WEAK CONTAINMENT AND INDUCED REPRESENTATIONS OF GROUPS

J. M. G. FELL

Introduction. Let G be a locally compact group and $G\dagger$ its dual space, that is, the set of all unitary equivalence classes of irreducible unitary representations of G. An important tool for investigating the group algebra of G is the so-called hull-kernel topology of $G\dagger$, which is discussed in **(3)** as a special case of the relation of weak containment. The question arises: Given a group G, how do we determine $G\dagger$ and its topology? For many groups G, Mackey's theory of induced representations permits us to catalogue all the elements of $G\dagger$. One suspects that by suitably supplementing this theory it should be possible to obtain the topology of $G\dagger$ at the same time. It is the purpose of this paper to explore this possibility. Unfortunately, we are not able to complete the programme at present. However, we shall derive a theorem (Theorem 4.3) which gives the topology of $G\dagger$ in many cases (including all the nilpotent groups which Dixmier has treated in **(2)**).

Let A be a C^*-algebra; in particular, A might be the group C^*-algebra of a locally compact group. We recall the definition of weak containment. If $\mathfrak{S}$ is a family of *-representations of A, and T a *-representation of A, T is *weakly contained* in $\mathfrak{S}$ if all positive functionals on A associated with T can be weakly *-approximated by sums of positive functionals associated with representations in $\mathfrak{S}$. When restricted to $A\dagger$, the relationship of weak containment gives the operation of closure in the hull-kernel topology.[1] However, for arbitrary representations it does not define the closure in any topology.[2]

Now it is sometimes convenient to have a genuine topology for the space of all *-representations of A, which will reduce to the hull-kernel topology on $A\dagger$. In §§ 1 and 2 we discuss two such topologies, the quotient topology (introduced in § 3 of **(4)**) and the inner hull-kernel topology. In § 3 we prove a theorem (Theorem 3.1) on direct integrals and weak containment. A corollary of Theorem 3.1 is the highly plausible result (Theorem 3.3) that, if A is separable and of Type I, and if μ is the measure on $A\dagger$ associated with the direct integral decomposition of a *-representation T of A into irreducible parts, then the spectrum of T (in the sense of **(3)**, § 5) is just the closed hull of μ.

In § 4, the principal section of this paper, it is shown (Theorems 4.1 and

Received January 3, 1961.

[1]See **(3**, Lemma 1.6**)**.

[2]Indeed, let S and T be two *-representations of A, neither of which is weakly contained in the other. Then $S \oplus T$ is weakly contained in the set $\{S, T\}$, but not in $\{S\}$ or in $\{T\}$.

4.2) that the process of taking induced representations of a separable locally compact group G is continuous. This result is clearly part of what is needed in order to obtain the topology of $G\dagger$ from the topologies of the dual spaces of subgroups. But we also need information of the converse kind: Suppose that $\mathfrak{S}$ is a family of unitary representations of a closed subgroup K of G, and T is a unitary representation of some other closed subgroup H; and suppose it is known that U^T is weakly contained in $\{U^S | S \in \mathfrak{S}\}$. Then how is T related to $\mathfrak{S}$? Theorem 4.3 answers this question in terms of the "orbits" of representations of K, for the case that K is normal and $H \supset K$.

In § 5 Theorem 4.3 is applied to deduce the topology of the dual spaces of certain groups, most of them nilpotent, whose irreducible representations are easily catalogued by Mackey's theory of induced representations. Most of these groups have been treated by Dixmier in **(2)** by a different method. However, Theorem 4.3 by no means suffices to deduce the topologies of the dual spaces of all groups whose irreducible representations are catalogued by Mackey's theory. For this more general theorems will be needed, of a kind indicated in the concluding remarks.

1. The quotient topology. In this paper representations of groups and algebras will be allowed to have null spaces. Let A be a C^*-algebra. A *representation* of A will be a *-homomorphism T of A into the *-algebra $B(H)$ of all bounded linear operators on some Hilbert space H, called the *space* of T. The *null space* $N(T)$ of T is defined as $\{\xi \in H | T_a\xi = 0 \text{ for all } a \text{ in } A\}$; the *essential space* $H(T)$ is $N(T)^\perp$, or, equivalently, the closed linear span of the ranges of the T_a; $P(T)$ is projection onto $H(T)$. If $N(T) = \{0\}$, that is, $H = H(T)$, T is *proper*; if $N(T) = H$, T is a *zero* representation.

Similarly, if G is a locally compact group, and K is a closed linear subspace of a Hilbert space H, a *representation* T of G with *space* H and *essential space* K $(= H(T))$ means a mapping $x \to S_x \oplus 0$ of G into $B(H)$, where S is a unitary representation of G on K (in the usual sense) and 0 here denotes the zero operator on $K^\perp$. We say T is *unitary* or *zero* if $K = H$ or $K = \{0\}$ respectively.

As in **(4)**, a representation T (of a C^*-algebra or group A) is *irreducible* if $H(T) \neq \{0\}$ and there are no closed invariant subspaces of $H(T)$ except $\{0\}$ and $H(T)$. Let T and T' be two representations of A; T and T' are *equivalent* $(T \sim T')$ if there is a linear isometry of $H(T)$ onto $H(T')$ carrying the restriction of T to $H(T)$ into the restriction of T' to $H(T')$.

If G is a locally compact group, we shall denote by $C^*(G)$ the *group C^*-algebra* of G, that is, the completion of $L_1(G)$ under its minimal regular norm. If H is a fixed Hilbert space, then by a trivial extension of a well-known theorem there is a natural one-to-one correspondence between representations of G whose space is H and representations of $C^*(G)$ whose space is H. Under this correspondence, proper representations of $C^*(G)$ go into unitary representations of G, and vice versa. It also preserves the notions of essential space, irreducibility, and equivalence. We shall usually denote corresponding representations of G and $C^*(G)$ by the same letter.

Let A be a C^*-algebra and H a Hilbert space, and denote by $\mathfrak{T}(A;H)$ the space of all representations of A with space H. We equip $\mathfrak{T}(A;H)$ with the topology defined in § 2 of **(4)**; in this topology a net $\{T^i\}$ converges to T (T^i, $T \in \mathfrak{T}(A;H)$) if and only if, for each a in A and ξ in $H(T)$,

$$||T^i_a\xi - T_a\xi|| \underset{i}{\to} 0, \tag{1}$$

and

$$||P(T^i)\xi - \xi|| \underset{i}{\to} 0. \tag{2}$$

As a matter of fact, condition (2) may be omitted without altering the topology. Indeed, suppose (1) holds (for all a in A and ξ in $H(T)$); and let ξ be in $H(T)$ and $\delta > 0$. By the existence of an approximate identity in A, there is an a in A such that $||a|| \leqslant 1$, $||T_a\xi - \xi|| < \delta$. Then, by (1), $||T_a{}^i\xi - T_a\xi|| < \delta$ for all large enough i; so that $||\xi|| \geqslant ||P(T^i)\xi|| \geqslant ||T_a{}^iP(T^i)\xi|| = ||T_a{}^i\xi|| > ||T_a\xi|| - \delta > ||\xi|| - 2\delta$, and hence $||\xi|| - ||P(T^i)\xi|| < 2\delta$. It follows that $||P(T^i)\xi - \xi||^2 < 4||\xi||\delta$ for all large enough i. The arbitrariness of δ now gives (2).

If G is a locally compact group and H a Hilbert space, $\mathfrak{T}(G;H)$ will mean the same as $\mathfrak{T}(C^*(G);H)$. Convergence in this space is described in terms of the group as follows:

LEMMA 1.1. *If $\{T^i\}$, T are in $\mathfrak{T}(G;H)$, then $T^i \to T$ if and only if, for each ξ in $H(T)$ and each compact subset C of G,*

$$||T^i_x\xi - T_x\xi|| \underset{i}{\to} 0 \tag{3}$$

uniformly on C.

Proof. The proof that (3) implies $T^i \to T$ is straightforward. We prove only the converse direction. Assume then that $T^i \to T$. Fix two elements ξ, η of $H(T)$, and a in $L_1(G)$. For each x in G, we have

$$T_xT_a = T_{a_x},\ T^i_xT^i_a = T^i_{a_x}, \tag{4}$$

where $a_x(y) = a(x^{-1}y)$. The functionals ϕ_i on $L_1(G)$, defined by

$$\phi_i(b) = ((T^i_b - T_b)\xi, \eta),$$

are bounded in norm uniformly in i, and converge weakly * to 0. So, by Gelfand's Lemma, the ϕ_i converge to 0 uniformly on any compact subset of $L_1(G)$. Now let C be a compact subset of G; then $\{a_x|x \in C\}$ is norm-compact in $L_1(G)$. So

$$\phi_i(a_x) \underset{i}{\to} 0 \tag{5}$$

uniformly on C. Now, by (4) and (5), $(T_x{}^iT_a{}^i\xi, \eta) \to (T_xT_a\xi, \eta)$ uniformly for x in C. Therefore, since $T_a{}^i\xi \to T_a\xi$,

$$(6)\quad (T_x^i T_a\xi, \eta) - (T_x T_a \xi, \eta)$$
$$= (T_x^i(T_a\xi - T_a^i\xi), \eta) + (T_x^i T_a^i \xi, \eta) - (T_x T_a\xi, \eta) \to 0$$

uniformly for x in C. Since linear combinations of the $T_a\xi$ are dense in $H(T)$, (6) implies that $(T_x{}^i\zeta, \eta) \to (T_x\zeta, \eta)$ uniformly on C for all ζ, η in $H(T)$. Now let ξ be an element of $H(T)$. We claim that $(T_x{}^i\xi, T_x\xi) \to ||\xi||^2$ uniformly on C. Indeed, given a positive ϵ, we can cover C with finitely many open sets $U_1, \ldots, U_n$, and take a point y_r in each U_r, such that

$$||T_x\xi - T_{y_r}\xi|| < \frac{\epsilon}{3}$$

for each r and each x in U_r. Now, by what we have already proved, there is an index i_0 such that, if $i > i_0$, $x \in C$, and $r = 1, \ldots, n$,

$$|(T_x^i\xi, T_{y_r}\xi) - (T_x\xi, T_{y_r}\xi)| < \frac{\epsilon}{3}.$$

Combining these facts, we see that, if $i > i_0$ and $x \in C$, $|(T_x{}^i\xi, T_x\xi) - (T_x\xi, T_x\xi)| < \epsilon$. This establishes the claim. Hence, multiplying out the inner product $||T_x{}^i\xi - T_x\xi||^2$, and applying what we have just proved, we obtain (3).

Let A be a C^*-algebra (or locally compact group) and H a Hilbert space; let us abbreviate $\mathfrak{T}(A; H)$ to $\mathfrak{T}$. If $T \in \mathfrak{T}$, we denote by $\tilde{T}$ the equivalence class of T under the relation $\sim$; if $\mathfrak{S} \subset \mathfrak{T}$, $\mathfrak{S}^\sim$ denotes $\{\tilde{S} | S \in \mathfrak{S}\}$. In particular, $\mathfrak{T}^\sim = \mathfrak{T}^\sim(A; H)$ will be the family of all such equivalence classes. As in **(4)**, p. 224, $\mathfrak{T}^\sim$ inherits from $\mathfrak{T}$ a quotient topology, in which a subset W is open if and only if $\{T | \tilde{T} \in W\}$ is open in $\mathfrak{T}$. The natural map of $\mathfrak{T}$ onto $\mathfrak{T}^\sim$ is continuous and open **(**see **4**, p. 224**)**.

A certain interest attaches to those subsets $\mathfrak{S}$ of $\mathfrak{T}$ such that the map $T \to \tilde{T}$ is open when restricted to $\mathfrak{S}$ (here we give to $\mathfrak{S}$ and $\mathfrak{S}^\sim$ the topologies relativized from $\mathfrak{T}$ and $\mathfrak{T}^\sim$ respectively); such $\mathfrak{S}$ we will call *representative.* If $\mathfrak{S}$ is representative, we can discover the quotient topology of $\mathfrak{S}^\sim$ (that is, the quotient topology of $\mathfrak{T}^\sim$ relativized to $\mathfrak{S}^\sim$) by looking at $\mathfrak{S}$ only. To obtain representative sets we shall need the following technical lemma. Note that $[X]$ means the closed linear span of the set of vectors X.

