3.6), it follows that

(3.7) $$\omega(\bar{u}s) = s^{-1}\omega(\bar{u})s \qquad \bar{u} \in T(P'), \qquad s \in G'.$$

Suppose

(3.8) $$\omega(\bar{u}) = 0,$$

which implies

(3.9) $$\theta(\bar{u}) = 0, \quad \omega'(\bar{u}) = 0.$$

The first means that $\delta\pi(\bar{u})$ is the zero vector, or that the vector $\bar{u}$ is vertical in the sense of Ambrose **(1)**, and the latter implies that the vector $\bar{u}$ is horizontal **(1)** with respect to the connection in P' defined by ω'. Therefore $\bar{u}$ is the zero vector.

We have proved the following

LEMMA 1. *Suppose $F = G/G'$ is weakly reductive. If ω' is a $\mathfrak{g}'$-valued linear differential form on P' defining a connection in P' and θ is an $\mathfrak{k}$-valued linear differential form on P' satisfying the conditions* $(\theta.1)$, $(\theta.1')$, $(\theta.3)$, *then the form $\omega = \theta + \omega'$ defines a connection of Cartan in P; that is, ω is the restriction of a form $\tilde{\omega}$ on P defining a connection of Cartan in P.*

Now, suppose that ω is a form on P' satisfying the conditions (c.1)–(c.3). Let θ(resp. ω') be the $\mathfrak{k}$ (resp. $\mathfrak{g}'$) component of ω:

(3.10) $$\omega = \theta + \omega',$$

(3.11) $$\theta(\bar{u}) \in \mathfrak{k}, \quad \omega'(\bar{u}) \in \mathfrak{g}' \qquad \bar{u} \in T(P').$$

We shall prove that θ satisfies the conditions $(\theta.1)$, $(\theta.2')$, $(\theta.3)$ and that ω' defines a connection in P'.

Suppose

(3.12) $$\theta(\bar{u}) = 0.$$

Then

(3.13) $$\omega(\bar{u}) = \omega'(u) \in \mathfrak{g}'.$$

Take an element $\bar{s} \in T_e(G')$ such that

(3.14) $$s = -\omega(\bar{u}).$$

($T_e(G')$ was identified with the Lie algebra $\mathfrak{g}'$ of G'.) Then[2]

(3.15) $$\omega(\bar{u}\bar{s}) = \omega(\bar{u}) + \bar{s} = 0.$$

From (c.3), it follows that $\bar{u}\bar{s}$ is the zero vector; hence

(3.16) $$\delta\pi(\bar{u}) = \delta\pi(\bar{u}\bar{s}) = 0,$$

which proves that θ satisfies $(\theta.1)$.

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[2]The conditions (c.1) and (c.2) are equivalent to the following single condition: $\omega(\bar{u}\bar{s}) = s^{-1}\bar{s} + s^{-1}\omega(\bar{u})s$, because $\omega(\bar{u}\bar{s}) = \omega(u\bar{s}) + \omega(\bar{u}s)$. Putting $s = e$, we obtain (3.15).

Since $\omega(u\bar{s}) = s^{-1}\bar{s}$ is contained in $\mathfrak{g}'$, $\theta(u\bar{s})$ vanishes for any $u \in P'$ and $\bar{s} \in T_s(G')$; hence

$$\omega(u\bar{s}) = \omega'(u\bar{s}) \qquad u \in P', \quad \bar{s} \in T_s(G'). \tag{3.17}$$

Therefore θ satisfies (θ.3) and ω' satisfies (c.1). We have

$$\omega(\bar{u}s) = s^{-1}(\theta(\bar{u}) + \omega'(\bar{u}))s = s^{-1}\theta(\bar{u})s + s^{-1}\omega'(\bar{u})s \qquad \bar{u} \in T(P'),\ s \in G'. \tag{3.18}$$

As the homogeneous space F is weakly reductive, $s^{-1}\theta(\bar{u})s$ is contained in $\mathfrak{k}$. Comparing (3.18) with the following equality

$$\omega(\bar{u}s) = \theta(\bar{u}s) + \omega'(\bar{u}s), \tag{3.19}$$

we obtain

$$\theta(\bar{u}s) = s^{-1}\theta(\bar{u})s, \qquad \omega'(\bar{u}s) = s^{-1}\omega'(\bar{u})s. \tag{3.20}$$

Therefore θ satisfies (θ.2′) and ω' satisfies (c.2).

