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ROSENTHAL SETS AND THE RADON-NIKODYM PROPERTY

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Abstract

Let X be a complex Banach space, G a compact abelian metrizable group and Λ a subset of $\widehat{G}$, the dual group of G. If X has the Radon-Nikodym property and $L^\infty_\Lambda(G; X)$ is separable, then $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property. One consequence of this is that $C_\Lambda(G, X)$ has the Radon-Nikodym property whenever X has the Radon-Nikodym property and the Schur property and Λ is a Rosenthal set. A partial stability property for products of Rosenthal sets is also obtained.

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1. Introduction

For Banach spaces X and Y we denote by $\mathscr{L}(X, Y)$ the Banach space of all bounded linear operators from X into Y. In this set-up, Pettis's last theorem states: if $\mathscr{L}(X, Y^*)$ is separable, then $\mathscr{L}(X, Y^*)$ has the Radon-Nikodym property [3, p. 165]. Notice that if $\mathscr{L}(X, Y^*)$ is separable, then X^* and Y^* are separable and so X^* and Y^* have the Radon-Nikodym property.

In this note, we will prove a result which is a variant of Pettis's last theorem. In our case, we will replace X by a particular quotient of L^1 and Y^* by an arbitrary Banach space with the Radon-Nikodym property. To be more specific, if G is a compact abelian metrizable group and Λ is a Rosenthal subset of $\widehat{G}$, then $L^\infty_\Lambda(G)$ is a separable dual space. If we let $\Lambda' = \{\gamma \in \widehat{G} : \overline{\gamma} \notin \Lambda\}$, then $L^\infty_\Lambda(G)$ is the dual of $L^1(G)/L^1_{\Lambda'}(G)$. We will show that $\mathscr{L}(L^1(G)/L^1_{\Lambda'}(G), X)$ has the Radon-Nikodym property whenever X has the Radon-Nikodym property and $\mathscr{L}(L^1(G)/L^1_{\Lambda'}(G), X)$ is separable. In fact, a close analysis of Lemma 4 of [2, p. 62] yields that proving the above result is equivalent to proving that $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property whenever X has the Radon-Nikodym property and $L^\infty_\Lambda(G, X)$ is separable. This is the first result we prove in the next section.

One consequence of these results is that if Λ is a Rosenthal subset of $\widehat{G}$ and X is a complex Banach space with the Radon-Nikodym property and the Schur property, then $C_\Lambda(G, X)$ has the Radon-Nikodym property. It is unknown, at the moment, if the restriction of X having the Schur property is necessary. Actually, a related question, which is also still unanswered, is the following: if G_1 and G_2 are two compact abelian metrizable groups and Λ_1 and Λ_2 are two Rosenthal subsets of $\widehat{G}_1$ and $\widehat{G}_2$, respectively, is $\Lambda_1 \times \Lambda_2$ a Rosenthal subset of $\widehat{G}_1 \times \widehat{G}_2$? However, we will be able to show that if Λ_1 is a Rosenthal set and Λ_2 is such that $L^\infty_{\Lambda_2}(G_2)$ has the Schur property, then $\Lambda_1 \times \Lambda_2$ is a Rosenthal set. The question of whether the product of two Rosenthal sets is again a Rosenthal set was first raised by F. Lust-Piquard.

2. The results

Throughout this section, G will denote a compact abelian metrizable group, $\mathscr{B}(G)$ denotes the σ-algebra of Borel subsets of G and λ is normalised Haar measure on G. The dual group of G is denoted by $\widehat{G}$. If f is a function defined on G and $\gamma \in \widehat{G}$, we define $\hat{f}(\gamma)$ by

$$\hat{f}(\gamma) = \int_G \overline{\gamma(g)} f(g) d\lambda(g).$$

For a subset Λ of $\widehat{G}$, we define

$$C_\Lambda(G) = \{f \in C(G) : \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\}$$

and

$$L^\infty_\Lambda(G) = \{f \in L^\infty(G) : \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\}.$$

A subset Λ of $\widehat{G}$ is called a Rosenthal set if $L^\infty_\Lambda(G) = C_\Lambda(G)$. A result of Lust-Piquard [5] says that Λ is a Rosenthal set if and only if $C_\Lambda(G)$ has the Radon-Nikodym property.

For a complex Banach space X, we define

$$L^\infty_\Lambda(G, X) = \{f \in L^\infty(G, X) : \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\},$$

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where $L^\infty(G, X)$ is the space of all (equivalence classes of) X-valued Bochner integrable functions defined on G that are essentially bounded. $C_\Lambda(G, X)$ can be defined in a similar manner.