LEMMA 1.2. *Let H be a Hilbert space, $x_1, \ldots, x_r$ a sequence of r vectors in H, and $\epsilon > 0$. Then there exists a $\delta > 0$ such that, for each closed linear subspace K of H containing all x_i, and each sequence of r vectors $y_1, \ldots, y_r$ such that $\dim[y_1, \ldots, y_r] \leqslant \dim K$ and $||y_i - x_i|| < \delta$ $(i = 1, \ldots, r)$ there is a linear isometry F of $[y_1, \ldots, y_r]$ into K for which $||F(y_i) - x_i|| < \epsilon$ $(i = 1, \ldots, r)$.*

Proof. Let K be a linear subspace containing all x_i; and, for each i, let $\{y_i{}^n\}$ be a sequence of elements of H such that $\dim[y_1{}^n, \ldots, y_r{}^n] \leqslant \dim K$ (for each n) and $y_i{}^n \to x_i$ (for each i). It is clearly sufficient to set up, for each n, a linear isometry F_n of $[y_1{}^n, \ldots, y_r{}^n]$ into K, such that

$$(7)\qquad F_n(y_i^n) \to x_i\ (i = 1, \ldots, r).$$

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

Assume that $x_1, \ldots, x_s$ are linearly independent, while $x_{s+1}, \ldots, x_r$ are in $[x_1, \ldots, x_s]$. Then, for all large enough n, the $y_1^n, \ldots, y_s^n$ are linearly independent; we shall in fact assume this for all n. Let $L^n = [y_1^n, \ldots, y_s^n]$, and let $e_1, \ldots, e_s$ be an orthonormal basis of $[x_1, \ldots, x_s]$ with

$$e_i = \sum_{j=1}^{s} a_{ij}x_j (i = 1, \ldots, s)$$

and

$$x_j = \sum_{i=1}^{s} b_{ji}e_i (j = 1, \ldots, r).$$

Setting

$$e_i^n = \sum_{j=1}^{s} a_{ij}y_j^n,$$

we have clearly

$$\lim_n (e_i^n, e_j^n) = \delta_{ij}. \tag{8}$$

For each n, let $d_1^n, \ldots, d_s^n$ be the orthonormal basis of L^n obtained by the Gram-Schmidt process from $e_1^n, \ldots, e_s^n$. Then (8) gives

$$\lim_n d_i^n = e_i. \tag{9}$$

Now let F_n be any linear isometry of $[y_1^n, \ldots, y_r^n]$ into K which carries d_i^n into e_i $(i = 1, \ldots, s)$; and set

$$z_j^n = \sum_{i=1}^{s} b_{ji}d_i^n (j = 1, \ldots, r),$$

so that

$$F_n(z_j^n) = x_j. \tag{10}$$

By (9), $z_j^n \to x_j$; and we know that $y_j^n \to x_j$ $(j = 1, \ldots, r)$. Combining these two, and remembering that z_j^n and y_j^n are in $[y_1^n, \ldots, y_r^n]$, we get $||F_n(z_j^n) - F_n(y_j^n)|| \to 0$, which by (10) implies (7).

LEMMA 1.3. *Let A be a C^*-algebra, H a Hilbert space, and K a closed linear subspace of H. Then $\mathfrak{S} = \{T \in \mathfrak{T}(A; H) | H(T) \subset K\}$ is a representative set.*

Proof. It will be sufficient to show that if $T \in \mathfrak{S}$ and U is a neighbourhood of T in $\mathfrak{T}(A; H)$, there is a neighbourhood V of T (in $\mathfrak{T}(A; H)$) such that

$$\tilde{V} \cap \mathfrak{S}^{\sim} \subset (U \cap \mathfrak{S})^{\sim}. \tag{11}$$

Without loss of generality we may suppose

$$U = \{S \in \mathfrak{T}(A; H) | \, ||S_{a_i}\xi_i - T_{a_i}\xi_i|| < \epsilon, \qquad i = 1, \ldots, r\},$$

where $\xi_1, \ldots, \xi_r \in H(T)$, $a_1, \ldots, a_r \in A$, $||a_i|| \leqslant 1$, and $\epsilon > 0$. Let δ be constructed from H, the vectors ξ_i and $T_{a_i}\xi_i$ $(i = 1, \ldots, r)$, and $\epsilon/2$, in accordance with Lemma 1.2; and let

$$(12) \qquad V = \{S \in \mathfrak{T}(A;H) \mid \|S_{a_i}\xi_i - T_{a_i}\xi_i\| < \delta, \quad \|P(S)\xi_i - \xi_i\| < \delta, \quad i = 1, \ldots, r\}.$$

Now take any element S' of V such that $\tilde{S}' \in \mathfrak{S}^{\sim}$. Then $H(S')$, which contains all $S_{a_i}{}'\xi_i$ and $P(S')\xi_i$, has dimension no greater than dim K. Hence, by Lemma 1.2 there is a linear isometry F of $H(S')$ into K such that for each i

$$(13) \qquad \|FS'_{a_i}\xi_i - T_{a_i}\xi_i\| < \frac{\epsilon}{2}, \; \|FP(S')\xi_i - \xi_i\| < \frac{\epsilon}{2}.$$

Defining S to be the representation (in $\mathfrak{S}$) whose essential space is $F(H(S'))$ and for which $S_a F\xi = FS_a{}'\xi$ ($a \in A$, $\xi \in H(S')$), we obtain by (12) and (13)

$$\|S_{a_i}\xi_i - T_{a_i}\xi_i\| \leqslant \|S_{a_i}(\xi_i - FP(S')\xi_i)\| + \|FS'_{a_i}P(S')\xi_i - T_{a_i}\xi_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $S \in U \cap \mathfrak{S}$, so $\tilde{S}' = \tilde{S} \in (U \cap \mathfrak{S})^{\sim}$. Since $\tilde{S}'$ was an arbitrary element of $\tilde{V} \cap \mathfrak{S}^{\sim}$, (11) is proved.

There is a natural correspondence between $\mathfrak{T}^{\sim}(A;H)$ and the family of all unitary equivalence classes of proper representations of A of dimension no greater than dim H. Hence, if K is a Hilbert space with dim $K \leqslant$ dim H, $\mathfrak{T}^{\sim}(A;K)$ can be considered as a subset of $\mathfrak{T}^{\sim}(A;H)$. Lemma 1.3 then asserts that the quotient topology of $\mathfrak{T}^{\sim}(A;K)$ is the same as that of $\mathfrak{T}^{\sim}(A;H)$ relativized to $\mathfrak{T}^{\sim}(A;K)$.

Let us fix a huge infinite cardinal γ, and agree once for all to consider only representations whose dimension does not exceed γ.[3] Denote by $\mathfrak{T}^{\sim}(A)$ the family of all unitary equivalence classes of proper representations of A of dimension no greater than γ. By the quotient topology of $\mathfrak{T}^{\sim}(A)$ we mean its quotient topology when it is identified with $\mathfrak{T}^{\sim}(A;H_\gamma)$, H_γ being a Hilbert space of dimension γ. If $\mathfrak{S} \subset \mathfrak{T}^{\sim}(A)$, then by Lemma 1.3 the quotient topology relativized to $\mathfrak{S}$ can be calculated in $\mathfrak{T}^{\sim}(A;H)$, where H is any Hilbert space big enough to contain everything in $\mathfrak{S}$.

It follows from Theorem 3.1 of **(4)** that the quotient topology relativized to $A\dagger$ is just the hull-kernel topology.

If we are concerned only with countably-dimensional representations, then a useful representative set can be obtained in which the essential space of each representation is uniquely specified (Lemma 1.5).

LEMMA 1.4. *For any (finite) integer n and Hilbert space H, $\{T \in \mathfrak{T}(A;H) \mid$ dim $H(T) \leqslant n\}$ is closed in $\mathfrak{T}(A;H)$.*

The proof is simple and we omit it.

LEMMA 1.5. *Let A be a C^*-algebra, $H = H_\infty$ a separable infinite-dimensional*

[3]We make this restriction in order to avoid paradoxically large sets, such as the set of *all* representations of A. In future it will be assumed without mention. In particular, γ must be greater than the dimensions of all elements of $A\dagger$ and greater than the cardinality of $A\dagger$ itself. The index sets in direct sums of representations must be of cardinality less than γ.

Hilbert space, and $H_0, H_1, H_2, H_3, \ldots,$ *an increasing sequence of linear subspaces of* H *such that* $\dim H_n = n$ *(for each* n*) and* $\bigcup_{n<\infty} H_n$ *is dense in* H*. Then* $\mathfrak{S} = \{T \in \mathfrak{T}(A;H) | H(T) = H_n$ *for some* $n = 0, 1, 2, \ldots, \infty\}$ *is a representative set.*

Proof. Clearly $\mathfrak{S}^{\sim} = \mathfrak{T}^{\sim}(A;H)$. Let T be in $\mathfrak{S}$, and U be an arbitrary open neighbourhood of T in $\mathfrak{T}(A;H)$. We must find a neighbourhood V of T (in $\mathfrak{T}(A;H)$) such that

(14) $$\tilde{V} \subset (U \cap \mathfrak{S})^{\sim}.$$

Suppose first that $H(T) = H_n$, n finite. By Lemma 1.4, we may as well assume that $\dim H(S) \geqslant n$ for all S in U. Then the essential spaces of elements S of $\mathfrak{S}$ for which $\tilde{S} \in \tilde{U}$ contain $H(T)$; and (14) may be obtained by an argument (based on Lemma 1.2) similar to the proof of Lemma 1.3.

Now suppose that $H(T) = H$, and

$$U = \{S \in \mathfrak{T}(A;H) | \, ||S_{a_i}\xi_i - T_{a_i}\xi_i|| < \epsilon \quad (i = 1, \ldots, r)\},$$

where $\xi_1, \ldots, \xi_r \in H$, $a_1, \ldots, a_r \in A$, $||a_i|| \leqslant 1$, and $\epsilon > 0$. Let n be such a large integer that $||\eta_i - \xi_i|| < \epsilon/3$, where η_i is the projection of ξ_i onto H_n, and set

$$W = \left\{S \in \mathfrak{T}(A;H) | \, ||S_{a_i}\eta_i - T_{a_i}\eta_i|| < \frac{\epsilon}{3} \,(i = 1, \ldots, r), \right.$$
$$\left. \text{and } \dim H(S) \geqslant n \right\}.$$

By Lemma 1.4, W is easily seen to be a neighbourhood of T contained in U. By an argument based on Lemma 1.2, similar to that used for the case $H(T) = H_n$, we obtain a neighbourhood V of T (in $\mathfrak{T}(A;H)$) such that

$$\tilde{V} \subset (W \cap \mathfrak{S})^{\sim} \subset (U \cap \mathfrak{S})^{\sim}.$$

The proof is now complete.

The following Lemmas 1.6–1.11 are easily verified.

LEMMA 1.6. *If* T *lies in the quotient closure of* $\mathfrak{S}$*, so does any subrepresentation*[4] *of* T*. If* T *lies in the quotient closure of* $\mathfrak{S}$*, and each* S *in* $\mathfrak{S}$ *is a subrepresentation of some* S' *in* $\mathfrak{S}'$*, then* T *lies in the quotient closure of* $\mathfrak{S}'$*.*

LEMMA 1.7. *For each* i *in an index set* I*, let* $T^i \in \mathfrak{T}^{\sim}(A)$*. Then*

$$\sum_{i \in F} \oplus T^i \underset{F}{\rightarrow} \sum_{i \in I} \oplus T^i$$

in the quotient topology, where F *runs over the directed set of all finite subsets of* I*.*

[4] T' is a *subrepresentation* of T if T' is unitarily equivalent to the restriction of T to some invariant subspace of $H(T)$.

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

LEMMA 1.8. *If, for each i in an index set I, $T^i \in \mathfrak{T}^\sim(A)$, and*

$$S_\lambda^i \underset{\lambda}{\to} T^i$$

in the quotient topology (the directed set $\{\lambda\}$ being independent of i), then

$$\sum_{i \in I} \oplus S_\lambda^i \underset{\lambda}{\to} \sum_{i \in I} \oplus T^i$$

in the quotient topology.

LEMMA 1.9. *Suppose A is a C^*-algebra with no unit element, and A_1 is the C^*-algebra obtained by adjoining a unit 1 to A. For each T in $\mathfrak{T}^\sim(A)$, let T' be the extension of T to an element of $\mathfrak{T}^\sim(A_1)$ obtained by setting T_1' equal to the identity operator on $H(T)$. Then $T \to T'$ is one-to-one and bicontinuous on $\mathfrak{T}^\sim(A)$ into $\mathfrak{T}^\sim(A_1)$.*

If B is any closed *-subalgebra of the C^*-algebra A, and $T \in \mathfrak{T}^\sim(A)$, $T|B$ as usual denotes the restriction of T to B. Let T_B be the result of restricting the operators of $T|B$ to $H(T|B)$. Then $T_B \in \mathfrak{T}^\sim(B)$.

LEMMA 1.10. *The restriction mapping $T \to T_B$ is continuous with the quotient topologies of $\mathfrak{T}^\sim(A)$ and $\mathfrak{T}^\sim(B)$.*

If G is a locally compact group, $\mathfrak{T}^\sim(G)$ will mean $\mathfrak{T}^\sim(C^*(G))$; or it may be identified with the class of all unitary equivalence classes of unitary representations of G (of dimension no greater than γ).

LEMMA 1.11. *If K is a closed subgroup of G, the restriction mapping $T \to T_K = T|K$ is continuous with the quotient topologies of $\mathfrak{T}^\sim(G)$ and $\mathfrak{T}^\sim(K)$.*

Next we investigate the relationship between the quotient topology and weak containment. We recall that a positive functional p on a C^*-algebra A is *associated* with a family $\mathfrak{S}$ of representations of A if, for some T in $\mathfrak{S}$ and some ξ in $H(T)$, $p(x) = (T_x\xi, \xi)$ (for all x in A).

Now the proof of Theorem 3.1 of **(4)** actually establishes the following stronger result:

LEMMA 1.12. *Let A be a C^*-algebra with unit, T a cyclic element of $\mathfrak{T}^\sim(A)$ with cyclic vector ξ, and $\mathfrak{S}$ a subfamily of $\mathfrak{T}^\sim(A)$. Then T belongs to the quotient closure of $\mathfrak{S}$ if and only if the positive functional $x \to (T_x\xi, \xi)$ is a weak * limit of positive functionals associated with $\mathfrak{S}$.*

THEOREM 1.1. *Suppose A is a C^*-algebra, $T \in \mathfrak{T}^\sim(A)$, and $\mathfrak{S} \subset \mathfrak{T}^\sim(A)$. Then T is weakly contained in $\mathfrak{S}$ if and only if T belongs to the quotient closure of the set $\mathfrak{S}_f$ of all finite direct sums*

$$\sum_{i=1}^{n} \oplus S^i$$

(where $n = 1, 2, \ldots,$ and each $S^i \in \mathfrak{S}$).

