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OPTIMALITY AND DUALITY WITHOUT A CONSTRAINT QUALIFICATION FOR MINIMAX PROGRAMMING

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Without the need of a constraint qualification, we establish the optimality necessary and sufficient conditions for generalised minimax programming. Using these optimality conditions, we construct a parametric dual model and a parameter-free mixed dual model. Several duality theorems are established.

1. INTRODUCTION

Consider the generalised minimax programming:

$$(P) \qquad \min_{x\in S} \sup_{y\in Y} \phi(x,y)$$

where S is nonempty subset of R^n defined by

$$S = \{x \in R^n : h_j(x) \leqslant 0,\ j = 1, 2, \ldots, m\}.$$

Y is a compact subset of R^n, $\phi(x,y) : R^n \times Y \to R$ is a convex function with respect to x, and $h_j(x) : R^n \to R$, $j = 1, \ldots, m$ are convex functions, for each $x \in R^n$, and $\phi(x,y)$ is upper semi-continuous with respect to y.

Ben-Israel, Ben-Tal, and Zlobec presented some necessary and sufficient optimality conditions for (scalar) convex programming problems without any constraint qualification in [**2**], and Egudo, Weir, and Mond gave some necessary and sufficient optimality conditions for multi-objective convex programming problems without any constraint qualification in [**4**]. Latter, Mond[**7**], Ben-Tal, and Zlobec [**2**], Egudo [**4**], Weir [**8, 9**] used these optimality conditions in designing various dual models and established several duality theorems without any constraint qualification for the (convex or generalised convex, single-objective or multi-objective) mathematical programming.

Motivated by [**3, 2**], Lai, Liu and Tanaka [**6**] established some optimality necessary and sufficient conditions for the generalised fractional programming without any constraint qualification and constructed a parametric dual model and two parameter-free dual models.

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In this paper, we establish the optimality necessary and sufficient conditions for generalised minimax programming (P) without the need of a constraint qualification. Using these optimality conditions and the ideas of Bector [1], we construct a parametric dual model and a parameter-free mixed dual model, whereas the latter model unifies the two parameter-free dual models of Lai, Liu andk Tanaka in [6]. Several duality theorems are established, subsequently, one of the problems posed by Lai, Liu and Tanaka [6] is solved.

2. Notations and preliminary results

In this section, we first introduce some notation. Let $J = \{1, 2, \ldots, m\}$, and define

$$J(x) = \{j \in J \mid h_j(x) = 0\}$$

as the set of active indices at x. The minimal index set of the binding constraints at x for S is

$$J^= = \{j \in J \mid h_j(x) = 0, \text{ for all } x \in S\}$$

We also denote

$$J^<(x) = J(x) \setminus J^= = \{j \in J(x) \mid \exists x_i \in S \text{ to satisfy } h_j(x_i) < 0\}.$$

Define the set

$$S^= = \{x \in R^n \mid h_j(x) \leqslant 0, j \in J^=\}$$

with the convention

$$S^= = R^n, \text{ if } J^= = \emptyset.$$

Since $J^= \subset J$, the set $S^= \supset S$. Obviously,

$$S^= = \{x \in R^n \mid h_j(x) = 0, j \in J^=\}.$$

Definition 2.1. ([6]) For a function $h : R^n \mapsto R$ and a point $x \in \mathrm{dom}(h)$, the cone of directions of constancy at x with respect to the function h is defined by

$$D_h^=(x) = \{d \in R^n \mid \text{there exists an } \overline{\alpha} \text{ such that } h(x + \alpha d) = h(x), \text{ for every } 0 < \alpha < \overline{\alpha}\}.$$

We use the following notation:

$$D_j^=(x) = D_{h_j}^=(x), \ D_J^=(x) = \bigcap_{j \in J} D_j^=(x).$$

If $J = \emptyset$, we define $D_\emptyset^=(x) = R^n$.