Theorem. *Let G be a compact abelian metrizable group, let Λ be a subset of $\widehat{G}$ and let X be a complex Banach space. If X has the Radon-Nikodym property and $L^\infty_\Lambda(G, X)$ is separable, then $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property.*

Proof. To show that $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property it suffices to show that every bounded linear operator from $L^1[0, 1]$ into $L^\infty_\Lambda(G, X)$ is Bochner representable [2]. So, let $T: L^1[0, 1] \to L^\infty_\Lambda(G, X)$ be a bounded linear operator. For a Lebesgue measurable subset A of $[0, 1]$ and $B \in \mathscr{B}(G)$, define

$$\mu(A \times B) = \int_B T(1_A)(g)d\lambda(g).$$

If we let $\mathscr{R}$ denote the algebra generated by sets of the form $A \times B$, where A is a Lebesgue measurable subset of $[0, 1]$ and $B \in \mathscr{B}(G)$, then it is easily shown that μ is a finitely additive X-valued measure on $\mathscr{R}$. Also,

$$\begin{aligned}\|\mu(A \times B)\| &\le \int_B \|T(1_A)(g)\|_X d\lambda(g) \le \int_B \|T(1_A)\|_{L^\infty_\Lambda(G,X)} d\lambda(g)\\ &= \lambda(B)\|T(1_A)\|_{L^\infty_\Lambda(G,X)}\\ &\le \lambda(B)\|T\|\,\|1_A\|_{L^1[0,1]} = \lambda(B)\|T\|m(A)\\ &= \|T\|(m \times \lambda)(A \times B),\end{aligned}$$

where m is Lebesgue measure on $[0, 1]$. Thus $\|\mu(R)\| \le \|T\|(m \times \lambda)(R)$ for all $R \in \mathscr{R}$.

If we let $\mathscr{A}$ denote the σ-algebra of Lebesgue measurable subsets of $[0, 1]$, then the product σ-algebra, $\mathscr{A} \times \mathscr{B}(G)$, is generated by $\mathscr{R}$. Also, $m \times \lambda$ is a non-negative finite countably additive measure on $\mathscr{A} \times \mathscr{B}(G)$. For $E, F \in \mathscr{A} \times \mathscr{B}(G)$, define $d(E, F) = (m \times \lambda)(E \Delta F)$. Then $(\mathscr{A} \times \mathscr{B}(G), d)$ is a pseudo-metric space and $\mathscr{R}$ is dense in $(\mathscr{A} \times \mathscr{B}(G), d)$. For E, $F \in \mathscr{R}$, $\mu(E) - \mu(F) = \mu(E \backslash F) - \mu(F \backslash E)$. Therefore,

$$\begin{aligned}\|\mu(E) - \mu(F)\| &\le \|\mu(E\backslash F)\| + \|\mu(F\backslash E)\|\\ &\le \|T\|(m \times \lambda)(E\backslash F) + \|T\|(m \times \lambda)(F\backslash E)\\ &= \|T\|(m \times \lambda)(E\Delta F)\\ &= \|T\|d(E, F).\end{aligned}$$

Consequently, $\mu: (\mathscr{R}, d) \to (X, \|\cdot\|)$ is a Lipschitz function and since $(\mathscr{R}, d)$ is dense in $\mathscr{A} \times \mathscr{B}(G), d)$ there is a Lipschitz extension of μ,

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$\overline{\mu}: (\mathscr{A} \times \mathscr{B}(G), d) \to (X, \|\cdot\|)$, such that $\overline{\mu}(R) = \mu(R)$ for all $R \in \mathscr{R}$. Thus, $\overline{\mu}$ is an X-valued countably additive measure on $\mathscr{A} \times \mathscr{B}(G)$ and $\|\overline{\mu}(S)\| \leq \|T\|(m \times \lambda)(S)$ for all $S \in \mathscr{A} \times \mathscr{B}(G)$. Hence, $\overline{\mu}$ is an X-valued measure of bounded average range and since X has the Radon-Nikodym property there exists $F \in L^\infty([0, 1] \times G, X)$ such that

$$\overline{\mu}(S) = \int_S F(t, g)d(m \times \lambda)(t, g)$$

for all $S \in \mathscr{A} \times \mathscr{B}(G)$.

