ONE-RELATOR GROUPS WITH CENTER

Dedicated to the memory of Hanna Neumann

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ABSTRACT. Many one-relator groups with center have been shown to be of the form $\langle x_1, x_2, \cdots, x_{t+1}; x_1^{P_1} = x_2^{Q_1}, x_2^{P_2} = x_3^{Q_2}, \cdots, x_t^{P_t} = x_{t+1}^{Q_t} \rangle$. A necessary and a sufficient condition for the sequence $(P_1, Q_1, P_2, Q_2, \cdots, P_t, Q_t)$ are given in order for groups of the above form to be one-relator groups.

1. Introduction

One relator groups with center have been discussed in [1], [2] and [4]. Recently Pietrowski [5] has shown that any non-abelian one-relator group G with a non-trivial center such that G/G' is not free abelian of rank 2 can be presented by

$$(1) \qquad G = \langle x_1, x_2, \cdots, x_{t+1}; x_1^{P_1} = x_2^{Q_1}, x_2^{P_2} = x_3^{Q_2}, \cdots, x_t^{P_t} = x_{t+1}^{Q_t} \rangle.$$

The groups G with G/G' free abelian of rank 2 imbed those of the form (1) in a natural way. Conversely, groups of the form (1) do have non-trivial centers. Thus we are now faced with a new problem; i.e., which of the groups (1) are one-relator groups.

In this note we present two partial results giving respectively a necessary and a sufficient numerical condition on the ordered set of integers $(P_1, Q_1, P_2, Q_2, \cdots, P_t, Q_t)$ for (1) to be a one-relator group. The gap between these results can be illustrated by the ordered set $(2, 2, 5, 5, 3, 3,)$ for which the authors cannot decide whether (1) is a one-relator group or not.

It will be convenient to assume in the discussion below that all the integers P_i and Q_i are strictly greater than 1.

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2. A necessary condition

In [4] it is shown that if the group G in (1) is a one-relator group then G can be generated by two of its elements. The following theorem translates this necessary condition into a numerical condition.

THEOREM 1. *Let G be presented by* (1). *Then the following statements are equivalent.*

(a) *G is a two generator group.*

(b) *$gcd\,(Q_i, P_j) = 1$ for all i,j, $1 \leqq i < j \leqq t$.*

(c) $G = < x_1, x_{t+1}; x_1^{P_1P_2\cdots P_t} = x_{t+1}^{Q_1Q_2\cdots Q_t}, [x_1^{P_1P_2\cdots P_{k-1}}, x_{t+1}^{Q_k\cdots Q_t}] = 1,$

$k = 2, \cdots, t >.$

PROOF. (c) $\Rightarrow$ (a) is obvious.

(b) $\Rightarrow$ (c). First of all (b) is equivalent to

$$gcd(Q_1Q_2\cdots Q_{k-1}, P_k \cdots P_t) = 1, k = 2, \cdots, t. \tag{2}$$

Thus for each k there exists integers a_k and b_k such that

$$1 = a_kQ_1Q_2\cdots Q_{k-1} + b_kP_k\cdots P_t. \tag{3}$$

The relations in (1) thus imply for $k = 2, \cdots, t$

$$x_k = x_k^{a_kQ_1Q_2\cdots Q_{k-1}} x_k^{b_kP_k\cdots P_t} = x_1^{a_kP_1P_2\cdots P_{k-1}} x_{t+1}^{b_kQ_k\cdots Q_t} \tag{4}$$

The relations in (1) also imply

$$[x_1^{P_1P_2\cdots P_{k-1}}, x_{t+1}^{Q_k\cdots Q_t}] = 1, k = 2, \cdots, t, \tag{5}$$

and

$$x_1^{P_1P_2\cdots P_t} = x_{t+1}^{Q_1\cdots Q_t}. \tag{6}$$

By using Tietze transformations (see [3], page 48), we can add relations (4), (5) and (6) to the relations in (1). We can now delete the original relations in (1), $x_i^{P_i} = x_{i+1}^{Q_i}$, $i = 1, 2, \cdots, t$, if we can show that they are implied by (4), (5), and (6). Having done this the relations (4) and the generators $x_2, \cdots, x_t$ may be deleted leaving us with the presentation (c).