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DEFINITION 2.2. ([6]) Let $M \subset R^n$ be a nonempty subset. The positive dual cone M^* of M is defined by

$$M^* = \{d \in R^n \mid d^T x \geqslant 0, \text{ for all } x \in M\}.$$

LEMMA 2.3. ([7]) *If $x \in S$ and $u \in S^=$, then*

$$(x-u)^T d \geqslant 0, \text{ for all } d \in \left[D^=_{\overline{J}^=}(x)\right]^*.$$

Let $A : R^n \mapsto R$ be a convex continuous function. We consider the following scalar minimisation problem:

$$\text{(SP)} \quad \min\{A(x) \mid x \in S\}.$$

LEMMA 2.4. ([2, Corollary 3.7]) *Let $x_0 \in S$. Then, x_0 is an optimal solution of (SP) if, and only if, there exist $\gamma_j \geqslant 0$, $j \in J^<(x_0)$, such that*

$$0 \in \partial A(x_0) + \sum_{j \in J^<(x_0)} \gamma_j \partial h_j(x_0) - \left[D^=_{\overline{J}^=}(x)\right]^*.$$

Let Y is a infinite compact set, $f_y, y \in Y$ are convex functions, we assume that

$$f(x) := \sup\{f_y(x) \mid y \in Y\} < +\infty \quad \text{for all } x \in R^n,$$
$$Y(x) = \{y \in Y \mid f_y(x) = f(x)\}.$$

LEMMA 2.5. ([5, Theorem 4.4.2]) *Assume that $f_y, y \in Y$ are convex functions, Y is a infinite compact set, on which the functions $y \mapsto f_y(x)$ are upper semi-continuous for each $x \in R^n$. Then*

$$\partial f(x) = co\{\cup \partial f_y(x), y \in Y(x)\}.$$

In the following, we shall consider elements of the set $R_+^{(Y)}$ defined as

$$R_+^{(Y)} := \{\lambda : Y \to R_+ \mid \lambda_y = 0, \quad \text{for all } y \text{ except for a finite number}\}.$$

LEMMA 2.6. *For each $x \in S$, one has*

$$f(x) = \sup_{y \in Y}[\phi(x,y)] = \sup_{\beta \in U}\left[\sum_{y \in D(\beta)} \beta_y \phi(x,y)\right],$$

where $U = \left\{\beta \in R_+^{(Y)} \mid \sum_{y \in D(\beta)} \beta_y = 1\right\}, D(\beta) = \{y \in Y \mid \beta_y \neq 0\}.$

PROOF: For arbitrary $y_0 \in Y$, let $\overline{\beta} = (\overline{\beta}_y)$, where $\overline{\beta}_y = 1$, if $y = y_0$; otherwise, $\overline{\beta}_y = 0$, thus $\overline{\beta} \in U$ and

$$\phi(x, y_0) = \sum_{y \in D(\overline{\beta})} \overline{\beta}_y \phi(x,y) \leqslant \sup_{\beta \in U}\left[\sum_{y \in D(\beta)} \beta_y \phi(x,y)\right].$$

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By the arbitrariness of $y_0 \in Y$, we derive

$$\sup_{y\in Y} \phi(x,y) \leqslant \sup_{\beta\in U}\Big[\sum_{y\in D(\beta)} \beta_y \phi(x,y)\Big].$$

On the other hand, For arbitrary $\overline{\beta} = (\overline{\beta}_y) \in U$, we have

$$\sum_{y\in D(\overline{\beta})} \overline{\beta}_y \phi(x,y) \leqslant \max\{\phi(x,y) \mid y \in D(\overline{\beta})\} \leqslant \sup_{y\in Y} \phi(x,y).$$

By the arbitrariness of $\overline{\beta} \in U$, it holds that

$$\sup_{\beta\in U}\Big[\sum_{y\in D(\beta)} \beta_y \phi(x,y)\Big] \leqslant \sup_{y\in Y}(\phi(x,y)).$$

Thus

$$\sup_{y\in Y} \phi(x,y) = \sup_{\beta\in U}\Big[\sum_{y\in D(\beta)} \beta_y \phi(x,y)\Big].$$

∎

3. Optimality necessary and sufficient condition for (P)

In this section, we shall establish optimality necessary and sufficient conditions for the generalised minmax programming problem (P).