In particular, for $A \in \mathscr{A}$ and $B \in \mathscr{B}(G)$,

$$\int_B T(1_A)(g)d\lambda(g) = \mu(A \times B) = \overline{\mu}(A \times B) = \int_{A \times B} F(t, g)d(m \times \lambda)(t, g)$$
$$= \int_B \int_A F(t, g)dm(t)d\lambda(g).$$

Now, for $x^* \in X^*$, define

$$T_{x^*}: L^1[0, 1] \to L^\infty_\Lambda(G) \quad \text{by } (T_{x^*}f)(g) = x^*((Tf)(g))$$

for $g \in G$ and $f \in L^1[0, 1]$. T_{x^*} is a bounded linear operator and $L^\infty_\Lambda(G)$ has the Radon-Nikodym property (this is so because $L^\infty_\Lambda(G)$ is a dual space which is separable since it is a subspace of the separable space $L^\infty_\Lambda(G, X)$). Therefore, there is $F_{x^*} \in L^\infty([0, 1], L^\infty_\Lambda(G))$ such that

$$T_{x^*}(f) = \int_{[0,1]} f(t)F_{x^*}(t)dm(t)$$

for all $f \in L^1[0, 1]$. In particular, if $A \in \mathscr{A}$ and $g \in G$, then

$$x^*(T(1_A)(g)) = (T_{x^*}(1_A))(g) = \left[\int_{[0,1]} 1_A(t)F_{x^*}(t)dm(t)\right](g)$$
$$= \int_A (F_{x^*}(t))(g)dm(t).$$

Thus for $A \in \mathscr{A}$, $B \in \mathscr{B}(G)$ and $x^* \in X^*$

$$\int_B x^*(T(1_A)(g))d\lambda(g) = \int_B \int_A (F_{x^*}(t))(g)dm(t)d\lambda(g).$$

Consequently, for all $A \in \mathscr{A}$, $B \in \mathscr{B}(G)$ and $x^* \in X^*$

$$\int_B \int_A x^*(F(t, g))dm(t)d\lambda(g) = x^* \left[\int_B \int_A F(t, g)dm(t)d\lambda(g)\right]$$
$$= x^*(\mu(A \times B))$$
$$= x^* \left[\int_B T(1_A)(g)d\lambda(g)\right] = \int_B x^*(T(1_A)(g)d\lambda(g)$$
$$= \int_B \int_A (F_{x^*}(t))(g)dm(t)d\lambda(g).$$

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The function $F_{x^*} \in L^\infty([0, 1], L^\infty_\Lambda(G))$, so by [4, p. 198], Theorem 17, there exists $H_{x^*}: [0, 1] \times G \to \mathbb{C}$ which is $m \times \lambda$-measurable and such that $F_{x^*}(t) = H_{x^*}(t, \cdot)$ for m-almost all $t \in [0, 1]$. From this we get that for all $A \in \mathscr{A}$ and $B \in \mathscr{B}(G)$

$$\int_B \int_A x^*(F(t, g)) dm(t) d\lambda(g) = \int_B \int_A H_{x^*}(t, g) dm(t) d\lambda(g).$$

Therefore $x^*(F(t, g)) = H_{x^*}(t, g)$ for $m \times \lambda$-almost all $(t, g) \in [0, 1] \times G$ (where the exceptional set of measure zero may vary with x^*). In particular, for m-almost all $t \in [0, 1]$, $x^*(F(t, g)) = H_{x^*}(t, g)$ for λ-almost all $g \in G$. This yields that for m-almost all $t \in [0, 1]$, $x^*(F(t, g)) = F_{x^*}(t))(g)$ for λ-almost all $g \in G$. Therefore, $x^*(F(t, \cdot))$ is λ-measurable for m-almost all $t \in [0, 1]$, where again we note that the exceptional set of measure zero may vary with x^*. However, since X is separable, there is a countable norming set $\{x_n^*\}_{n=1}^\infty$ in the unit ball of X^*. From this we have that for all $n \in \mathbb{N}$, $x_n^*(F(t, \cdot)$ is λ-measurable for m-almost all $t \in [0, 1]$. By Corollary 4 of [2, p. 42] we have that $F(t, \cdot)$ is λ-measurable for almost all $t \in [0, 1]$.

$$\text{Define } K(t) = \begin{cases} F(t, \cdot) & \text{if } F(t, \cdot) \text{ is } \lambda\text{-measurable}, \\ 0 & \text{otherwise.} \end{cases}$$