We prove inductively that for each integer n, $1 \leqq n \leqq t$, (4), (5) and (6) imply

$$x_i^{P_i} = x_{i+1}^{Q_i} \text{ for all } i < n \tag{7}$$

and

$$x_n^{P_nP_{n+1}\cdots P_t} = x_{t+1}^{Q_n\cdots Q_t}. \tag{8}$$

The result we wish is the case $n = t$.

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Statements (7) and (8) clearly hold when $n = 1$.

Suppose that $n > 1$. Then by induction we have

$$x_1^{P_1P_2\cdots P_{n-2}} = x_{n-1}^{Q_1Q_2\cdots Q_{n-2}} \quad \text{and} \quad x_{n-1}^{P_{n-1}P_n\cdots P_t} = x_{t+1}^{Q_{n-1}\cdots Q_t}$$

Using these in connection with (3), (5) and (4) we have

$$x_{n-1}^{P_{n-1}} = x_{n-1}^{a_nQ_1\cdots Q_{n-1}P_{n-1}} x_{n-1}^{b_nP_{n-1}P_n\cdots P_t} = (x_1^{a_nP_1P_2\cdots P_{n-1}} x_{t+1}^{b_nQ_n\cdots Q_t})^{Q_{n-1}} = x_n^{Q_{n-1}}$$

In a similar manner

$$x_n^{P_n\cdots P_t} = x_1^{a_nP_1P_2\cdots P_t} x_{t+1}^{b_nQ_n\cdots Q_tP_n\cdots P_t} = x_{t+1}^{(a_nQ_1Q_2\cdots Q_{n-1}+b_nP_n\cdots P_t)Q_n\cdots Q_t} = x_{t+1}^{Q_n\cdots Q_t}.$$

This completes the induction and the proof that (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b). Again we proceed inductively and show that for each integer n, $1 \leqq n \leqq t$,

$$(9) \qquad gcd\,(Q_i, P_j) = 1 \text{ for all } i,j, \quad 1 \leqq i < j \leqq n.$$

Again the result we are after is the case $n = t$.

Statement (9) holds vacuously when $n = 1$.

Suppose that $n > 1$. Then by induction and by using (b) $\Rightarrow$ (c) we see that G can be presented by

$$(10) \quad G = \langle x_1, x_n, x_{n+1}, \cdots, x_{t+1}; x_1^{P_1P_2\cdots P_{n-1}} = x_n^{Q_1\cdots Q_{n-1}},$$
$$x_n^{P_n} = x_{n+1}^{Q_n}, \cdots, x_t^{P_t} = x_{t+1}^{Q_t},$$
$$[x_1^{P_1P_2\cdots P_{k-1}}, x_n^{Q_k\cdots Q_{n-1}}] = 1, k = 2, \cdots, n-1 \rangle.$$

Now we add to (10) the relations $x_1^{P_1} = 1$ and $x_{n+1} = x_{n+2} = \cdots = x_t = 1$ and obtain a homomorphic image $\bar{G}$ of G which is the free product of three groups

$$G_1 = \langle x_1; x_1^{P_1} = 1\rangle, G_2 = \langle x_n; x_n^{Q_1Q_2\cdots Q_{n-1}} = 1, x_n^{P_n} = 1\rangle$$

and

$$G_3 = \langle x_{t+1}; x_{t+1}^{Q_t} = 1\rangle.$$

Since G is a two generator group so is $\bar{G}$. But the number of generators needed for $\bar{G}$ is the sum of the numbers needed for G_1, G_2 and G_3 (see [3], page 192). Since G_1 and G_3 are clearly non-trivial, G_2 must be trivial which implies $gcd\,(Q_1Q_2\cdots Q_{n-1}, P_n) = 1$. The result follows and Theorem 1 is proved.

3. A sufficient condition

We will show in Lemma 2 that for $t = 2$ the necessary condition above is also sufficient. Using that as a starting point we can, by using Lemma 1, add new generators one at a time to (1) to obtain new one-relator groups.