In the following, we suppose that for each $v \in R_+$,

$$f(x) = \sup\{\phi(x,y) \mid y \in Y\} < +\infty \quad \text{for all} \quad x \in R^n,$$

and denote:

$$L(x_0) = \Big\{y \in Y \mid \phi(x_0, y) = \sup_{y\in Y} \phi(x_0, y)\Big\},$$

$$R_+^{(L(x_0))} = \{\alpha : L(x_0) \to R_+ \mid \alpha_y = 0, \quad \text{for all } y \text{ except for a finite number}\},$$

and $U_0 = \Big\{\alpha \in R_+^{(L(x_0))} \mid \sum_{y\in D_0(\alpha)} \alpha_y = 1\Big\}$, $D_0(\alpha) = \{y \in L(x_0) \mid \alpha_y \neq 0\}$.

Based on Lemmas 2.4 and 2.5, we can get the following result.

Theorem 3.1. *$x_0 \in S$ is an optimal solution of (P) if, and only if, there exist $\alpha \in U_0$, and $\gamma_j \geqslant 0, j \in J^<(x_0)$, such that*

$$0 \in \sum_{y\in D_0(\alpha)} \alpha_y[\partial\phi(\cdot,y)(x_0)] + \sum_{j\in J^<(x_0)} \gamma_j \partial h_j(x_0) - [D_{\overline{J}^=}(x_0)]^*.$$

Proof: If x_0 is an optimal solution of (P), then, by lemma 2.4, there exist $\gamma_j \geqslant 0, j \in J^<(x_0)$, such that

$$0 \in \partial \sup_{y\in Y}[\phi(\cdot,y)](x_0) + \sum_{j\in J^<(x_0)} \gamma_j \partial h_j(x_0) - [D_{\overline{J}^=}(x_0)]^*.$$

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It follows from Lemma 2.5 that there exists $\alpha \in U_0$, such that

$$0 \in \sum_{y \in D_0(\alpha)} \alpha_y \big[\partial \phi(\cdot, y)\big](x_0) + \sum_{j \in J^<(x_0)} \gamma_j \partial h_j(x_0) - \big[D_{\overline{J}=}^{=}(x_0)\big]^*.$$

Conversely, if there exist $\alpha \in U_0$, and $\gamma_j \geqslant 0, j \in J^<(x_0)$, such that

$$0 \in \sum_{y \in D_0(\alpha)} \alpha_y \big[\partial \phi(\cdot, y)\big](x_0) + \sum_{j \in J^<(x_0)} \gamma_j \partial h_j(x_0) - \big[D_{\overline{J}=}^{=}(x_0)\big]^*,$$

then from Lemma 2.5, it follows that

$$\sum_{y \in D_0(\alpha)} \alpha_y \big[\partial \phi(\cdot, y)\big](x_0) \subset \operatorname{co}\Big\{ \bigcup_{y \in L(x_0)} \partial\big[\phi(\cdot, y)\big](x_0) \Big\} = \partial \sup_{y \in Y} \big[\phi(\cdot, y)\big](x_0).$$

So we obtain

$$0 \in \partial \sup_{y \in Y} \big[\phi(\cdot, y)\big](x_0) + \sum_{j \in J^<(x_0)} \gamma_j \partial h_j(x_0) - \big[D_{\overline{J}=}^{=}(x_0)\big]^*.$$