Proof. By Lemma 1.7 and **(3)**, Lemma 1.3, we may as well assume that A has a unit. Write T as a direct sum $\sum_\alpha \oplus T^\alpha$ of cyclic representations T^α; and observe that a sum of positive functionals associated with $\mathfrak{S}$ is associated with $\mathfrak{S}_f$.

Let T be weakly contained in $\mathfrak{S}$. Then so is each T^α; and the positive functional associated with a cyclic vector of T^α is a weak * limit of positive functionals associated with $\mathfrak{S}_f$. So, by Lemma 1.12, each T^α is in the quotient closure of $\mathfrak{S}_f$. Applying Lemmas 1.7 and 1.8, we deduce that T is in the quotient closure of $\mathfrak{S}_f$.

The converse implication—that if T belongs to the quotient closure of $\mathfrak{S}_f$ it is weakly contained in $\mathfrak{S}$—is almost trivial.

2. The inner hull-kernel topology.

Let X be an arbitrary topological space, and $E(X)$ the family of all closed subsets of X. For each finite set $\{A_1, A_2, \ldots, A_n\}$ of non-void open subsets of X, let $U(A_1, \ldots, A_n) = \{Y \in E(X) | Y \cap A_i$ is non-void for every $i\}$. A subset of $E(X)$ will be called *open* if it is a union of certain of the $U(A_1, \ldots, A_n)$. This clearly defines a topology for $E(X)$, called the *inner topology derived from* X.

Now fix a C^*-algebra A. As before, $\mathfrak{T}^\sim = \mathfrak{T}^\sim(A)$ is the set of all unitary equivalence classes of proper representations of A. If $T \in \mathfrak{T}^\sim$, the *spectrum* $\mathrm{Sp}(T)$ of T will be the family of all S in $A^\dagger$ which are weakly contained in T (see **3**, § 5). Clearly $\mathrm{Sp}(T)$ is closed in $A^\dagger$.

DEFINITION. *If* $T \in \mathfrak{T}^\sim$ *and* $\mathfrak{S} \subset \mathfrak{T}^\sim$, *we define* T *to be in the inner hull-kernel closure of* $\mathfrak{S}$ *if* $\mathrm{Sp}(T)$ *belongs to the closure of* $\{\mathrm{Sp}(S) | S \in \mathfrak{S}\}$ *with respect to the inner topology of* $E(A^\dagger)$ (*derived from the hull-kernel topology of* $A^\dagger$). *This closure operation defines the* inner hull-kernel *topology of* $\mathfrak{T}^\sim(A)$.

From Theorem 1.6 of **(3)**, we see that the inner hull-kernel topology does not distinguish between elements of $\mathfrak{T}^\sim$ which are weakly equivalent,[5] since such elements have the same spectrum. In particular, it does not distinguish between a representation T and a multiple $\alpha \cdot T$ of T. Clearly, the inner hull-kernel topology, like the quotient topology, becomes the hull-kernel topology when relativized to $A^\dagger$.

LEMMA 2.1. *Let* A *have no unit; and adjoin a unit* 1 *to get the* C^*-*algebra* A_1. *For each* T *in* $\mathfrak{T}^\sim$, *let* T' *be the extension to* A_1 *obtained by setting* T_1' *equal to the identity operator on* $H(T)$. *Then the map* $T \to T'$ *is one-to-one and bicontinuous in the inner hull-kernel topologies of* $\mathfrak{T}^\sim(A)$ *and* $\mathfrak{T}^\sim(A_1)$.

This follows by an easy argument based on Lemmas 1.3 and 1.8 of **(3)**.

We shall next obtain an equivalent definition of the inner hull-kernel topology in terms of positive functionals. For this we remark the following minor generalization of Theorem 1.4 of **(3)**:

[5]Two families $\mathfrak{S}$ and $\mathfrak{S}'$ of proper representations of A are *weakly equivalent* if each is weakly contained in the other. See **(3**, § 5**)**.

LEMMA 2.2.[6] *Suppose* $\mathfrak{S} \subset \mathfrak{T}^{\sim}(A)$, $T \in A^{\dagger}$. *Then the following three conditions are equivalent:*

(i) *T is weakly contained in $\mathfrak{S}$;*

(ii) *some non-zero positive functional associated with T is a weak * limit of finite linear combinations of positive functionals associated with $\mathfrak{S}$;*

(iii) *every non-zero positive functional ϕ associated with T is the weak * limit of positive functionals ψ associated with $\mathfrak{S}$ such that $||\psi|| \leqslant ||\phi||$.*

Lemma 2.2 differs from Theorem 1.4 of **(3)** in that the elements of $\mathfrak{S}$ are not assumed irreducible. But their irreducibility was nowhere used in the proof of **(3)**, Theorem 1.4.

Now let T be an element of $\mathfrak{T}^{\sim}(A)$, ϵ a positive number, $\phi_1, \ldots, \phi_n$ positive functionals on A associated with T, and $a_1, \ldots, a_m$ elements of A. We define

$$(1) \qquad U(T) = U(\phi_1, \ldots, \phi_n; \; a_1, \ldots, a_m; \; \epsilon; \; T)$$

to be the set of all those S in $\mathfrak{T}^{\sim}$ such that there exist $\phi_1', \ldots, \phi_n'$, each of which is a sum of positive functionals associated with S, for which

$$(2) \qquad |\phi_i(a_j) - \phi_i'(a_j)| < \epsilon \; (i = 1, \ldots, n; j = 1, \ldots, m),$$

$$(3) \qquad |\,||\phi_i|| - ||\phi_i'||\,| < \epsilon \; (i = 1, \ldots, n).$$

THEOREM 2.1. *For each T in $\mathfrak{T}^{\sim}(A)$, the $U(T)$ just defined form a basis for the neighbourhoods of T in the inner hull-kernel topology.*

Proof. (A) It is sufficient to assume that A has a unit. Indeed, assume it does not, and adjoin 1 to get A_1; we shall identify T with T' (see Lemma 2.1). The inner hull-kernel topology of $\mathfrak{T}^{\sim}(A)$ is by Lemma 2.1 just that of $\mathfrak{T}^{\sim}(A_1)$ relativized to $\mathfrak{T}^{\sim}(A)$. On the other hand, by well-known facts about positive functionals on C^*-algebras,[7] $U(T)$ can be described as the set of all S in $\mathfrak{T}^{\sim}(A)$ such that there exist positive functionals $\phi_1', \ldots, \phi_n'$ on A_1, each a sum of positive functionals associated with S', for which $|\phi_i(a_j) - \phi_i'(a_j)| < \epsilon$, $|\phi_i(1) - \phi_i'(1)| < \epsilon$ $(i = 1, \ldots, n; \; j = 1, \ldots, m)$. That is, $U(T)$ is the intersection of $\mathfrak{T}^{\sim}(A)$ with the set $U_1(T)$ defined using A_1 instead of A. Thus we may assume a unit element, and omit (3) from the definition of $U(T)$.

(B) Assume that $T \in \mathfrak{T}^{\sim}(A)$, $\mathfrak{S} \subset \mathfrak{T}^{\sim}(A)$, and that every $U(T)$ intersects $\mathfrak{S}$. We shall show that T lies in the inner hull-kernel closure of $\mathfrak{S}$, and hence that every inner hull-kernel neighbourhood of T contains some $U(T)$.

---

[6]At the time of writing **(3)**, the author was unaware that Lemma 1.7 of that paper, the main tool in the proof of Theorem 1.4, is merely a special case of Proposition 4, § 4, chapter II of Bourbaki, *Espaces vectoriels topologiques.*

[7]We are using the fact that, if $\phi(x) = (T_x\xi, \xi)$ for $x \in A$, where T is a proper representation of A and $\xi \in H(T)$, then $||\phi|| = ||\xi||^2$; also that, if $\phi_1, \ldots, \phi_n$ are positive functionals on A, then $||\sum_{i=1}^{n}\phi_i|| = \sum_{i=1}^{n}||\phi_i||$. Both of these facts follow immediately from the existence of an approximate identity on A.

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

Since the $U(T)$ form a directed set under inverse inclusion, there is a net $\{S^\lambda\}$ of elements of $\mathfrak{S}$ such that, for each $U(T)$, $S^\lambda \in U(T)$ for all large enough λ. Now let $B_1, \ldots, B_n$ be open subsets of $A\dagger$ each of which intersects $\mathrm{Sp}(T)$; and choose representations T_i in $\mathrm{Sp}(T) \cap B_i$.

Consider a (non-zero) positive functional ϕ associated with T_1. Since T_1 is weakly contained in T, Lemma 2.2 assures us that ϕ may be weakly * approximated by positive functionals ϕ' associated with T. By the definition of $U(T)$, for each λ_0 ϕ' may be weakly * approximated by a sum $\phi'' = \psi_1 + \ldots + \psi_r$, where each ψ_i is a positive functional associated with S^λ for a suitable $\lambda > \lambda_0$ (the same λ for all i). Further, since S^λ is weakly contained in $\mathrm{Sp}(S^\lambda)$ (see **3**, § 5), each ψ_i can be weakly * approximated by a sum of positive functionals associated with $\mathrm{Sp}(S^\lambda)$. Combining these facts, we see that, for each λ_0, ϕ is weakly * approximated by sums of positive functionals associated with

$$\bigcup_{\lambda > \lambda_0} \mathrm{Sp}(S^\lambda).$$

Hence T_1 belongs to the hull-kernel closure of

$$\bigcup_{\lambda > \lambda_0} \mathrm{Sp}(S^\lambda).$$

It follows that there is a subnet $\{S'^\mu\}$ of $\{S^\lambda\}$ such that each neighbourhood of T_1 intersects $\mathrm{Sp}(S'^\mu)$ for all large enough μ. We now repeat the considerations of the preceding paragraph, replacing $\{S^\lambda\}$ by $\{S'^\mu\}$ and T_1 by T_2. After n repetitions of this argument we arrive at a subnet $\{S''^\nu\}$ of $\{S^\lambda\}$ such that, for each $i = 1, \ldots, n$, every neighbourhood of T_i intersects $\mathrm{Sp}(S''^\nu)$ for all large enough ν. In particular, there is an S''^ν (in $\mathfrak{S}$) such that $\mathrm{Sp}(S''^\nu)$ intersects all the B_i.

But $\{S \in \mathfrak{T}^\sim(A) | \mathrm{Sp}(S)$ intersects all B_i $(i = 1, \ldots, n)\}$ is a typical inner hull-kernel neighbourhood of T. Thus T belongs to the inner hull-kernel closure of $\mathfrak{S}$.

(C) Suppose $T \in \mathfrak{T}^\sim(A)$, $\mathfrak{S} \subset \mathfrak{T}^\sim(A)$, and T belongs to the inner hull-kernel closure of $\mathfrak{S}$. We shall complete the proof by showing that every $U(T)$ intersects $\mathfrak{S}$.

Let $U(T)$ be as in (1). Since T is weakly equivalent to its spectrum (**3** Theorem 1.6), each ϕ_i is weakly * approximated by a sum ψ_i of positive functionals ψ_i^k associated with $\mathrm{Sp}(T)$:

$$\psi_i = \sum_{k=1}^{r_i} \psi_i^k, \tag{4}$$

$$|\psi_i(a_j) - \phi_i(a_j)| < \frac{\epsilon}{2} \; (i = 1, \ldots, n; j = 1, \ldots, m). \tag{5}$$

Let ψ_i^k be associated with the representation S_i^k in $\mathrm{Sp}(T)$; and define W_i^k to be the set of all R in $A\dagger$ such that there is a positive functional χ associated with R satisfying:

$$|\chi(a_j) - \psi_i^k(a_j)| < \delta (j = 1, \ldots, m),$$

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

where

$$4r_i\delta < \epsilon \text{ for all } i. \tag{6}$$

Then W_i^k is a hull-kernel neighbourhood of S_i^k; indeed, if Z is a subset of $A\dagger$ with S_i^k in its closure, Lemma 2.2 assures us that Z intersects W_i^k.

Now $V = \{S \in \mathfrak{T}^\sim(A) | \text{Sp}(S) \text{ intersects } W_i^k \text{ for all } i = 1, \ldots, n; k = 1, \ldots, r_i\}$ is an inner hull-kernel neighbourhood of T; and so contains an element S of $\mathfrak{S}$. Choose $R_i^k \in \text{Sp}(S) \cap W_i^k$, and let χ_i^k be such a positive functional associated with R_i^k that

$$|\chi_i^k(a_j) - \psi_i^k(a_j)| < \delta (j = 1, \ldots, m). \tag{7}$$

Now R_i^k is irreducible and weakly contained in S; and so there is a positive functional π_i^k associated with S such that

$$|\pi_i^k(a_j) - \chi_i^k(a_j)| < \delta (j = 1, \ldots, m). \tag{8}$$

Put

$$\pi_i = \sum_{k=1}^{r_i} \pi_i^k.$$

Combining (5), (6), (7), and (8), we get $|\pi_i(a_j) - \phi_i(a_j)| < \epsilon$. So $S \in U(T) \cap \mathfrak{S}$.

Remark. If in Theorem 2.1 T is irreducible, then a basis for the inner hull-kernel neighbourhoods of T is formed by the sets $V(T)$ defined in the same way as the $U(T)$, except that each ϕ_i' must be a positive functional associated with S (rather than merely a sum of such).

Indeed, if T is irreducible, by Lemma 2.2 we do not need sums of more than one term in (4). Otherwise the proof of Theorem 2.1 goes through as before.

Now let G be a locally compact group with unit e. It will be useful to express the inner hull-kernel topology of $\mathfrak{T}^\sim(G)$ in terms of functions on G, after the pattern of Theorem 1.3 of **(3)**.