For each $t \in [0, 1]$, $K(t): G \to X$ is λ-measurable and for m-almost all $t \in [0, 1]$, $x^*(K(t)(g)) = x^*(F(t, g)) = (F_{x^*}(t))(g)$ for λ-almost all $g \in G$.

$$\|F_{x_n^*}\|_{L^\infty([0, 1], L^\infty_\Lambda(G))} = \|T_{x_n^*}\| \le \|x_{n^*}\| \, \|T\| \le \|T\|.$$

Therefore, for m-almost all $t \in [0, 1]$ and for all $n \in \mathbb{N}$, $\|F_{x_n^*}(t)\|_{L^\infty_\Lambda(G)} \le \|T\|$. Consequently, for m-almost all $t \in [0,1]$ and for all $n \in \mathbb{N}$, $|F_{x_n^*}(t)(g)| \le \|T\|$ for λ-almost all $g \in G$. From this we get that for m-almost all $t \in [0, 1]$ and for all $n \in \mathbb{N}$, $|x_n^*(K(t)(g))| \le \|T\|$ for λ-almost all $g \in G$. Since $\{x_n^*\}_{n=1}^\infty$ is a norming set we have that for m-almost all $t \in [0, 1]$, $\|K(t)(g)\|_X \le \|T\|$ for λ-almost all $g \in G$. Thus, for m-almost all $t \in [0, 1]$, $\|K(t)\|_{L^\infty(G, X)} \le \|T\|$.

Now define $K_1: [0, 1] \to L^\infty(G, X)$ by

$$K_1(t) = \begin{cases} K(t) & \text{if } \|K(t)\|_{L^\infty(G, X)} \le \|T\|, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \in [0, 1]$, $\gamma \notin \Lambda$ and $x^* \in X^*$

$$x^* \left[\widehat{K_1(t)}(\gamma)\right] = \int_G x^*(K_1(t)(g)) \overline{\gamma(g)} d\lambda(g).$$

If $\|K(t)\|_{L^\infty(G, X)} \le \|T\|$, then

$$x^*(\widehat{K_1(t)}(\gamma)) = \int_G x^*(K(t))(g)) \overline{\gamma(g)} d\lambda(g) = \int_G (F_{x^*}(t))(g) \overline{\gamma(g)} d\lambda(g)$$
$$= \widehat{F_{x^*}(t)}(\gamma) = 0$$

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since $F_{x^*}(t) \in L^\infty_\Lambda(G)$. If $\|K(t)\|_{L^\infty(G,X)} > \|T\|$, then $x^*(\widehat{K_1(t)}(\gamma)) = 0$. Hence $K_1 : [0, 1] \to L^\infty_\Lambda(G, X)$.

At this stage, we need to show K_1 is m-measurable. Since $L^\infty_\Lambda(G, X)$ is separable, it suffices to show that K_1 is scalarly measurable on a total set of linear functionals on $L^\infty_\Lambda(G, X)$ [1]. Note that for $\gamma \in \widehat{G}$ and $x^* \in X^*$, the map $x^* \otimes \gamma : L^\infty_\Lambda(G, X) \to \mathbb{C}$ defined by $(x^* \otimes \gamma)(f) = x^*\widehat{(f(\cdot))}(\gamma)$ is a bounded linear functional. The total set we will use is the set $\{x_n^* \otimes \gamma : n \in \mathbb{N}, \gamma \in \Lambda\}$ where $\{x_n^*\}$ is a norming set in X^*. Note that this set is countable since Λ is countable.

For $n \in \mathbb{N}$ and $\gamma \in \Lambda$,

$$(x_n^* \otimes \gamma)(K_1(t)) = \int_G x_n^*(K_1(t)(g))\overline{\gamma(g)}d\lambda(g) = \int_G x_n^*(F(t, g))\overline{\gamma(g)}d\lambda(g).$$

However, since $x_n^*(F(t, g))$ is a measurable function of t, so is the above integral and hence $(x_n^* \otimes \gamma)(K_1(t))$ is a measurable function of t. Therefore, K_1 is an m-measurable function.

To complete the proof we must show that for all $A \in \mathscr{A}$, $T(1_A) = \int_A K_1(t)dm(t)$. For $A \in \mathscr{A}$ and $B \in \mathscr{B}(G)$,

$$\int_B T(1_A)(g)d\lambda(g) = \int_B \int_A F(s, t)dm(t)d\lambda(g) = \int_B \int_A K_1(t)(g)dm(t)d\lambda(g).$$

Thus, for λ-almost all $g \in G$,

$$T(1_A)(g) = \int_A K_1(t)(g)dm(t)$$

and so as elements of $L^\infty_\lambda(G,X)$, $T(1_A) = \int_A K_1(t)dm(t)$. Hence $L^\infty_\Lambda(G,X)$ has the Radon-Nikodym property.