By lemma 2.4 we can deduce that x_0 is an optimal solution of (P). ∎

For $\alpha \in U$, we denote by

$$\alpha^T \Phi(x, y) = \sum_{y \in D(\alpha)} \alpha_y \phi(x, y), \quad \gamma^T H(x) = \sum_{j=1}^{m} \gamma_j h_j(x),$$

$$\partial\big[\alpha^T \Phi(\cdot, y)\big](x) = \sum_{y \in D(\alpha)} \alpha_y \big[\partial \phi(\cdot, y)\big](x), \quad \partial[\gamma^T H](x) = \sum_{j=1}^{m} \gamma_j \partial h_j(x).$$

COROLLARY 3.2. (Optimality Necessary Condition) *If $x_0 \in S$ is an optimal solution of (P), then there exist $\alpha \in U$, and $\gamma \in R_+^m$, such that*

$$0 \in \partial\big[\alpha^T \Phi(\cdot, y)\big](x_0) + \partial[\gamma^T H](x_0) - \big[D_{\overline{J}=}^{=}(x_0)\big]^*, \tag{3.1}$$

$$\alpha^T \phi(x_0, y) = f(x_0), \tag{3.2}$$

$$\gamma^T H(x_0) = 0. \tag{3.3}$$

PROOF: In the proof of Theorem 3.1, if we set

$$\alpha_y = 0, \; y \in Y \setminus L(x_0), \quad \text{and} \quad \gamma_j = 0, \; j \in J \setminus J^<(x_0),$$

then (3.1)–(3.3) hold. ∎

THEOREM 3.3. (Optimality Sufficient Condition) *Let $x_0 \in S$ and $f(x_0) = \sup_{y \in Y}\big[\phi(x_0, y)\big]$. Assume that there exist $\alpha \in U$ and $\gamma \in R_+^m$ such that the expressions (3.1)-(3.3) hold. Then, x_0 is an optimal solution of (P).*

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PROOF: Suppose to the contrary that x_0 is not an optimal solution of (P). Then there exists a feasible solution $x_1 \in S$ such that

$$\sup_{y\in Y}\big[\phi(x_1,y)\big] < f(x_0).$$

From Lemma 2.6, we get

$$\alpha^T\Phi(x_1,y) \leqslant \sup_{\beta\in U}\Big[\sum_{y\in D(\beta)} \beta_y\phi(x,y)\Big] < f(x_0).$$

Combining with relation (3.2) yields

$$\alpha^T\Phi(x_1,y) < \alpha^T\Phi(x_0,y).$$

Since $x_1 \in S$, $\gamma \in R^m_+$, using (3.3), we have

$$\gamma^T H(x_1) \leqslant 0 = \gamma^T H(x_0).$$

Hence,

$$\alpha^T\Phi(x_1,y) + \gamma^T H(x_1) < \alpha^T\Phi(x_0,y) + \gamma^T H(x_0). \tag{3.4}$$

By (3.1), there exist

$$\xi \in \partial\big[\alpha^T\Phi(\cdot,y)\big](x_0), \sigma \in \partial[\gamma^T H](x_0), \tau \in \big[D^{=}_{J^=}(x_0)\big]^*,$$

such that

$$\xi + \sigma - \tau = 0.$$

By the convexity of functions $\alpha^T\Phi(\cdot,y), \gamma^T H$, we have

$$\alpha^T\Phi(x_1,y) - \alpha^T\Phi(x_0,y) + \gamma^T H(x_1) - \gamma^T H(x_0) \geqslant (x_1 - x_0)^T(\xi+\sigma) = (x_1 - x_0)^T\tau \geqslant 0.$$