A function f of positive type on G is *associated with* a family $\mathfrak{S}$ of unitary representations of G if there is an S in $\mathfrak{S}$ and a vector ξ in $H(S)$ such that $f(x) = (S_x\xi, \xi)$ $(x \in G)$.

Let T be a unitary representation of G, ϵ a positive number, $\phi_1, \ldots, \phi_n$ functions of positive type associated with T, and C a compact subset of G. We define $W(T) = W(\phi_1, \ldots, \phi_n; C; \epsilon; T)$ to be the set of all unitary representations S of G such that there exist $\phi_1', \ldots, \phi_n'$, each of which is a sum of functions of positive type associated with S, for which $|\phi_i(x) - \phi_i'(x)| < \epsilon$ for all $i = 1, \ldots, n$ and all x in C.

THEOREM 2.2. *For each T in $\mathfrak{T}^\sim(G)$, the set of all such $W(T)$ forms a basis for the inner hull-kernel neighbourhoods of T.*

Proof. We shall denote by the same letter a function of positive type on

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

G and the corresponding positive functional on $C^*(G)$; and similarly for representations.

(A) Let $U(T) = U(\phi_1, \ldots, \phi_n; a_1, \ldots, a_m; \epsilon; T)$ be one of the sets of Theorem 2.1. We shall show that some $W(T)$ is contained in $U(T)$. Without loss of generality assume $||\phi_i|| \leqslant 1$, $a_j \in L_1(G)$, $||a_j||_{L_1(G)} \leqslant 1$. (By **(3**, Lemma 1.4**)**, $||\phi_i||$ is the same whether ϕ_i is considered as belonging to $(L_1(G))^*$ or to $(C^*(G))^*$.)

Choose a compact subset C of G containing e such that

$$\int_{G-C} |a_i(x)|dx < \frac{\epsilon}{3}$$

for all i (dx being left Haar measure); and let $W(T) = W(\phi_1, \ldots, \phi_n; C; \epsilon/3; T)$. If $S \in W(T)$, there are sums $\phi_1', \ldots, \phi_n'$ of functions of positive type associated with S such that $|\phi_i'(x) - \phi_i(x)| < \epsilon/3$ for x in C. In particular, since $e \in C$, this gives

$$(9) \qquad | \, ||\phi_i'|| - ||\phi_i|| \, | < \frac{\epsilon}{3}.$$

Further, by a simple calculation,

$$(10) \qquad |\phi_i'(a_j) - \phi_i(a_j)| < \epsilon \; (i = 1, \ldots, n; j = 1, \ldots, m).$$

By (9) and (10), $S \in U(T)$. So $W(T) \subset U(T)$.

(B) Let $W(T) = W(\phi_1, \ldots, \phi_n; C; \epsilon; T)$. We shall complete the proof by finding a $U(T)$ contained in $W(T)$.

Assume that $W(T)$ contains no $U(T)$. Then by Theorem 2.1 we can find (i) a net of unitary representations $\{S^\lambda\}$, all outside $W(T)$, and (ii) for each $i = 1, \ldots, n$ and each λ, a sum ϕ_i^λ of positive functionals on $C^*(G)$ associated with S^λ, such that for each i

$$\phi_i^\lambda \underset{\lambda}{\rightarrow} \phi_i$$

weakly * on $C^*(G)$ and

$$||\phi_i^\lambda|| \underset{\lambda}{\rightarrow} ||\phi_i||.$$

In particular the $||\phi_i^\lambda||$ may be assumed bounded in λ; so by Gelfand's Lemma,

$$(11) \qquad \phi_i^\lambda(a) - \phi_i(a) \underset{\lambda}{\rightarrow} 0$$

uniformly on any norm-compact subset of $L_1(G)$. Now let $\phi_i(x) = (T_x\xi_i, \xi_i)$ ($x \in G$), where $\xi_i \in H(T)$. Using the approximate identity in $L_1(G)$, we may find an element b in $L_1(G)$ such that $T_b\xi_i$ is close to ξ_i, in fact, such that for all i and all x in G

$$(12) \qquad |(T_xT_b\xi_i, T_b\xi_i) - \phi_i(x)| < \frac{\epsilon}{2}.$$

Now as x ranges over C, b^*b_x ranges over a compact subset of $L_1(G)$ (here $b_x(y) = b(x^{-1}y)$). So, by (11), for each i

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

$$(13)\qquad \phi_i^\lambda(b^*b_x) - \phi_i(b^*b_x) \underset{\lambda}{\rightarrow} 0$$

uniformly for x in C. Since $\phi_i(b^*b_x) = (T_xT_b\xi_i, T_b\xi_i)$, (12) and (13) combine to show that there is a λ such that, for each i,

$$(14)\qquad |\phi_i^\lambda(b^*b_x) - \phi_i(x)| < \epsilon \qquad \text{for all } x \text{ in } C.$$

But the function $x \rightarrow \phi_i^\lambda(b^*b_x)$, like ϕ_i^λ, is a sum of positive functionals associated with S^λ. Therefore (14) asserts that $S^\lambda \in W(T)$. This contradicts the definition of $\{S^\lambda\}$.

Remark. Just as in the case of Theorem 2.1, if T is irreducible, a basis for the inner hull-kernel neighbourhoods of T is formed by the family of all $W'(T)$, where $W'(T)$ is defined just like $W(T)$ except that each ϕ_i' must be a function of positive type associated with S (rather than merely a sum of such).

What is the relation between the quotient and inner hull-kernel topologies of representations of a C^*-algebra A?

LEMMA 2.3. *The quotient topology contains the inner hull-kernel topology* (*that is, the inner hull-kernel open sets are quotient open*).

This follows almost immediately from the definitions, and the openness of the mapping $T \rightarrow \tilde{T}$ on $\mathfrak{T}(A; H)$ (see § 1).

The quotient and inner hull-kernel topologies are different. Indeed, the latter cannot tell the difference between T and a multiple of T, while the former in general can. However, when relativized to $A\dagger$, both topologies coincide with the hull-kernel topology. At the other extreme, we have the following lemma:

LEMMA 2.4. *The quotient and inner hull-kernel topologies coincide when relativized to the set $\mathfrak{J}$ of those T in $\mathfrak{T}^\sim(A)$ for which $T \cong \aleph_0 \cdot T$.*

Proof. By Lemmas 1.9 and 2.1 we may assume that A has a unit 1. By Lemma 2.3 it is enough to show that, if $T \in \mathfrak{T}^\sim(A)$, $\mathfrak{S} \subset \mathfrak{T}^\sim(A)$, and T lies in the inner hull-kernel closure of $\mathfrak{S}$, then $\aleph_0 \cdot T$ lies in the quotient closure of $\aleph_0 \cdot \mathfrak{S} = \{\aleph_0 \cdot S | S \in \mathfrak{S}\}$.

Let $\aleph_0 . T$ be written as a direct sum of cyclic subrepresentations T^i of T:

$$\aleph_0 \cdot T = \sum_{i \in I} \oplus T^i.$$

By Lemma 1.7, it is sufficient to show that

$$\sum_{i \in F} \oplus T^i$$

belongs to the quotient closure of $\aleph_0 \cdot \mathfrak{S}$ for each finite subset F of I.

Let ϕ_i be the positive functional associated with a cyclic vector for T^i. Since T lies in the inner hull-kernel closure of $\mathfrak{S}$, by Theorem 2.1 there is a net $\{S^\lambda\}$ of elements of $\mathfrak{S}$ and, for each i in F and each λ, a positive functional ψ_i^λ associated with $\aleph_0 . S^\lambda$, such that

$$\psi_i^\lambda \underset{\lambda}{\rightarrow} \phi_i$$

weakly * for each i in F. Then by Lemma 1.12

$$\aleph_0 \cdot S^\lambda \underset{\lambda}{\rightarrow} T^i$$

(quotientwise) for each i in F. It follows by Lemma 1.8 that

$$\aleph_0 \cdot S^\lambda \underset{\lambda}{\rightarrow} \sum_{i \in F} \oplus T^i;$$

so that

$$\sum_{i \in F} \oplus T^i$$

belongs to the quotient closure of $\aleph_0 \cdot \mathfrak{S}$.

Recalling that the inner hull-kernel topology cannot distinguish T from $\aleph_0 \cdot T$, and combining Lemmas 1.10, 1.11, and 2.4, and Theorem 1.1, we obtain:

THEOREM 2.3. *Lemmas* 1.10 *and* 1.11, *and also Theorem* 1.1, *remain valid when the quotient topology is replaced by the inner hull-kernel topology.*

LEMMA 2.5. *If A is a separable C^*-algebra* (*or a separable*[8] *locally compact group*), *the family of all countably-dimensional representations in $\mathfrak{T}^\sim(A)$ has a countable base for its open sets with respect to both the* (*relativized*) *quotient and inner hull-kernel topologies.*

Proof. If H is a separable infinite-dimensional Hilbert space, $\mathfrak{T}(A;H)$ has a countable base for its open sets (**4**, Lemma 2.1). From this, together with the openness of the map $T \rightarrow \tilde{T}$, follows the conclusion for the quotient topology. Relativizing the quotient topology to $\{T | T \cong \aleph_0 \cdot T\}$ and using Lemma 2.4, we obtain the conclusion for the inner hull-kernel topology.

3. Direct integrals and weak containment.

Mackey in (**9**) has investigated what can be accomplished by imposing on the space of representations of a group or algebra not a topological structure, but merely a structure of Borel sets. In the present section we shall make considerable use of his definitions and results.

Throughout this section we fix a separable C^*-algebra A, and a separable infinite-dimensional Hilbert space $H = H_\infty$. Let $H_0, H_1, H_2, \ldots$, be a fixed increasing sequence of linear subspaces of H, such that $\dim H_n = n$ and $\bigcup_{n<\infty} H_n$ is dense in H. A representation T belonging to $\mathfrak{T} = \mathfrak{T}(A;H)$ will be said to be in *standard position* if $H(T) = H_n$ for some $n = 0, 1, 2, \ldots, \infty$;

[8]In this paper a topological space is *separable* if it has a countable base for its open sets.

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

and the set of all T which are in standard position will be called $\mathfrak{T}_s$. Clearly $\mathfrak{T}^{\sim}{}_s = \mathfrak{T}^{\sim}$. We saw in Lemma 1.5 that $\mathfrak{T}_s$ is a representative set; that is, a subset $\mathfrak{S}$ of $\mathfrak{T}^{\sim}$ is open in the quotient topology of $\mathfrak{T}^{\sim}$ if and only if $\{T \in \mathfrak{T}_s | \tilde{T} \in \mathfrak{S}\}$ is open relative to $\mathfrak{T}_s$.

We equip $\mathfrak{T}_s$ with the smallest Borel structure[9] such that, for each fixed a in A and ξ, η in H, the function $T \to (T_a \xi, \eta)$ $(T \in \mathfrak{T}_s)$ is a Borel function. Using the separability of A and H we verify without difficulty that, for each $n = 0, 1, 2, \ldots, \infty$, $\{T \in \mathfrak{T}_s | H(T) = H_n\}$ is a Borel set. It follows that our definition of the Borel structure of $\mathfrak{T}_s$ is equivalent to that of (**9**, p. 149); and hence that $\mathfrak{T}_s$ is a standard Borel space (**9**, Theorem 8.1). By other routine considerations we see that the Borel structure just defined for $\mathfrak{T}_s$ is the same as that generated by the topology of $\mathfrak{T}_s$ (relativized from that of $\mathfrak{T}$).

The quotient Borel structure of $\mathfrak{T}^{\sim}$, derived from the Borel structure of $\mathfrak{T}_s$ (a subset W of $\mathfrak{T}^{\sim}$ is a Borel set if and only if $\{T \in \mathfrak{T}_s | \tilde{T} \in W\}$ is a Borel subset of $\mathfrak{T}_s$) will be called the *Mackey Borel structure* of $\mathfrak{T}^{\sim}$. It clearly contains the Borel structure generated by the quotient topology of $\mathfrak{T}^{\sim}$.

Now let X be a fixed Borel space, and μ a fixed finite standard measure on X (see **9**, p. 142). Suppose further that $y \to \mathfrak{L}^y$ is a Borel function on X to $\mathfrak{T}_s$. We shall denote by L^y the unitary equivalence class of $\mathfrak{L}^y$, and by M the direct integral

$$M = \int_X L^y d\mu y,$$

formed as on (**9**, p. 156). If $U \subset \mathfrak{T}^{\sim}$, $L^{-1}(U)$ will mean $\{y \in X | L^y \in U\}$; if U is a Mackey Borel subset of $\mathfrak{T}^{\sim}$, $L^{-1}(U)$ is clearly a Borel subset of X.

LEMMA 3.1. *If z is an element of X such that $\mu(L^{-1}(U)) > 0$ for each open quotient neighbourhood U of L^z, then L^z is weakly contained in M.*

Proof.[10] Let p be a positive functional associated with L^z: $p(x) = (\mathfrak{L}_x{}^z \xi, \xi)$ $(x \in A)$, where $\xi \in H(\mathfrak{L}^z)$. Suppose $a_1, \ldots, a_r \in A$ and $\epsilon > 0$. It is enough to find a positive functional q associated with M such that

$$|q(a_i) - p(a_i)| < \epsilon \qquad (i = 1, \ldots, r). \tag{1}$$

Let

$$V = \{T \in \mathfrak{T}_s | |(T_{a_i} \xi, \xi) - p(a_i)| < \epsilon \quad (i = 1, \ldots, r)\}.$$

Then V is an open neighbourhood of $\mathfrak{L}^z$ in $\mathfrak{T}_s$. Since $\mathfrak{T}_s$ is a representative set (Lemma 1.5), it follows that $\tilde{V}$ is an open quotient neighbourhood of L^z. So by hypothesis $\mu(L^{-1}(\tilde{V})) > 0$.