LEMMA 2. *If a $\mathfrak{g}$-valued linear differential form ω on P' satisfies the conditions* (c.1)–(c.3), *then ω is the direct sum of an $\mathfrak{k}$-valued form θ satisfying* (θ.1), (θ.2′), (θ.3) *and a form ω' defining an infinitesimal connection in P'.*

Theorem 1 justifies the following definition: An $\mathfrak{k}$-valued linear differential form θ is called a *form of "soudure,"* if θ satisfies the conditions (θ.1)–(θ.3).

THEOREM 2. *Suppose $F = G/G'$ is weakly reductive. Then, to every pair of a soudure of B and a connection in P', there corresponds a unique connection of Cartan in P. Conversely, to each connection of Cartan in P, there corresponds a unique pair of a soudure of B and a connection in P'. If we denote by θ, ω', ω a form of soudure, a form of connection in P', a form (restricted on P') of Cartan connection in P respectively, then the correspondence is given by $\omega = \theta + \omega'$.*

The Theorem follows immediately from Lemmas 1 and 2.

4. Structure equations. Let ω be a form on P defining a connection of Cartan in P. Then we have

$$d\tilde{\omega} = -\tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] + \tilde{\Omega}, \tag{4.1}$$

where $\tilde{\Omega}$ is the curvature form **(1; 3)**.

Consider the restricted form ω on P'. Then we have

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega, \tag{4.2}$$

where Ω is the restriction of $\tilde{\Omega}$ on P'.

Assuming the homogeneous space $F = G/G'$ is weakly reductive, we substitute $\omega = \theta + \omega'$ in (4.2) and we obtain

$$d\theta + d\omega' = -\tfrac{1}{2}([\theta, \omega'] + [\omega', \theta] + [\omega', \omega'] + [\theta, \theta]) + \Omega. \tag{4.3}$$

We decompose $[\theta, \theta]$ and Ω into two components as follows:

$$[\theta, \theta] = [\theta, \theta]_{\mathfrak{k}} + [\theta, \theta]_{\mathfrak{g}'}, \quad \Omega = \Omega_{\mathfrak{k}} + \Omega_{\mathfrak{g}'},$$

where, for any $\bar{u}$, $\bar{u}' \in T(P')$ tangent at the same point,

$$[\theta(\bar{u}), \theta(\bar{u}')]_{\mathfrak{f}} \in \mathfrak{f}, \qquad [\theta(\bar{u}), \theta(\bar{u}')]_{\mathfrak{g}'} \in \mathfrak{g}'$$
$$\Omega_{\mathfrak{f}}(\bar{u}, \bar{u}') \in \mathfrak{f}, \qquad \Omega_{\mathfrak{g}'}(\bar{u}, \bar{u}') \in \mathfrak{g}'.$$

Then we obtain from (4.3) the following equalities:

$$d\theta = -\tfrac{1}{2}([\theta, \omega'] + [\omega', \theta] + [\theta, \theta]_{\mathfrak{f}}) + \Omega_{\mathfrak{f}}. \tag{4.4}$$

$$d\omega' = -\tfrac{1}{2}([\omega', \omega'] + [\theta, \theta]_{\mathfrak{g}'}) + \Omega_{\mathfrak{g}'}. \tag{4.5}$$