This contradicts inequality (3.4). So, x_0 is an optimal solution of (P). ∎

4. Parametric dual model

In this section, we consider the following parametric dual problem:

$$(D1) \quad \max \lambda$$

$$\text{subject to} \quad 0 \in \partial\big[\alpha^T\Phi(\cdot,y)\big](u) + \partial[\gamma^T H](u) - \big[D^{=}_{J^=}(u)\big]^*, \tag{4.1}$$

$$\alpha^T\Phi(u,y) + \gamma^T H(u) \geqslant \lambda, \tag{4.2}$$

$$\alpha \in U, \gamma \in R^m_+, \tag{4.3}$$

$$u \in S^=. \tag{4.4}$$

We denote S_1 the set of all feasible solutions $(u,\alpha,\gamma,\lambda) \in S^= \times U \times R^m_+ \times R_+$ of Problem (D1). Then, a weak duality theorem is established as follows.

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Theorem 4.1. (Weak Duality) *Let $x \in S$ and $(u, \alpha, \gamma, \lambda) \in S_1$. Then*

$$\lambda \leqslant \sup_{y \in Y} \phi(x, y).$$

Proof: By relation (4.1), there exist

$$\xi \in \partial\big[\alpha^T \Phi(\cdot, y)\big](u), \quad \sigma \in \partial[\gamma^T H](u), \quad \tau \in \big[D_{\overline{J}^{=}}(u)\big]^*,$$

such that

$$\xi + \sigma - \tau = 0.$$

By the convexity of functions $\alpha^T \Phi(\cdot, y), \gamma^T H$, we have

$$\alpha^T \Phi(x, y) - \alpha^T \Phi(u, y) + \gamma^T H(x) - \gamma^T H(u) \geqslant (x - u)^T(\xi + \sigma) = (x - u)^T \tau \geqslant 0.$$

It follows that

$$\alpha^T \Phi(x, y) + \gamma^T H(x) \geqslant \alpha^T \Phi(u, y) + \gamma^T H(u). \tag{4.5}$$

From (4.2), (4.5) and $\gamma^T H(x) \leqslant 0$, we get

$$\alpha^T \Phi(x, y) \geqslant \alpha^T \Phi(u, y) + \gamma^T H(u) \geqslant \lambda.$$

Hence, from Lemma 2.6, one has

$$\sup_{y \in Y} \phi(x, y) = \sup_{\beta \in U} \Bigg[\sum_{y \in D(\beta)} \beta_y \phi(x, y)\Bigg] \geqslant \alpha^T \Phi(x, y) \geqslant \lambda. \tag{4.6}$$

∎

Theorem 4.2. (Strong Duality) *Let $\overline{u} \in S$ be an optimal solution of (P). Then there exist $\overline{\alpha} \in U, \overline{\gamma} \in R_+^m, \overline{\lambda} \in R_+$ such that $(\overline{u}, \overline{\alpha}, \overline{\gamma}, \overline{\lambda}) \in S_1$ is an optimal solution of (D1), and the optimal values of (P) and (D1) are equal.*

Proof: By Corollary 3.2, there exist $\overline{\alpha} \in U, \overline{\gamma} \in R_+^m, \overline{\lambda} \in R_+$ such that $(\overline{u}, \overline{\alpha}, \overline{\gamma}, \overline{\lambda})$ is a feasible solution for (D1) and $\overline{\lambda} = f(\overline{u})$, by Theorem 4.1, we derive that $(\overline{u}, \overline{\alpha}, \overline{\gamma}, \overline{\lambda})$ is an optimal solution for (D1), and the optimal values of (P) and (D1) are equal. ∎

Theorem 4.3. (Strict Converse Duality) *Let $\overline{x} \in S$ and $(\overline{u}, \overline{\alpha}, \overline{\gamma}, \overline{\lambda}) \in S_1$ be optimal solution of (P) and (D1), respectively. If $\overline{\alpha}^T \Phi(\cdot, y)$, $\overline{\gamma}^T H(\cdot)$ are convex and one of them is strictly convex at $\overline{u}$, then $\overline{x} = \overline{u}$; that is, $\overline{u}$ is an optimal solution of (P) and $f(\overline{x}) = \overline{\lambda}$.*