We shall now prove the following Proposition (P):

(P) There is a μ null set N (contained in X) and a Borel function $\mathfrak{P}$ on $L^{-1}(\tilde{V}) - N$ to V such that $\mathfrak{P}^y$ and $\mathfrak{L}^y$ are unitarily equivalent for each y in $L^{-1}(\tilde{V}) - N$.

[9]For the definition of a Borel structure, see (**9**).

[10]The idea of this proof was taken from (**9**, Theorem 10.1, proof).

Let $X_n = \{y \in L^{-1}(\tilde{V}) | \dim H(\mathfrak{L}^y) = n\}$ $(n = 0, 1, \ldots, \infty)$. Since $\bigcup_n X_n = L^{-1}(\tilde{V})$, we may find and fix an n for which $\mu(X_n) > 0$. It will be sufficient to prove (P) for X_n instead of $L^{-1}(\tilde{V})$. Let Q be the set of all pairs (y, T), where $y \in X_n$ and T is a unitary operator on H_n such that $T^{-1}\mathfrak{L}^y T \in V$. We claim that

(2) $$Q \text{ is a Borel subset of } X_n \times B(H_n).$$

(Here $B(H_n)$ is the algebra of all bounded linear operators on H_n; we equip it with the smallest Borel structure making $a \to (a\xi, \eta)$ a Borel function for each fixed ξ, η in H_n.)

Indeed, $Q = Q_1 \cap Q_2$, where $Q_1 = \{(y, T) | y \in X_n,\ T \text{ is unitary in } B(H_n)\}$, $Q_2 = \{(y, T) | y \in X_n, |(\mathfrak{L}_{a_i}{}^y T\xi, T\xi) - p(a_i)| < \epsilon\ (i = 1, \ldots, r)\}$; and a routine argument shows that both Q_1 and Q_2 are Borel sets.

Further, by the definition of Q and X_n,

(3) $$\{T \in B(H_n) | (y, T) \in Q\}$$

is non-void for each y in X_n.

Now μ is standard on X_n since it is standard on X. Also $B(H_n)$ is a standard Borel space (see **9**, Theorem 8.1, proof). So, by (2), (3), and the Borel Choice Theorem (**9**, Theorem 6.3), there is a μ null subset N of X_n, and a Borel function $y \to T_y$ on X_n to $B(H_n)$, such that $(y, T_y) \in Q$ for all y in $X_n - N$. Defining $\mathfrak{P}_a{}^y = T_y{}^{-1}\mathfrak{L}_a{}^y T_y\,(a \in A)$, we see that $\mathfrak{P}^y$ is unitarily equivalent to $\mathfrak{L}^y$ and belongs to V. Thus $\mathfrak{P}$ is the desired function on X_n, and Proposition (P) is proved.

Now form the direct integral

$$P = \int_{X_n} \mathfrak{P}^y\, d\mu y;$$

then P is a non-zero subrepresentation of M. If η is the projection of ξ onto H_n, the constant function f on X_n with value $(\mu(X_n))^{-\frac{1}{2}}\eta$ belongs to the space of P. Setting, for a in A,

$$q(a) = (P_a f, f) = (\mu(X_n))^{-1}\int_{X_n} (\mathfrak{P}_a^y \xi, \xi) d\mu y,$$

and recalling that $\mathfrak{P}^y \in V$, we find that $|q(a_i) - p(a_i)| < \epsilon$ for each i, which is (1). Since q is associated with P, and hence with M, the lemma is proved.

A function L on X to $\mathfrak{T}^\sim$ is said to be *integrable* (**9**, p. 157) if there is a Borel function $\mathfrak{L}$ on X to $\mathfrak{T}_s$ such that $\mathfrak{L}^y$ belongs to the class L^y for μ—almost all y. In that case the direct integral $\int_X L^y d\mu y$ has a well-defined meaning.

THEOREM 3.1. *Let A be a separable C^*-algebra, μ a σ-finite standard measure on a Borel space X, and $y \to L^y$ an integrable function on X to $\mathfrak{T}^\sim(A; H)$, where H is a separable infinite-dimensional Hilbert space. Then $M = \int_X L^y d\mu y$ is weakly equivalent to the set of all those L^y $(y \in X)$ such that*

(4) $\mu(L^{-1}(U)) > 0$ *for each open quotient neighbourhood U of L^y.*

Exactly the same is true if in (4) "*quotient*" *is replaced by* "*inner hull-kernel*".

We shall denote by (4′) the condition obtained from (4) by replacing "quotient" by "inner hull-kernel."

Proof. We may clearly assume without loss of generality that A has a unit and that μ is finite. Since (4′) is weaker than (4) (Lemma 2.3), it is enough to show that M weakly contains every L^y satisfying (4′), and that M is weakly contained in the set of all L^y satisfying (4).

Suppose that L^z satisfies (4′); and let U be an open quotient neighbourhood of $\aleph_0 \cdot L^z$. By Lemma 2.4 there is an open inner hull-kernel neighbourhood V of $\aleph_0 \cdot L^z$ whose intersection with $\{T | T \cong \aleph_0 \cdot T\}$ coincides with that of U. But V is also an inner hull-kernel neighbourhood of L^z (since the inner hull-kernel topology does not distinguish L^y from $\aleph_0 \cdot L^y$). Hence by (4′) $\mu(L^{-1}(V)) > 0$. It follows that $\{y \in X | \aleph_0 \cdot L^y \in U\}$ is of positive μ measure. So, by Lemma 3.1 and the arbitrariness of U, $\aleph_0 \cdot M = \int_X \aleph_0 \cdot L^y d\mu y$ weakly contains $\aleph_0 \cdot L^z$. That is, M weakly contains L^z.

Next we claim that M is weakly contained in the set of all L^y $(y \in X)$. Indeed, let $a_1, \ldots, a_n$ be elements of A, ϵ a positive number, and f any vector in $H(M)$ such that $||f(y)||$ is bounded in y. Then for each $i = 1, \ldots, n$ the function

$$y \to (L^y_{a_i} f(y), f(y)) \tag{5}$$

is bounded on X; and there exists a partition $X_1, \ldots, X_m$ of X into disjoint Borel sets such that, on each X_j, each function (5) has oscillation less than $\epsilon/\mu(X)$. Choosing an element y^j in each X_j, and setting $\phi^j(a) = (L_a^{y^j} f(y^j), f(y^j))$, and $\phi = \sum_j \mu(X_j)\phi^j$, we find that

$$|\phi(a_i) - (M_{a_i} f, f)| < \epsilon$$

for all $i = 1, \ldots, n$. Thus the positive functional $a \to (M_a f, f)$ associated with M is weakly * approximated by sums ϕ of positive functionals associated with the L^y. Since the elements f for which $||f(y)||$ is bounded in y are dense in $H(M)$, we have proved that M is weakly contained in the set of all L^y.

Now the direct integral M is not altered by eliminating a μ null set from X. Hence, by the preceding paragraph, M will be weakly contained in the set of all L^y satisfying (4), provided that the set of y for which (4) fails is a μ null set. Let W be the set of all T in $\mathfrak{T}^\sim(A; H)$ having a quotient neighbourhood U_T such that $\mu(L^{-1}(U_T)) = 0$. By Lemma 2.5 the covering $\{U_T\}$ of W can be reduced to a countable covering; from which it follows that $L^{-1}(W)$ is μ null. But $L^{-1}(W)$ is just the set of y for which (4) fails.

Remark. The correspondence between the unitary representations of a separable locally compact group and the proper representations of its (separable) group C^*-algebra preserves the notion of direct integral, as well as the Borel structure of the space of countably-dimensional representations (see **9**, especially Theorem 9.1). It also (by definition) preserves the quotient and inner hull-kernel topologies. Hence Theorem 3.1 remains valid when A is a separable locally compact group.

The following consequence of the preceding theorem is a considerable generalization of (3, Theorem 1.7) (at least for the separable case).

THEOREM 3.2. *Let A be a separable C^*-algebra (or a separable locally compact group), X a separable locally compact Hausdorff space, and μ a σ-finite measure on the Borel subsets of X. Further let $y \to L^y$ be an integrable function on X to $\mathfrak{T}^\sim(A; H)$ (where H is a Hilbert space of countable dimension), which is continuous with respect to the inner hull-kernel topology of $\mathfrak{T}^\sim(A)$. Then $M = \int_X L^y d\mu y$ is weakly equivalent to the set of all L^y, where y ranges over the closed hull[11] of μ.*

Proof. Since the Borel structure of X is standard (9, p. 138), μ is a standard measure. We may assume without loss of generality that the closed hull of μ is X. The assertion then follows immediately from Theorem 3.1 (and the Remark following it).

Suppose now that A is a separable C^*-algebra or a separable locally compact group which is of Type I (and hence by (5) has a smooth dual). If M is a proper (or unitary) countably-dimensional representation of A, it is shown in (9, § 10) that there is a σ-finite measure μ on the Borel subsets of $A^\dagger$ and a Borel function α on $A^\dagger$ to the non-zero countable cardinals such that

$$M \cong \int_{A^\dagger} \alpha(L) \cdot L \, d\mu L.$$

THEOREM 3.3. *If A, M, and μ are as in the preceding paragraph, M is weakly equivalent to the closed hull of μ (in $A^\dagger$).*

Proof. Since $A^\dagger$ is smooth, μ is standard. We now apply Theorem 3.1 (and the Remark following it).

On examining Theorem 3.2, one naturally asks under what conditions the continuity of the map $y \to L^y$ implies its integrability.

PROPOSITION. *If in Theorem 3.2 A is of Type I and if all the L^y are irreducible, the continuity of the map $y \to L^y$ (with respect to the hull-kernel topology) implies its integrability.*

Proof. By (5, Theorem 1), the hull-kernel topology separates points of $A^\dagger$; hence the Mackey Borel structure of $A^\dagger$ coincides with the Borel structure generated by the hull-kernel topology (see 5 or 4, Theorem 4.1). So the map $y \to L^y$, being continuous, is a Borel map. The assertion now follows from the Corollary (9, p. 157).

[11]The closed hull of μ is the (closed) set of all points y in X such that $\mu(U) > 0$ for every open neighbourhood U of y.

Suppose, on the other hand, that the L^y are not irreducible. Then, even assuming that A is of Type I and that the map $y \to L^y$ is continuous, we cannot conclude its integrability. Indeed, suppose for simplicity that A is abelian and let μ_1 and μ_2 be two Borel measures on $A\dagger$ which have the same compact closed hull, but are of different measure classes (that is, have different null sets). Then $T_1 = \int_{A\dagger}\chi d\mu_1\chi$ and $T_2 = \int_{A\dagger}\chi d\mu_2\chi$ are unitarily inequivalent, but it is easily verified (using Lemma 1.12) that even the quotient topology, let alone the inner hull-kernel topology, cannot distinguish them. Hence, if $y \to L^y$ has range equal to $\{T_1, T_2\}$, it is automatically quotient-continuous; but the values T_1 and T_2 can in general be distributed so irregularly that $y \to L^y$ is not integrable.

4. Induced representations. We remind the reader of the definition and the notation for induced representations.[12] Let G be a separable locally compact group with unit e, K a closed subgroup, G/K the space of right K cosets $\tilde{x} = Kx$ $(x \in G)$, ρ a non-zero quasi-invariant[13] Borel measure on G/K, and λ the Radon-Nikodym derivative of ρ considered as defined on $G \times G$:

$$\lambda(x, y) = \frac{d\rho(\tilde{x}y)}{d\rho\tilde{x}}.$$

Throughout this section all representations of G will be assumed countably-dimensional.

Let T be a unitary representation of K. We define $H(U^T)$ as the space of all measurable[14] functions f on G to $H(T)$ such that

(i) $$f(\xi x) = T_\xi(f(x))\,(x \in G, \xi \in K),$$

(ii) $$\int_{G/K} ||f(x)||^2 d\rho\tilde{x} < \infty.$$ [15]

This is a Hilbert space under the inner product

$$(f, g) = \int_{G/K} (f(x), g(x))d\rho\,\tilde{x}.$$

Now U^T (or ${}_GU^T$ if it is necessary to specify the larger group) is defined by

$$(U_y^T f)(x) = f(xy)\sqrt{(\lambda(x, y))}\,(x, y \in G, f \in H(U^T)).$$

This U^T is the representation of G *induced from* T. The reader will be assumed familiar with the elementary theory of induced representations as presented in **(7)**.

Let H be a countably-dimensional Hilbert space, and T a representation of K with space H (in the sense of § 1, with perhaps a null space). By U^T

---

[12]See **(7)**.

[13]That is, the elements of G, acting on G/K, carry ρ null sets into ρ null sets. See **(7**, p. 103**)**.

[14]f is measurable if $(f(x), \xi)$ is measurable in x for each ξ in $H(T)$.

[15]By (i) $||f(x)||^2$ depends on $\tilde{x}$ only.

we shall mean the same as $U^{T'}$, where T' is the restriction of T to its essential space $H(T)$. Let ϕ be any norm-continuous function on G to H with compact support, and define a function ψ on G to $H(T)$ by [16]

$$\psi(x) = \int_K T_{\xi^{-1}}(\phi(\xi x))d\nu\xi,$$

where ν is right Haar measure on K. We easily verify that ψ belongs to $H(U^T)$; indeed ψ is norm-continuous on G, and the function $\tilde{x} \to ||\psi(x)||$ has compact support on G/K. The function ψ will be called f_ϕ (or $f_\phi^{(T)}$ if we desire to emphasize its dependence on T).