Putting

$$\Theta = \Omega_{\mathfrak{f}} - \tfrac{1}{2}[\theta, \theta]_{\mathfrak{f}}, \tag{4.6}$$

we call Θ the *torsion form* of the connection of Cartan. As the curvature form Ω' of the connection in P' defined by ω' is given by

$$d\omega' = -\tfrac{1}{2}[\omega', \omega'] + \Omega', \tag{4.7}$$

we obtain from (4.5) the following equality.

$$\Omega' = \Omega_{\mathfrak{g}'} - \tfrac{1}{2}[\theta, \theta]_{\mathfrak{g}'}. \tag{4.8}$$

Now we obtain the following

THEOREM 3. *Let*

$$\Theta = \Omega_{\mathfrak{f}} - \tfrac{1}{2}[\theta, \theta]_{\mathfrak{f}}$$

be the torsion form and Ω' the curvature form of the connection in P' defined by ω'. Then we have

$$d\theta = -\tfrac{1}{2}([\theta, \omega'] + [\omega', \theta]) + \Theta,$$
$$\Omega' = \Omega_{\mathfrak{g}'} - \tfrac{1}{2}([\theta, \theta]_{\mathfrak{g}'}).$$

(1) *If the homogeneous space $F = G/G'$ satisfies furthermore the condition*

$$[\mathfrak{f}, \mathfrak{f}] \subseteq \mathfrak{g}',$$

then we have

$$\Theta = \Omega_{\mathfrak{f}}, \qquad \Omega' = \Omega_{\mathfrak{g}'} - \tfrac{1}{2}[\theta, \theta].$$

(2) *If the homogeneous space $F = G/G'$ satisfies the stronger condition*

$$[\mathfrak{f}, \mathfrak{f}] = 0,$$

then we have

$$\Theta = \Omega_{\mathfrak{f}}, \qquad \Omega' = \Omega_{\mathfrak{g}'}.$$

Remarks. A homogeneous space F is called *symmetric*, if it satisfies the assumption of (1) in Theorem 3. On such a space F, there exists **(8)** an affine symmetric connection invariant under G. If F is an affine space and G is the affine transformation group, then F satisfies the assumption of (2) in Theorem 3. In this case, a connection in P' is called a *linear connection* (because the structure group G' is the general linear group). If F is an affine space and B is the tangent bundle $T(M)$, then there is a canonical soudure in B. If we take always this canonical soudure, then Theorem 2 says that, to each linear connection in P', there corresponds a unique connection of Cartan in P, which will be called an *affine connection*. Part (2) of Theorem 3 implies that the restriction on P' of the curvature form of an affine connection is the sum of the torsion form and the curvature form of the corresponding linear connection (which is usually called the curvature form of the affine connection).

It will not be useless to point out that the holonomy group of the linear connection corresponding to an affine connection is usually called the homogeneous holonomy group of the affine connection. If the torsion form of an affine connection vanishes, then the form $\Omega_{\mathfrak{f}}$ vanishes also ((2) of Theorem 3). But this does not imply that the form $\tilde{\Omega}_{\mathfrak{f}}$, f-component of the curvature form of the affine connection (of which $\Omega_{\mathfrak{f}}$ is the restriction on P') vanishes. That is why the holonomy group of an affine connection without torsion contains the translation part **(7)**. And we shall see easily that, if the non-homogeneous holonomy group coincides with the homogeneous holonomy group, then our affine connection is flat.

5. Invariant connections of Cartan. Consider a homogeneous space $F = G/G'$. G is considered as a principal fibre bundle over the base manifold F, with structure group G' and with the natural projection **(9)**

$$\pi: G \to F = G/G'.$$

Let P be the fibre bundle with fibre G (on which G' acts on the left)associated to the principal fibre bundle G described above. P is defined as follows. We shall say two elements (s_1, s_2) and (s_3, s_4) of $G \times G$ are equivalent if there is an element s' of G' such that

$$s_1 s' = s_3, \qquad s'^{-1} \cdot s_2 = s_4. \tag{5.1}$$

P is the set of these equivalence classes with the natural structure of fibre bundle; the projection of P onto the base manifold F is induced from the mapping of $G \times G$ onto F:

$$(s_1, s_2) \to \pi(s_1), \tag{5.2}$$

where π is the natural projection of G onto F. The operation of G on $G \times G$ on the right given by

$$(s_1, s_2)\, s = (s_1, s_2 s) \tag{5.3}$$

induces the operation of G on P on the right. In this way, P can be considered as a principal fibre bundle with group G.