Proof: Suppose to the contrary that $\overline{x} \neq \overline{u}$. From Theorem 3.2, we know that there exist $\overline{\alpha}_1 \in U, \overline{\gamma}_1 \in R_+^m, \overline{\lambda}_1 \in R_+$ such that $(\overline{x}, \overline{\alpha}_1, \overline{\gamma}_1, \overline{\lambda}_1) \in S_1$ is an optimal solution of (D1) with the optimal value$\overline{\lambda}_1 = f(\overline{x})$. Similar to the proof of Theorem 4.1, we can obtain the strict inequality $f(\overline{x}) > \overline{\lambda}$ which contradicts that $f(\overline{x}) = \overline{\lambda}_1 = \overline{\lambda}$. The proof is complete. ∎

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5. Mixed-type dual model

In this section, we shall introduce the parameter-free mixed type duality for (P) and establish several mixed duality theorems. The following dual problem is called a Mixed-type dual problem:

$$(MD) \qquad \sup\ \alpha^T\Phi(u,y)+\gamma_{J_1}^T H_{J_1}(u)$$

(5.1) $$\text{subject to}\quad 0\in\partial\big[\alpha^T\Phi(\cdot,y)\big](u)+\partial[\gamma^T H](u)-\big[D_{\overline{J}^=}^{=}(u)\big]^*,$$

(5.2) $$\gamma_j h_j(u)\geqslant 0, j\in J_2,$$

(5.3) $$\alpha\in U,\gamma_j\geqslant 0, j\in J, u\in S^=,$$

where J_1 is a subset of $J=\{1,\dots,m\}$, $J_2=J\backslash J_1$.

We denote the set of all feasible solution $(u,\alpha,\gamma)\in S^=\times U\times R_+^m$ of problem (MD) by S_2. In the following, we shall prove the weak duality, strong duality, and strict converse duality theorems.

Theorem 5.1. (Weak Duality) *Let $x\in S$ and $(u,\alpha,\gamma)\in S_2$. Then*

$$f(x)\geqslant\alpha^T\Phi(u,y)+\gamma_{J_1}^T H_{J_1}(u).$$

Proof: By (5.1), there exist

$$\begin{aligned} &q\in\partial\big[\alpha^T\Phi(\cdot,y)\big](u), && r\in\partial\big[\gamma_{J_1}^T H_{J_1}\big](u),\\ &e\in\partial[\gamma_{J_2}^T H_{J_2}](u), && d\in\big[D_{\overline{J}^=}^{=}(u)\big]^*,\end{aligned}$$

such that

(5.4) $$q+r+e-d=0.$$

Using the characterisation of subgradients, (5.4), the fact that $\gamma^T H(x)\leqslant 0$,$\forall\ x\in S$, and (5.2), we have

$$\begin{aligned}&\alpha^T\Phi(x,y)-\big[\alpha^T\Phi(u,y)+\gamma_{J_1}^T H_{J_1}(u)\big]\\ &\qquad\geqslant\big[\alpha^T\Phi(x,y)-\alpha^T\Phi(u,y)\big]+\big[\gamma_{J_1}^T H_{J_1}(x)-\gamma_{J_1}^T H_{J_1}(u)\big]+\big[\gamma_{J_2}^T H_{J_2}(x)-\gamma_{J_2}^T H_{J_2}(u)\big]\\ &\qquad\geqslant(x-u)^T[q+r+e]=(x-u)^T d\geqslant 0.\end{aligned}$$

So

$$\alpha^T\Phi(x,y)\geqslant\alpha^T\Phi(u,y)+\gamma_{J_1}^T H_{J_1}(u).$$

By Lemma 2.6, we get

$$f(x)=\sup_{\beta\in U}\big[\beta^T\Phi(x,y)\big]\geqslant\alpha^T\Phi(x,y)\geqslant\alpha^T\Phi(u,y)+\gamma_{J_1}^T H_{J_1}(u).$$