Let us denote by $\mathfrak{L}$ the linear space of all norm-continuous functions on G to H with compact support, and by $\mathfrak{L}_0$ the linear subspace of $\mathfrak{L}$ generated by functions of the form $x \to \alpha(x)\xi$, where $\xi \in H$ and α is a continuous complex-valued function with compact support on G.

Lemma 4.1. *The set of all f_ϕ, where ϕ runs over $\mathfrak{L}_0$, is dense in $H(U^T)$.*

Proof. It is proved in **(7,** Lemma 3.5**)** that the f_ϕ are dense in $H(U^T)$ when ϕ runs over $\mathfrak{L}$. Furthermore, approximating elements of $\mathfrak{L}$ by elements of $\mathfrak{L}_0$, we easily verify that $\{f_\phi|\phi \in \mathfrak{L}_0\}$ is dense (in the $H(U^T)$ norm) in $\{f_\phi|\phi \in \mathfrak{L}\}$. These two facts give the required result.

Theorem 4.1. *If G is a separable locally compact group, and K is a closed subgroup of G, the map $T \to {}_GU^T$ (T ranging over all unitary representations of K) is continuous with respect to the inner hull-kernel topologies of unitary representations of K and G.*

Proof. Let $\mathfrak{S}$ be a family of unitary representations of K and T a unitary representation of K belonging to the inner hull-kernel closure of $\mathfrak{S}$. It is enough to prove that U^T belongs to the inner hull-kernel closure of

$$U^{\mathfrak{S}} = \{U^S|S \in \mathfrak{S}\}.$$

Since the inner hull-kernel topology does not separate S from $\aleph_0 \cdot S$, and since $U^{\aleph_0 \cdot S} \cong \aleph_0 \cdot U^S$, we may assume without loss of generality that $T \cong \aleph_0 \cdot T$ and $S \cong \aleph_0 \cdot S$ for all S in $\mathfrak{S}$. Then by Lemma 2.4 T belongs to the quotient closure of $\mathfrak{S}$. Thus we may assume that T and the representations in $\mathfrak{S}$ are all situated in the same separable Hilbert space H (with null spaces perhaps, see § 1); and that there is a net $\{S^i\}$ of elements of $\mathfrak{S}$ converging to T in $\mathfrak{T}(K; H)$.

Let $\mathfrak{L}_0$ be defined as before, using the above Hilbert space H. We shall prove that for each fixed ϕ in $\mathfrak{L}_0$ whose range is contained in $H(T)$,

(1) $$(U_x^{S^i} f_\phi^{(S^i)}, f_\phi^{(S^i)}) \to (U_x^T f_\phi^{(T)}, f_\phi^{(T)})$$

uniformly in x on compact subsets of G.

[16]This construction is given in **(7,** § 3**)**.

Suppose

$$\varphi(x) = \sum_{i=1}^{r} \lambda_i(x)\alpha_i,$$

where the λ_i are complex continuous functions on G with compact supports C_i, and $\alpha_i \in H(T)$. We shall assume without loss of generality that

$$||\alpha_i|| \leqslant 1, \sup_x |\lambda_i(x)| \leqslant 1.$$

Let us put

$$C = \bigcup_{i=1}^{r} C_i \cup \{e\}.$$

Fix a positive number ϵ, and a compact subset D of G; and define E to be the compact subset $K \cap CC^{-1}CDC^{-1}$ of K. Further, let U be the neighbourhood

$$\{S \in \mathfrak{T}(K; H) |\, ||S_x\alpha_i - T_x\alpha_i|| < \epsilon \text{ for all } i = 1, \ldots, r \text{ and all } x \text{ in } E\}$$

of T in $\mathfrak{T}(K; H)$ (see Lemma 1.1).

Now, if $S \in \mathfrak{T}(K; H)$ and $x, y \in G$,

$$\begin{aligned}(U_y^S f_\phi^{(S)})(x) &= f_\phi^{(S)}(xy)\sqrt{(\lambda(x, y))} \\ &= \sqrt{(\lambda(x, y))} \sum_{i=1}^{r} \int_K \lambda_i(\xi xy) S_{\xi^{-1}}\alpha_i d\nu\xi;\end{aligned}$$

whence

$$\begin{aligned}(U_y^S f_\phi^{(S)}, f_\phi^{(S)}) &= \int_{G/K} ((U_y^S f_\phi^{(S)})(x), f_\phi^{(S)}(x)) d\rho\, \tilde{x} \\ &= \sum_{i,j=1}^{r} \int_{G/K} \sqrt{(\lambda(\tilde{x}, y))} d\rho\, \tilde{x} \int_K \int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta x)}(S_{\xi^{-1}}\alpha_i, S_{\eta^{-1}}\alpha_j)\, d\nu\eta\, d\nu\, \xi \\ &= \sum_{i,j=1}^{r} \int_{G/K} \sqrt{(\lambda(\tilde{x}, y))} d\rho\, \tilde{x} \int_K \int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta\xi x)}(S_\eta\alpha_i, \alpha_j) d\nu\eta\, d\nu\, \xi \\ &= \sum_{i,j=1}^{r} \int_{G/K} \sqrt{(\lambda(\tilde{x}, y))} d\rho\, \tilde{x} \int_K (S_\eta\alpha_i, \alpha_j) d\nu\eta \int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta\xi x)} d\nu\, \xi,\end{aligned}$$

and similarly for T. Thus

$$\begin{aligned}&(U_y^S f_\phi^{(S)}, f_\phi^{(S)}) - (U_y^T f_\phi^{(T)}, f_\phi^{(T)}) \\ (2) \qquad &= \sum_{i,j=1}^{r} \int_{G/K} \sqrt{(\lambda(\tilde{x}, y))} d\rho\, \tilde{x} \int_K ((S_\eta\alpha_i, \alpha_j) - (T_\eta\alpha_i, \alpha_j)) d\nu\eta \int_K \lambda_i(\xi xy) \\ &\qquad \times \overline{\lambda_j(\eta\xi x)} d\nu\xi.\end{aligned}$$

If $\tilde{x}$ is a coset having no element in common with C, then $\lambda_j(\eta\xi x) = 0$ for all η, ξ, and j; so that

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

$$\int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta\xi x)}d\nu\xi = 0$$

for all η in K and y in G. It follows that the integral over G/K in (2) need be taken only over $\tilde{C}$, and in the inner integrals x may be taken to lie in C.

If $x \in C$ and $y \in D$, $\lambda_i(\xi xy) = 0$ unless $\xi \in CD^{-1}C^{-1}$; so the innermost integral in (2) need be taken only over $K \cap CD^{-1}C^{-1}$. If $x \in C$ and $\xi \in CD^{-1}C^{-1}$, then $\lambda_j(\eta\xi x) = 0$ unless $\eta \in E = CC^{-1}CDC^{-1} \cap K$. So the middle integral in (2) may be taken over E. Note that, if $x \in C$, $y \in D$, and $\eta \in E$,

$$\left|\int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta\xi x)}d\nu\xi\right| \leqslant \nu(K \cap CD^{-1}C^{-1}) \tag{3}$$

(since $|\lambda_i| \leqslant 1$).

Now assume that $S \in U$; then $||S_\eta\alpha_i - T_\eta\alpha_i|| < \epsilon$ for all $i = 1, \ldots, r$, and all η in E. Since $||\alpha_i|| \leqslant 1$, this implies $|(S_\eta\alpha_i, \alpha_j) - (T_\eta\alpha_i, \alpha_j)| < \epsilon$ for all i, j, and all η in E. Thus, for each x in C, y in D, and each i, j, we have by (3)

$$\left|\int_K ((S_\eta\alpha_i, \alpha_j) - (T_\eta\alpha_i, \alpha_j))d\nu\eta \int_K \lambda_i(\xi xy)\overline{\lambda_j(\eta\xi x)}d\nu\xi\right| \leqslant \epsilon\nu(E)\nu(K \cap CD^{-1}C^{-1}). \tag{4}$$

Further,

$$\int_{\tilde{C}} \sqrt{(\lambda(\tilde{x}, y))}d\rho\,\tilde{x} \leqslant \int_{\tilde{C}} (1 + \lambda(\tilde{x}, y))d\rho\,\tilde{x} = \rho(\tilde{C}) + \rho(\tilde{C}D). \tag{5}$$

Combining (2), (4), and (5), we obtain:

$$|(U_y^S f_\phi^{(S)}, f_\phi^{(S)}) - (U_y^T f_\phi^{(T)}, f_\phi^{(T)})| \leqslant \epsilon r^2(\rho(\tilde{C}) + \rho(\tilde{C}D))\nu(E)\nu(K \cap CD^{-1}C^{-1}) \tag{6}$$

for all y in D and S in U. By choosing ϵ properly, the right side of (6) may be made arbitrarily small. Since $\{S^i\}$ is eventually in U, this proves (1).

Now, by Lemma 4.1, as ϕ ranges over those elements of $\mathfrak{L}_0$ whose ranges are contained in $H(T)$, f_ϕ ranges over a dense subset of $H(U^T)$. This together with (1) shows that, to every function ϕ of positive type on G associated with U^T and every compact subset D of G, there is a neighbourhood W_ϕ of T (in $T(K; H)$) such that, if $S \in W_\phi$, there is a function of positive type associated with U^S differing from ϕ on D by less than ϵ.

If $\phi_1, \ldots, \phi_n$ are functions of positive type associated with U^T, there is an S belonging to

$$\mathfrak{S} \cap W_{\phi_1} \cap \ldots \cap W_{\phi_n}.$$

We may then find functions of positive type $\psi_1, \ldots, \psi_n$ associated with U^S such that for each i $|\phi_i(x) - \psi_i(x)| < \epsilon$ for all x in D. By Theorem 2.2, this implies that U^T is in the inner hull-kernel closure of $U^{\mathfrak{S}}$.

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

From Theorem 4.1 and Theorem 2.3 we obtain almost immediately the following theorem.

THEOREM 4.2. *If K is a closed subgroup of the separable locally compact group G, $\mathfrak{S}$ is a family of unitary representations of K, and T is a unitary representation of K weakly contained in $\mathfrak{S}$, then U^T is weakly contained in $\{U^S | S \in \mathfrak{S}\}$.*

Theorems 4.1 and 4.2 suggest a converse question: What can be said about T and $\mathfrak{S}$ if it is known that U^T is weakly contained in $\{U^S | S \in \mathfrak{S}\}$? Theorem 4.4 answers this question in case K is normal.

Assume that K is a closed normal subgroup of the separable locally compact group G (with unit e). If S is a unitary representation of K and $x \in G$, S^x will be the unitary representation of K for which $S_\xi^x = S_{x\xi x^{-1}}$ ($\xi \in K$). The set of all S^x, where x runs over G, is the *orbit of S under G*, and is denoted by $\theta(S)$; if $\mathfrak{S}$ is a family of unitary representations of K, $\theta(\mathfrak{S})$ will mean

$$\bigcup_{S \in \mathfrak{S}} \theta(S).$$

It is evident that, for fixed x, the mapping $S \to S^x$ leaves unaltered the relation of weak containment. We recall also that ${}_GU^S \cong {}_GU^{S^x}$ for all x in G and all unitary representations S of K.

We shall denote by $T \otimes S$ the Kronecker product of the two representations T and S of G (see **7**, p. 114), and by $S|H$ the restriction of S to a subgroup H.

LEMMA 4.2. *Let H be a closed subgroup of G. If S and T are unitary representations of G and H respectively, then*

$$U^T \otimes S \cong U^{T \otimes (S|H)}.$$

Proof. G and H, as subgroups of G, are regularly related (**7**, p. 127). The Lemma is now an immediate consequence of (**7**, Theorem 12.2).

COROLLARY 1. *Let H be a closed subgroup of G, and I and J the one-dimensional identity representations of G and H respectively. If S is any unitary representation of G, then*

$$U^{S|H} \cong U^J \otimes S. \tag{7}$$

Further, if U^J weakly contains I, then $U^{S|H}$ weakly contains S.

Proof. We obtain (7) by setting $T = J$ in the Lemma. To obtain the last statement, we apply to (7) the following easily verifiable proposition:

If T and W are unitary representations of G, and $\mathfrak{S}$ is a family of unitary representations of G which weakly contains T, then $\{S \otimes W | S \in \mathfrak{S}\}$ weakly contains $T \otimes W$.

If in Corollary 1 we take $H = \{e\}$, we obtain the fact, already noted by Godement (**6**, p. 77), that, if the regular representation of G weakly contains the identity representation, then it weakly contains all unitary representations of G.

We note, as a digression, a further immediate consequence of Corollary 1:

COROLLARY 2.[17] *Let K be a closed normal subgroup of G. For each unitary representation T of G/K, let T' be the corresponding representation of G ($T_x' = T_{xK}$). Suppose further that the right regular representation R of G/K is written as a direct integral of unitary representations R^α of G/K: $R \cong \int R^\alpha d\mu\alpha$. Then for each unitary representation S of G,*

$$U^{S|K} \cong \int (R^{\alpha'} \otimes S) d\mu\, \alpha.$$

In particular, if G is abelian, χ is a character of K, and χ_0 is a character of G which extends χ, then

$$U^\chi \cong \int_H (\phi\chi_0) d\mu\, \phi,$$

where $H = \{\phi \in G\dagger | \phi \equiv 1 \text{ on } K\}$ and μ is Haar measure on H.