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The injection of G into $G \times G$ such that $s \to (s, e)$, where e is the unit of G, defines the injection of G into P. The submanifold G of P is stable under the operation of G' on the right; that is, if $u \in P$ belongs to the submanifold G, then us belongs to G for any $s \in G'$.

LEMMA 3. *The princpal fibre bundle P is trivial; P is the direct product of the base space F and the structure group G.*

Proof. Define a mapping j of $G \times G$ onto $F \times G$ as follows:

$$j(s_1, s_2) = (\pi(s_1), s_1 s_2). \tag{5.4}$$

Then j induces a mapping j° of P onto $F \times G$, which commutes obviously with the operation of G on the right, proving the Lemma.

As P is trivial, the fibre bundle B with fibre F associated to the principal fibre bundle P is also trivial:

$$B = F \times F. \tag{5.5}$$

LEMMA 4. *The fibre bundle B with fibre F associated to P is soudé* **(3)** *to the base manifold F.*

Proof. The conditions (s.1) and (s.2) of §1 are apparently satisfied. We take the cross-section σ defined as follows:

$$F \ni x \to (x, x) \in F \times F = B. \tag{5.6}$$

The identification of $T(F)$ with $T_F(B)$ is given by

$$T(F) \ni \bar{x} \to (x, \bar{x}) \in T_F(B). \tag{5.7}$$

If we reduce the structure group G of P to G', we obtain the principal fibre bundle G, from which we started.

The fibre bundle G corresponds to the fibre bundle P' in §2. Therefore *we denote by P' the fibre bundle G.*

A connection of Cartan in P is given by a $\mathfrak{g}$-valued linear differential form ω on $P'(=G)$ satisfying the conditions (c.1)–(c.3). As P' is a group space G, G acts on P' on the left as well as on the right. We shall define a left invariant connection of Cartan; that is, we shall define a g-valued form ω on P' such that

$$\omega(s\bar{u}) = \omega(\bar{u}) \qquad \bar{u} \in T(P'), \qquad s \in G. \tag{5.8}$$

It is clear that such a form ω is unique and must be defined by

$$\omega(u\bar{s}) = \bar{s} \qquad u \in P', \quad \bar{s} \in T_e(G). \tag{5.9}$$

In this case the structure equation of E. Cartan reduces to the equation of Maurer-Cartan:

$$d\omega = -\tfrac{1}{2}[\omega, \omega]. \tag{5.10}$$

THEOREM 4. *There is a unique left invariant connection of Cartan in P. It is given by a g-valued form ω on $P'(=G)$ defined as follows:*

$$\omega(u\bar{s}) = \bar{s} \qquad u \in P', \qquad \bar{s} \in T_e(G).$$

The curvature form of the connection vanishes on P', hence on P, too.

Proof. From (5.10), it follows that the curvature form vanishes on P'. Let $\tilde{\Omega}$ be the curvature form. Then we have

$$(5.11) \qquad \tilde{\Omega}(\bar{u}s, \bar{u}'s) = s^{-1}\tilde{\Omega}(\bar{u}, \bar{u}')s \qquad \bar{u}, \bar{u}' \in T_u(P), \qquad s \in G.$$

Since $\tilde{\Omega}$ vanishes on P', it follows easily from (5.11) that $\tilde{\Omega}$ vanishes on P.