Then, the desired result is obtained. ∎

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THEOREM 5.2. (Strong Duality) *Let $\overline{x} \in S$ be an optimal solution of (P). Then there exist $\overline{\alpha} \in U, \overline{\gamma} \in R_+^m$, such that $(\overline{x}, \overline{\alpha}, \overline{\gamma}) \in S_2$ is an optimal solution of (MD), and the optimal values of (P) and (MD) are equal.*

PROOF: By Corollary 3.2, there exist $\overline{\alpha} \in U, \gamma \in R_+^m$, such that

$$0 \in \partial\left[\overline{\alpha}^T \Phi(\cdot, y)\right](\overline{x}) + \partial[\gamma^T H](\overline{x}) - \left[D_{J^=}^=(\overline{x})\right]^*, \tag{5.5}$$

$$\overline{\alpha}^T \phi(\overline{x}, y) = f(\overline{x}), \tag{5.6}$$

$$\gamma^T H(\overline{x}) = 0. \tag{5.7}$$

In fact, from the proof of Theorem 3.1, we know that $\gamma_j h_j(\overline{x}) = 0$, for all $j \in J$. Therefore, $(\overline{x}, \overline{\alpha}, \overline{\gamma})$ is an feasible solution of (MD), and

$$f(\overline{x}) = \overline{\alpha}^T \Phi(\overline{x}, y) = \overline{\alpha}^T \Phi(\overline{x}, y) + \overline{\gamma}_{J_1}^T H_{J_1}(\overline{x}).$$

Hence, $(\overline{x}, \overline{\alpha}, \overline{\gamma})$ is an optimal solution of (MD), and the optimal values of (P) and (MD) are equal. ∎

THEOREM 5.3. (Strict Converse Duality) *Let $\overline{x}$ and $(\overline{u}, \overline{\alpha}, \overline{\gamma})$ be optimal solution of (P) and (MD), respectively. If $\overline{\alpha}^T \Phi(\cdot, y)$, $\overline{\gamma}_{J_1}^T H_{J_1}(\cdot)$, $\overline{\gamma}_{J_2}^T H_{J_2}(\cdot)$, are convex and one of them is strictly convex at $\overline{u}$, then $\overline{x} = \overline{u}$; that is, $\overline{u}$ is an optimal solution of (P) and*

$$f(\overline{x}) = \overline{\alpha}^T \Phi(\overline{u}, y) + \overline{\gamma}_{J_1}^T H_{J_1}(\overline{u}).$$

PROOF: Suppose to the contrary that $\overline{x} \neq \overline{u}$. From Theorem 5.2, we know that there exist $\alpha \in U, \gamma \in R_+^m$, such that $(\overline{x}, \alpha, \gamma)$ is an optimal solution of (MD) with the optimal value

$$f(\overline{x}) = \alpha^T \Phi(\overline{x}, y) + \gamma_{J_1}^T H_{J_1}(\overline{x}).$$

Similar to the proof of Theorem 5.1, we can obtain the strict inequality

$$f(\overline{x}) > \overline{\alpha}^T \Phi(\overline{u}, y) + \overline{\gamma}_{J_1}^T H_{J_1}(\overline{u}).$$

This contradicts that

$$f(\overline{x}) = \alpha^T \Phi(\overline{x}, y) + \gamma_{J_1}^T H_{J_1}(\overline{x}) = \overline{\alpha}^T \Phi(\overline{u}, y) + \overline{\gamma}_{J_1}^{T'} H_{J_1}(\overline{u}).$$

Therefore,

$$\overline{x} = \overline{u}, \text{ and } f(\overline{x}) = \overline{\alpha}^T \Phi(\overline{u}, y) + \overline{\gamma}_{J_1}^T H_{J_1}(\overline{u}).$$

The proof is complete. ∎

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