THEOREM 4.3. *Let G be a separable locally compact group, K a closed normal subgroup of G, and H a closed subgroup of G containing K. Suppose further that the regular representation of H/K weakly contains the identity representation (hence all representations) of H/K. Let $\mathfrak{S}$ be a family of unitary representations of K, and T a unitary representation of H. Then*

$${}_GU^{\mathfrak{S}} = \{{}_GU^S | S \in \mathfrak{S}\}$$

weakly contains ${}_GU^T$ if and only if the orbit $\theta(\mathfrak{S})$ of $\mathfrak{S}$ (under G) weakly contains $T|K$.

Proof. (A) Suppose that $\theta(\mathfrak{S})$ weakly contains $T|K$. Then by Theorem 4.2, applied with H as the larger group,

(8) $${}_HU^{\theta(\mathfrak{S})} \text{ weakly contains } {}_HU^{T|K}.$$

By Corollary 1 of Lemma 4.2 (together with the hypothesis on the regular representation of H/K) ${}_HU^{T|K}$ weakly contains T. Hence by (8),

(9) $${}_HU^{\theta(\mathfrak{S})} \text{ weakly contains } T.$$

Inducing (9) up to G, and applying Theorem 4.2 (as well as (**7**, Theorem 4.1) on inducing in stages), we find that ${}_GU^{\theta(\mathfrak{S})}$ (which is the same as ${}_GU^{\mathfrak{S}}$) weakly contains U^T.

(B) Conversely suppose that $U^{\mathfrak{S}}$ weakly contains U^T. By (**3**, Theorem 1.3, Corollary),

(10) $$U^{\mathfrak{S}}|K \text{ weakly contains } U^T|K$$

[17]The last statement of this Corollary is also an immediate consequence of Mackey's generalized Frobenius Reciprocity Theorem (see **8**).

(where $U^{\mathfrak{S}}|K = \{U^S|K|S \in \mathfrak{S}\}$). But by Theorem 12.1 of **(7)**,

$$U^S|K \cong \int_{G/K} S^x d\rho\, \tilde{x}, \tag{11}$$

$$U^T|K \cong \int_{G/H} (T|K)^x d\sigma\, \check{x} \tag{12}$$

(where $\tilde{x} = Kx$, $\check{x} = Hx$, and ρ and σ are quasi-invariant measures on G/K and G/H respectively).

Now from the existence of a Borel cross-section for the coset spaces G/K and G/H **(7**, Lemma 1.1**)** we easily deduce that the maps $\tilde{x} \to S^x$ and $\check{x} \to (T|K)^x$ are integrable. Further, since $x \to S^x$ and $x \to (T|K)^x$ are continuous in the inner hull-kernel topology of representations of K, the same is true of the maps $\tilde{x} \to S^x$ and $\check{x} \to (T|K)^x$. Hence, applying Theorem 3.2 to (11) and (12), we find that $U^S|K$ is weakly equivalent to $\theta(S)$ (for each S in $\mathfrak{S}$) and $U^T|K$ is weakly equivalent to $\theta(T|K)$. So, by (10), $\theta(\mathfrak{S})$ weakly contains $\theta(T|K)$ and hence $T|K$. The proof is now complete.

The condition that the regular representation of a group weakly contains all other representations has been studied by Takenouchi in **(11)**; he derives a necessary and sufficient condition for it to hold, in terms of the structure of the group. It is well known, even without Takenouchi's result, that it holds for all compact and all abelian locally compact groups.

Putting $H = K$ in Theorem 4.3, we have:

THEOREM 4.4. *Let G be a separable locally compact group, K a closed normal subgroup of G, T a unitary representation of K, and $\mathfrak{S}$ a family of unitary representations of K. Then U^T is weakly contained in $U^{\mathfrak{S}}$ if and only if $\theta(\mathfrak{S})$ weakly contains T.*

We observe the following fact, which was demonstrated in the course of the proof of Theorem 4.3.

THEOREM 4.5. *If T is a unitary representation of a closed normal subgroup K of the separable locally compact group G, then ${}_GU^T|K$ is weakly equivalent to the orbit of T (under G).*

5. Examples. Theorem 4.3 constitutes the main step in determining the topology of the dual space of many solvable groups. Before giving examples, we mention two further facts. The first is the same as **(3**, Theorem 1.3, Corollary**)**; the second also follows directly from **(3**, Theorem 1.3**)**. We fix a separable locally compact group G.

LEMMA 5.1. *If K is a closed subgroup of G, and $\mathfrak{S}$ is a family of unitary representations of G which weakly contains a unitary representation T of G, then $T|K$ is weakly contained in $\{S|K|S \in \mathfrak{S}\}$.*

LEMMA 5.2. *Let K be a closed normal subgroup of G. For each unitary representation T of G/K, let T' be the corresponding representation of G ($T_x' = T_{xK}$). Then, if $\mathfrak{S}$ is a family of unitary representations of G/K, T is weakly contained in $\mathfrak{S}$ if and only if T' is weakly contained in $\mathfrak{S}' = \{S' | S \in \mathfrak{S}\}$. Further, if R is a unitary representation of G which is weakly contained in $\mathfrak{S}'$, R must be of the form T' for some unitary representation T of G/K.*

Example 1. *The "ax + b" group.* This is the group G of all pairs (x, y), where x is real and y is positive, with

$$(x, y)(x', y') = (xy' + x', yy').$$

Let K be the closed normal subgroup of all $(x, 1)$ (x real). Then $K^\dagger$ consists of all χ^k (k real), where $\chi^k(x, 1) = e^{ikx}$, and contains three orbits under G, namely (i) θ_+, consisting of all χ^k with $k > 0$, (ii) θ_- consisting of all χ^k with $k < 0$, and (iii) $\{\chi^0\} = \theta_0$. By the theory of induced representations (see **7**), the irreducible representation of G are U^{χ^1}, $U^{\chi^{-1}}$, and the one-dimensional ψ_r (r real), where $\psi_r(x, y) = y^{ir}$.

Now the closure (in $K^\dagger$) of θ_+ contains θ_0 but not θ_-; and similarly for θ_-. Hence, applying Theorem 4.3, we see that U^{χ^1} weakly contains all ψ_r but not $U^{\chi^{-1}}$; while $U^{\chi^{-1}}$ weakly contains all ψ_r but not U^{χ^1}. By Lemma 5.2, the set of all ψ_r (r real) is closed in $G^\dagger$, and the topology of $G^\dagger$ relativized to this set is just the ordinary topology of the parameter r. We therefore have:

THEOREM 5.1. *Let G be the "ax + b" group. A subset W of $G^\dagger$ is closed in $G^\dagger$ if and only if it satisfies the following two conditions:*

(i) *If either U^{χ^1} or $U^{\chi^{-1}}$ is in W, then all ψ_r are in W.*

(ii) *$\{r \text{ real} | \psi_r \in W\}$ is closed in the ordinary topology of the reals.*

Thus, the topology of $G^\dagger$ is not only not Hausdorff, but fails to be even T_1 (indeed, U^{χ^1} and $U^{\chi^{-1}}$ are not closed). This shows, incidentally, that U^{χ^1} and $U^{\chi^{-1}}$ are not completely continuous (see (**3**, Lemma 1.11)).[18]

Example 2. *The euclidean group of the plane.* This is the group G of all pairs (z, u), where z and u are complex and $|u| = 1$, with

$$(z, u)(z', u') = (z + uz', uu').$$

Let K be the closed normal subgroup of all $(z, 1)$. The dual $K^\dagger$ consists of all χ^w (w complex), where $\chi^w(z, 1) = e^{iRe(z\bar{w})}$. The orbit of χ^w under G consists of all those χ^v for which $|v| = |w|$. For $r \geqslant 0$ let θ_r be the orbit $\{\chi^w | |w| = r\}$. The distinct irreducible representations of G are:

(i) the $U^{\chi^r} (r > 0)$;

(ii) the one-dimensional ψ_n (n an integer), where $\psi_n(z, u) = u^n$.

Now if S is a subset of the positive reals and $r > 0$, then

$$\theta_r \subset \bigcup_{s \in S} \theta_s$$

[18]This fact has also been observed in (**10**, p. 556).

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

if and only if $r \in \bar{S}$ (in the ordinary topologies of plane and line). It follows from Theorem 4.4 that the topology of $G\dagger$ relativized to the set of all U^{χ^r} is just the ordinary topology of the parameter r. By Theorem 4.3, if S is a subset of the positive reals, the closure in $G\dagger$ of $\{U^{\chi^s} | s \in S\}$ contains all the ψ_n if $0 \in \bar{S}$, and none of the ψ_n if $0 \notin \bar{S}$. As in the previous example, $\{\psi_n | n \text{ an integer}\}$ is closed in $G\dagger$ and its relativized topology is the discrete topology. We therefore have:

THEOREM 5.2. *Let G be the Euclidean group of the plane. A subset W of $G\dagger$ is closed if and only if the following conditions hold:*

(*i*) $S = \{s > 0 | U^{\chi^s} \in W\}$ *is closed in the ordinary topology of the positive reals;*

(ii) *if $0 \in \bar{S}$, then all ψ_n are in W.*

Thus $G\dagger$ in this case is T_1 but not Hausdorff.

Dixmier **(1)** has studied the irreducible representations of all simply connected nilpotent Lie groups of dimension no greater than five. Apart from the additive group of reals and obvious direct sums, there are eight of these, which he calls Γ_3, Γ_4, $\Gamma_{5,1}$, $\Gamma_{5,2}$, $\Gamma_{5,3}$, $\Gamma_{5,4}$, $\Gamma_{5,5}$, and $\Gamma_{5,6}$. In **(2)** he has worked out the topologies of the duals of these groups (except for $\Gamma_{5,4}$ and, in part, $\Gamma_{5,5}$), by computations based on the detailed form of the representations. It turns out that one can obtain all of Dixmier's results by arguments based on Theorem 4.3. For $\Gamma_{5,4}$ these arguments do not seem to suffice; but the dual of this group can be obtained by a method peculiar to this case. The dual of $\Gamma_{5,5}$ offers no difficulties in principle; but the complicated nature of the orbits makes explicit calculation difficult.

We shall sketch the application of our method to Γ_3, Γ_4, and $\Gamma_{5,3}$; and conclude with a sketch of the argument for $\Gamma_{5,4}$.

If $\mathfrak{g}$ is the Lie algebra of the nilpotent group G, we denote by $\langle x_1, \ldots, x_r\rangle$ the linear span of the vectors $x_1, \ldots, x_r$ in $\mathfrak{g}$, and by E the exponential mapping. If $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, $E(\mathfrak{k})$ is the corresponding closed subgroup of G. If K is a closed normal subgroup of G, we shall sometimes denote by the same letter a representation of G/K and the corresponding lifted representation of G. By R^n we mean Euclidean n-space; R is the additive group of reals.

Example 3. *The group* Γ_3. This is the three-dimensional simply connected Lie group whose Lie algebra has basis $\{x_1, x_2, x_3\}$ with $[x_1, x_2] = x_3$, $[x_1, x_3]=0$, $[x_2, x_3] = 0$. Then $K = E(\langle x_2, x_3\rangle)$ is a closed normal abelian subgroup of Γ_3. Let $\chi_{u,v}$ (u, v real) be the character of K sending $E(rx_2 + sx_3)$ into $e^{i(ur+vs)}$. We verify that the orbits in $K\dagger$ (under Γ_3) are the following: (i) $\theta_u{}^0 = \{\chi_{u,0}\}$ for real u, and (ii) $\theta_v{}^1 = \{\chi_{t,v} | t \text{ real}\}$ for v real, $v \neq 0$. The stationary subgroup for $\theta_u{}^0$ is Γ_3; that for $\theta_v{}^1$ is K. Hence, by Mackey's theory of induced representations, the distinct elements of $\Gamma_3\dagger$ are:

(A) the one-dimensional $\psi_{u,v}(u, v \text{ real})$, where

$$\psi_{u,v}(E(r_1x_1 + r_2x_2 + r_3x_3)) = e^{i(ur_1+vr_2)},$$

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

(B) the representations T^v (v real $\neq 0$), where T^v is induced from the character $\chi_{0,v}$ of K.

Just as in Examples 1 and 2, we obtain:

THEOREM 5.3.[19] *A subset W of $\Gamma_3\dagger$ is closed in $\Gamma_3\dagger$ if and only if:*

(I) $\{(u, v)|\psi_{u,v} \in W\}$ *and* $\{v|T^v \in W\}$ *are closed in* R^2 *and* $R - \{0\}$ *respectively;*

(II) *If* 0 *is a limit point of* $\{v|T^v \in W\}$, *all* $\psi_{u,v}$ *are in* W.

Example 4. *The group* Γ_4. This is the four-dimensional simply connected Lie group whose Lie algebra has basis x_1, x_2, x_3, x_4 with $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, and $[x_i, x_j] = 0$ for all other $i < j$. Let K be the closed normal abelian subgroup $E(\langle x_2, x_3, x_4\rangle)$; and $\chi_{u,v,w}$ (for u, v, w real) the character of K sending $E(rx_2 + sx_3 + tx_4)$ into $e^{i(ur+vs+wt)}$. We verify that the orbits in $K\dagger$ (under Γ_4) are the following: (i) $\theta_u{}^0 = \{\chi_{u,0,0}\}$ (for u real); (ii) $\theta_v{}^1 = \{\chi_{t,v,0}|t$ real$\}$ (for v real $\neq 0$); and (iii) $\theta_{s,w}{}^2 = \{\chi_{u,v,w}|u, v$ are real and $uw - (v^2/2) = s\}$ (for s, w real, $w \neq 0$). Let $Q_0 = \{\theta_u{}^0|u$ real$\}$, $Q_1 = \{\theta_v{}^1|v$ real $\neq 0\}$, $Q_2{}^+ = \{\theta_{s,w}{}^2|s$ real, $w > 0\}$, $Q_2{}^- = \{\theta_{s,w}{}^2|s$ real, $w < 0\}$, $Q_2 = Q_2{}^+ \cup Q_2{}^-$. In view of Theorem 4.3, we must calculate the quotient topology of the orbit space $Q = Q_0 \cup Q_1 \cup Q_2$. It is not hard to verify:

PROPOSITION. (I) *Relativized to* Q_0, Q_1, *or* Q_2, *the quotient topology of* Q *is the natural topology of the parameters* u, v, s, w.