LEMMA 1. *Suppose $x^P = y^Q$ in the one-relator group $\langle x, y; R(x,y) = 1\rangle$, $P_0 = \pm 1 \bmod Q, Q_0$ is any integer, and*

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(11) $$G = \langle x, y, z;\ R(x,y) = 1, y^{P_0} = z^{Q_0}\rangle.$$

Then $x^{PP_0} = z^{QQ_0}$ *in* G *and for some integer* n,

(12) $$G = \langle x, z;\ R(x, x^{nP}z^{\pm Q_0}) = 1\rangle;$$

i.e., G *is also a one-relator group.*

PROOF. Since $1 = \pm P_0 + nQ$ for some integer n, it follows that

(13) $$y = y^{nQ}y^{\pm P_0} = x^{nP}z^{\pm Q_0},$$

(14) $$R(x, x^{nP}z^{\pm Q_0}) = 1$$

and

(15) $$(x^{nP}z^{\pm Q_0})^{P_0} = z^{Q_0}$$

are relations in G. Hence, we can adjoin (13), (14) and (15) to the relations in (11) We may now delete the original relations from (11) and then (13) along with the generator y. If we can show that (14) implies (15) then (15) can also be deleted and we will have the presentation (12).

Since $R(x,y) = 1$ implies $x^P = y^Q$, it follows that (14) implies

(16) $$x^P = (x^{nP}z^{\pm Q_0})^Q.$$

Now (16) implies that x^P is a power of $x^{nP}z^{\pm Q_0}$. Therefore x^P commutes with $x^{nP}z^{\pm Q_0}$ and hence also with z^{Q_0}. Thus, from (16) we obtain $x^{P(1-nQ)} = z^{\pm QQ_0}$; hence $x^{PP_0} = z^{QQ_0}$. However (15) is just a rearrangement of $x^{nPP_0} = z^{nQQ_0} = z^{Q_0(1\mp P_0)}$ and the conclusion follows.

Suppose that $gcd\ (L, M) = 1$. In the free group on free generators a and b, let $p_{L,M}(a,b)$ be the unique primitive, up to conjugacy, with exponent sum L on a and M on b. Thus $\langle a, b; p_{L,M}(a,b) = 1\rangle$ is infinite cyclic. Hence $p_{L,M}(a,b) = 1$ implies that a and b commute and thus $p_{L,M}(a,b) = 1$ also implies $a^L = b^{-M}$. Conversely $[a,b] = 1$ and $a^L = b^{-M}$ imply $p_{L,M}(a,b) = 1$.

Now suppose G is as in with (1) $t = 2$ and $gcd\ (Q_1, P_2) = 1$. Then by Theorem 1

$$G = \langle x_1, x_3; x_1^{P_1P_2} = x_3^{Q_1Q_2}, [x_1^{P_1}, x_3^{Q_2}] = 1\rangle.$$

By the above discussion it follows that

$$G = \langle x_1, x_3; p_{Q_2,Q_1}(x_1^{P_1}, x_3^{-Q_2}) = 1\rangle.$$

Thus we have proved

LEMMA 2. *If* G *is given by* (1) *and* $t = 2$ *then* G *is a one-relator group if and only if* $gcd\ (Q_1, P_2) = 1$.

By combining Lemmas 1 and 2 we have

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THEOREM 2. *Suppose G is given by (1). Then G is a one-relator group if there exists a sequence of pairs of integers,*

$$(\lambda_1, \mu_1), \cdots, (\lambda_{t-1}, \mu_{t-1}), \lambda_i, \mu_i \in \{l, \cdots, t\} \text{ for all } i = 1, \cdots, t-1,$$

such that

$$\lambda_1 + 1 = \mu_1 \text{ and } \gcd (Q_{\lambda_1}, P_{\mu_1}) = 1$$

and if $t > 2$ then for each $i = 1, \cdots, t-2$,

either

$$\lambda_{i+1} = \lambda_i - 1, \mu_{i+1} = \mu_i \text{ and } Q_{\lambda_{i+1}} = \pm 1 \mod (P_{\lambda_i} P_{\lambda_i + 1} \cdots P_{\mu_i})$$

or

$$\lambda_{i+1} = \lambda_i, \mu_{i+1} = \mu_i + 1 \text{ and } P_{\mu_{i+1}} = \pm 1 \mod (Q_{\lambda_i} Q_{\lambda_i + 1} \cdots Q_{\mu_i}).$$

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