Suppose the homogeneous space $F = G/G'$ is weakly reductive. Let

$$(5.12) \qquad \omega = \theta + \omega'$$

be the decomposition of the form ω into an $\mathfrak{f}$-valued form θ and into a $\mathfrak{g}'$-valued form ω'. The $\mathfrak{g}'$-valued form ω' defines a connection in the principal fibre bundle $P'(=G)$ with group G'. Let Θ be the torsion form of the connection of Cartan defined by ω and Ω' the curvature form of the connection in P' defined by ω'. From Theorems 3 and 4, it follows that

$$(5.13) \qquad \Theta = -\tfrac{1}{2}[\theta, \theta]_{\mathfrak{f}},$$

$$(5.14) \qquad \Omega' = -\tfrac{1}{2}[\theta, \theta]_{\mathfrak{g}'}.$$

THEOREM 5. *Let ω be the $\mathfrak{g}$-valued form on $P'(=G)$ defining the invariant connection of Cartan in P. Suppose the homogeneous space $F = G/G'$ is weakly reductive and let $\omega = \theta + \omega'$ be the decomposition corresponding to a decomposition of the Lie algebra $\mathfrak{g}$ satisfying* (2.2). *Then*

(1) *The torsion form of the connection of Cartan defined by ω is given by*

$$\Theta = -\tfrac{1}{2}[\theta, \theta]_{\mathfrak{f}}.$$

(2) *The curvature form of the connection in P' defined by ω' is given by*

$$\Omega' = -\tfrac{1}{2}[\theta, \theta]_{\mathfrak{g}'}.$$

(3) *The torsion form vanishes, if and only if the homogeneous space F is symmetric; that is,*

$$[\mathfrak{f}, \mathfrak{f}] \subseteq \mathfrak{g}'.$$

(4) *The restricted holonomy group of the connection defined by ω' is an invariant subgroup of the connected component of the unit of G'. And the Lie algebra of the holonomy group is the linear closure of*

$$\{[f_1, f_2]_{\mathfrak{g}'};\ f_1, f_2 \in \mathfrak{f}\}.$$

Proof. We have only to prove (3) and (4). From (θ.1) it follows that, for any f_1, f_2 and $u \in P'$, there are $\bar{u}_1, \bar{u}_2 \in T_u(P')$ such that

$$(5.15) \qquad \theta(\bar{u}_1) = f_1, \qquad \theta(\bar{u}_2) = f_2.$$

Therefore, in order that the homogeneous space F be symmetric, it is necessary that the torsion form vanishes. It is evident that, if F is symmetric, the torsion form vanishes.

Now we shall prove (4). Take an arbitrary point u_0 in P' and let P° be the set of all points in P' which can be joined to u_0 by horizontal curves **(1)** (with respect to the connection defined by ω'). In other words, we reduce the structure group of P' to the holonomy group of the connection defined by ω', and we obtain the principal fibre bundle P° whose structure group is the holonomy group. Then the Lie algebra of the holonomy group is the linear closure of **(1)**.

(5.16) $$\{\Omega'(\bar{u}_1, \bar{u}_2);\ \bar{u}_1, \bar{u}_2 \in T_u(P^\circ),\ u \text{ running through } P^\circ\},$$

which is equal to

(5.17) $$\{[\theta(\bar{u}_1), \theta(\bar{u}_2)]_{\mathfrak{g}'};\ \ \bar{u}_1, \bar{u}_2 \in T_u(P^\circ)\}.$$

Since, for any $f_1, f_2 \in \mathfrak{f}$ and $u \in P^\circ$, there are $\bar{u}_1, \bar{u}_2 \in T_u(P^\circ)$ satisfying (5.15), the set (5.17) is equal to the set

(5.18) $$\{[f_1, f_2]_{\mathfrak{g}'};\ \ f_1, f_2 \in \mathfrak{f}\}.$$

Using the Jacobi's identity, we see easily that the linear closure of the set (5.15) is an ideal of the Lie algebra $\mathfrak{g}'$ of G'. This completes the proof of (4).

Remark. The results in this section are closely related to those of Nomizu on invariant affine connections **(8)**. The relation between them will be discussed in another paper.

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University of Washington, Seattle