(II) *The closure of a subset* W *of* Q_1 *contains all or none of* Q_0 *according as* 0 *is or is not a limit point of* $\{v|\theta_v{}^1 \in W\}$.

(III) *An element* $\theta_v{}^1$ *of* Q_1 *belongs to the closure of a subset* W *of* Q_2 *if and only if* $(-(v^2/2), 0)$ *is a limit point of* $\{(s, w)|\theta_{s,w}{}^2 \in W\}$.

(IV) *If* $W \subset Q_2{}^+$ [*or* $Q_2{}^-$], *the closure of* W *contains* $\theta_u{}^0$ *if and only if* (a) $(0, 0)$ *is a limit point of* $\{(s, w)|\theta_{s,w}{}^2 \in W\}$, *and* (b)

$$\varliminf_{(s,w)\to(0,0),\theta^2_{s,w}\in W}\left(\frac{s}{w}\right) \leqslant u$$

$$\left[\text{or} \quad \varlimsup_{(s,w)\to(0,0),\theta^2_{s,w}\in W}\left(\frac{s}{w}\right) \geqslant u\right].$$

Now the stationary subgroup of any orbit in Q_0 is Γ_4; that of any other orbit is K. Thus the distinct elements of $\Gamma_4\dagger$ are the following:

(A) the one-dimensional ψ_{u_1,u_2} (u_1, u_2 real), where

$$\psi_{u_1,u_2}\left(E\left(\sum_{i=1}^{4} r_i x_i\right)\right) = e^{i(u_1r_1+u_2r_2)};$$

(B) the T^v (v real $\neq 0$), where T^v is the representation induced from the character $\chi_{0,v,0}$ of K;

(C) the $S^{s,w}$ (s, w real, $w \neq 0$),where $S^{s,w}$ is induced from the character $\chi_{s/w,0,w}$ of K.

[19]See (2, § 2, Proposition 1).

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

Combining the above proposition with the arguments of the preceding examples, we obtain Dixmier's result for $\Gamma_4\dagger$:

THEOREM 5.4.[20] *A subset W of $\Gamma_4\dagger$ is closed if and only if:*

(I) *the sets* $\{(u_1, u_2)|\psi_{u_1,u_2} \in W\}$, $\{v|T^v \in W\}$, *and* $\{(s, w)|S^{s,w} \in W\}$ *are closed in* R^2, $R - \{0\}$, *and* $R \times (R - \{0\})$ *respectively;*

(II) *if* $\{v|T^v \in W\}$ *has* 0 *as a limit point, all* ψ_{u_1,u_2} *are in* W;

(III) *if* v *is real,* $v \neq 0$, *and* $(-(v^2/2), 0)$ *is a limit point of* $\{(s, w)|S^{s,w} \in W\}$, *then* $T^v \in W$;

(IV) *if* $(0, 0)$ *is a limit point of* $\{(s, w)|w > 0, S^{s,w} \in W\}$ *and*

$$\varliminf_{(s,w)\to(0,0),\, w>0,\, S^{s,w}\in W} (s/w) \leqslant u,$$

then $\psi_{r,u} \in W$ *for all real* r;

(V) *if* $(0, 0)$ *is a limit point of* $\{(s, w)|w < 0, S^{s,w} \in W\}$ *and*

$$\varlimsup_{(s,w)\to(0,0),\, w<0,\, S^{s,w}\in W} (s/w) \geqslant u,$$

then $\psi_{r,u} \in W$ *for all real* r.

Example 5. *The group* $\Gamma_{5,3}$. This is the five-dimensional simply-connected group whose Lie algebra has basis x_1, x_2, x_3, x_4, x_5, with $[x_1, x_2] = x_4$, $[x_1, x_4] = x_5$, $[x_2, x_3] = x_5$, and $[x_i, x_j] = 0$ for all other $i < j$. Let $C = E(\langle x_5\rangle)$ be the centre of $\Gamma_{5,3}$, and $K = E(\langle x_3, x_4, x_5\rangle)$; K is a closed normal abelian subgroup of $\Gamma_{5,3}$. By the usual method of induced representations, we find that the elements of $\Gamma_{5,3}$ fall into two classes:

(A) Those which are the identity on C, that is, which are lifted from irreducible representations of $\Gamma_{5,3}/C \cong \Gamma_3 \times R$ (this class will be referred to as $(\Gamma_{5,3}/C)^\wedge$);

(B) the S^w (w real $\neq 0$), where S^w is induced from the character of K which sends $E(r_3x_3 + r_4x_4 + r_5x_5)$ into e^{iwr_5}.

Let $\chi_{s,t,w}$ be the character of K sending $E(r_3x_3 + r_4x_4 + r_5x_5)$ into $e^{i(sr_3+tr_4+wr_5)}$. The orbit in $K\dagger$ corresponding to S^w ($w \neq 0$) is $\{\chi_{s,t,w}|s, t \text{ real}\}$. Further, we easily verify that each representation in $(\Gamma_{5,3}/C)^\wedge$, when restricted to K, is weakly contained in $\{\chi_{s,t,0}|s, t \text{ real}\}$. Thus applying Theorem 4.3 as before, we obtain Dixmier's result:

THEOREM 5.5.[21] *A subset W of $\Gamma_{5,3}\dagger$ is closed if and only if*

(I) $\{w|S^w \in W\}$ *is closed in* $R - \{0\}$;

(II) $W \cap (\Gamma_{5,3}/C)^\wedge$ *is closed in* $(\Gamma_{5,3}/C)^\wedge$ (*the topology of the latter is known from Example* 3, *since* $\Gamma_{5,3}/C \cong \Gamma_3 \times R$);

(III) *if* $\{w|S^w \in W\}$ *has* 0 *as a limit point,* W *contains* $(\Gamma_{5,3}/C)^\wedge$.

Example 6. *The group* $\Gamma_{5,4}$. This is the five-dimensional simply-connected group whose Lie algebra has basis x_1, x_2, x_3, x_4, x_5, with $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_2, x_3] = x_5$, and $[x_i, x_j] = 0$ for all other $i < j$.

[20]See (2, § 3, Proposition 2).
[21]See (2, § 6, Proposition 5).

The dual of this group does not yield to Theorem 4.3 for the following reason. In the previous examples there was a small number of normal subgroups $K_1, K_2, \ldots$, such that each irreducible representation of the whole group G was either induced from a character of some K_i, or lifted from some G/K_i. In the case of $\Gamma_{5,4}$, however, it is found that the subgroups K, from which the elements of $\Gamma_{5,4}\dagger$ are obtained by lifting or inducing, vary continuously—a situation definitely not covered by Theorem 4.3. However, the day is saved by the existence of enough automorphisms of $\Gamma_{5,4}$ to carry these different subgroups into one fixed subgroup.

The centre of $\Gamma_{5,4}$ is $C = E(\langle x_4, x_5\rangle)$. Every T in $\Gamma_{5,4}\dagger$ reduces on C to a character times the identity operator, hence reduces to the identity on some one-dimensional subgroup of C. Now, for θ real, let F_θ be the automorphism of $\Gamma_{5,4}$ corresponding to that automorphism of its Lie algebra which sends x_i into x_i', where: $x_1' = x_1 \cos\theta + x_2 \sin\theta$, $x_2' = -x_1 \sin\theta + x_2\cos\theta$, $x_3' = x_3$, $x_4' = x_4\cos\theta + x_5 \sin\theta$, $x_5' = -x_4\sin\theta + x_5\cos\theta$. Every one-dimensional subgroup of C is carried by a suitable F_θ into $E(\langle x_5\rangle)$. We now verify the following propositions:

(1) The map $(S, \theta) \to S \bigcirc F_\theta$ (where $\bigcirc$ denotes composition) is continuous on $\Gamma_{5,4}\dagger \times R$ to $\Gamma_{5,4}\dagger$;

(2) every T in $\Gamma_{5,4}\dagger$ is of the form $S\bigcirc F_\theta$, where θ is real and S is lifted from an element of the dual of $\Gamma_{5,4}/E(\langle x_5\rangle)$.

Observe that $\Gamma_{5,4}/E(\langle x_5\rangle) \cong \Gamma_4$. We shall identify the dual of $\Gamma_{5,4}/E(\langle x_5\rangle)$ with $\Gamma_4\dagger$; thus ψ_{u_1,u_2}, T^v, $S^{s,w}$ (see Example 4) become irreducible representations of $\Gamma_{5,4}$, reducing to the identity on $E(\langle x_5\rangle)$.

(3) $S^{s,w}\bigcirc F_\theta \cong S^{s,-w}\bigcirc F_{\theta+\pi}$. Thus we lose none of the $S^{s,w}\bigcirc F_\theta$ by assuming $w > 0$. Assuming this, we have $S^{s,w}\bigcirc F_\theta \cong S^{s',w'}\bigcirc F_{\theta'}$ if and only if $s = s'$, $w = w'$, and $\theta \equiv \theta' \pmod{2\pi}$.

(4) For all real θ, u_1, u_2, and all real $v \neq 0$,

$$T^v \bigcirc F_\theta \cong T^v,$$

$$\psi_{u_1,u_2} \bigcirc F_\theta = \psi_{u_1\cos\theta + u_2 \sin\theta, -u_1\sin\theta+u_2\cos\theta}.$$

In view of these results, the distinct elements of $\Gamma_{5,4}\dagger$ are the following:

(A) The ψ_{u_1,u_2} (u_1, u_2 real);

(B) the T^v (v real $\neq 0$);

(C) the $S^{s,w}\bigcirc F_\theta$ (s, w real, $w > 0$, θ in the reals modulo 2π).

Using the above propositions (1)–(4), the compactness of the group of the F_θ, and the known topology of $\Gamma_4\dagger$, we obtain the following result:

THEOREM 5.6. *A subset W of $\Gamma_{5,4}\dagger$ is closed if and only if:*

(I) $\{(u_1, u_2) | \psi_{u_1,u_2} \in W\}$, $\{v | T^v \in W\}$, *and* $\{(s, w, \theta) | S^{s,w}\bigcirc F_\theta \in W\}$ *are closed in* R^2, $R - \{0\}$, *and* $R \times (R - \{0\}) \times (R \bmod 2\pi)$ *respectively;*

(II) *if* 0 *is a limit point of* $\{v | T^v \in W\}$, *W contains all* ψ_{u_1,u_2};

(III) *if* $(-(v^2/2), 0)$ *is a limit point of* $N = \{(s, w) | S^{s,w}\bigcirc F_\theta \in W$ *for some* $\theta\}$, *then* $T^v \in W$;

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press

(IV) *if* $(0,0)$ *is a limit point of* N *and if*

$$\lim_{(s,w)\to(0,0),\,w>0,\,S^{s,w}\bigcirc F_\theta\in W}\left(\frac{s}{w}-u_1\sin\theta-u_2\cos\theta\right)\leqslant 0,$$

then $\psi_{u_1,u_2}\in W$.

In conclusion, we repeat that the procedure used for $\Gamma_{5,4}$ was quite special, and depended on the existence of "enough" automorphisms of the group. There surely exists a general inductive procedure, perhaps similar to that of **(12)**, guaranteed to give the topology of the dual of any simply connected nilpotent group. Such a procedure might well require a generalization of Theorem 4.3 which, in line with the remark at the beginning of Example 6, would cover the case of continuously varying subgroups.

Indeed, the following more comprehensive problem arises naturally: How can we generalize Theorem 4.3 so that, whenever Mackey's theory of induced representations permits us to catalogue the elements of the dual $G\dagger$ of a group G, we obtain at the same time the topology of $G\dagger$?

References

1. J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents*, III, Can. J. Math., *10* (1958), 321–348.

2. ——— *Sur les représentations unitaires des groupes de Lie nilpotents*, VI, Can. J. Math., *12* (1960), 324–352.

3. J. M. G. Fell, *The dual spaces of C*-algebras*, Trans. Amer. Math. Soc., *94* (1960), 365–403.

4. ——— *C*-algebras with smooth dual*, Illinois J. Math., *4* (1960), 221–230.

5. J. C. Glimm, *Type I C*-algebras*, Ann. Math. *73* (1961), 572–612.

6. R. Godement, *Les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc., *63* (1948), 1–84.

7. G. W. Mackey, *Induced representations of locally compact groups* I, Ann. Math., *55* (1952), 101–139.

8. ——— *Induced representations of locally compact groups* II. *The Frobenius reciprocity theorem*, Ann. Math., *58* (1953), 193–220.

9. ——— *Borel structure in groups and their duals*, Trans. Amer. Math. Soc., *85* (1957), 134–165.

10. Edward Nelson and W. Forrest Stinespring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math., *81* (1959), 547–559.

11. O. Takenouchi, *Sur une classe de fonctions continues de type positif sur un groupe localement compact*, Math. J. Okayama Univ., *4* (1955), 143–173.

12. ——— *Sur la facteur-représentation d'un groupe de Lie resoluble de type (E)*, Math. J. Okayama Univ., *7* (1957), 151–161.

University of Washington

https://doi.org/10.4153/CJM-1962-016-6 Published online by Cambridge University Press