Remark. The converse of the theorem is also true: if $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property and X is separable, then $L^\infty_\Lambda(G, X)$ is separable. The proof can essentially be found in [5].

Corollary 1. *Let G be a compact abelian metrizable group, Λ a subset of $\widehat{G}$ and X a complex Banach space with the Radon-Nikodym property. If $\mathscr{L}(L^1(G)/L^1_{\Lambda'}(G), X)$ is separable, then it has the Radon-Nikodym property* ($\Lambda' = \{\gamma : \overline{\gamma} \notin \Lambda\}$).

Proof. Since X has the Radon-Nikodym property, $\mathscr{L}(L^1(G)/L^1_{\Lambda'}(G), X)$ is isometrically isomorphic to $L^\infty_\Lambda(G, X)$ [2, p. 63]. Now apply the Theorem.

Remark. A close analysis of the proof of Theorem 1 allows one to generalise Corollary 1 to obtain the following:

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Let (Ω, Σ, μ) be a probability space such that $L^1(\mu)$ is separable. Let Y be a closed subspace of $L^1(\mu)$. If X is a Banach space with the Radon-Nikodym property and if $\mathscr{L}(L^1(\mu)/Y, X)$ is separable, then $\mathscr{L}(L^1(\mu)/Y, X)$ has the Radon-Nikodym property.

COROLLARY 2. *Let G be a compact abelian metrizable group, let Λ be a subset of $\widehat{G}$ and let X be a complex Banach space. If Λ is a Rosenthal set and X has the Radon-Nikodym property and the Schur property, then $C_\Lambda(G, X)$ has the Radon-Nikodym property.*

PROOF. It suffices to show that every separable subspace of $C_\Lambda(G, X)$ has the Radon-Nikodym property. Notice that each separable subspace of $C_\Lambda(G, X)$ is a subspace of $C_\Lambda(G, Y)$ for some separable subspace Y of X. We will be finished if we can show that $C_\Lambda(G, Y)$ has the Radon-Nikodym property for every separable subspace Y of X.

Let Y be a separable subspace of X. We will show that $C_\Lambda(G, Y)$ is isomorphic to $L^\infty_\Lambda(G, Y)$. By Theorem 5 of [2, p. 63], $L^\infty_\Lambda(G, Y)$ is isomorphic to $\mathscr{L}(L^1(G)/L_{\Lambda'}(G), Y)$. However, since $L^\infty_\Lambda(G)$ is a separable dual, $L^1(G)/L^1_{\Lambda'}(G)$ does not contain a copy of ℓ^1. Hence every bounded linear operator from $L^1(G)/L^1_{\Lambda'}(G)$ to Y is compact because Y has the Schur property. $L^\infty_\Lambda(G)$ has the approximation property so the space of all compact operators from $L^1(G)/L^1_{\Lambda'}(G)$ to Y is isomorphic to $L^\infty_\Lambda(G)\check{\otimes}Y$. But $L^\infty_\Lambda(G)\check{\otimes}Y = C_\Lambda(G)\check{\otimes}Y$, which in turn is isomorphic to $C_\Lambda(G, Y)$. Thus we have shown that $C_\Lambda(G, Y)$ is isomorphic to $L^\infty_\Lambda(G, Y)$. $C_\Lambda(G, Y)$ is separable since Y is separable and so $L^\infty_\Lambda(G, Y)$ has the Radon-Nikodym property by the Theorem. Hence $C_\Lambda(G, Y)$ has the Radon-Nikodym property and so the proof is complete.

REMARK. In [6], Lust-Piquard shows that if G is a compact abelian metrizable group and Λ is a subset of $\widehat{G}$ is such that $L^\infty_\Lambda(G)$ has the Schur property, then Λ is a Rosenthal set.

COROLLARY 3. *Let G_1 and G_2 be two compact abelian metrizable groups and let Λ_1 and Λ_2 be subsets of $\widehat{G}_1$ and $\widehat{G}_2$, respectively. If Λ_1 is a Rosenthal set and $L^\infty_{\Lambda_2}(G_2)$ has the Schur property, then $\Lambda_1 \times \Lambda_2$ is a Rosenthal subset of $\widehat{G}_1 \times \widehat{G}_2$.*

PROOF. To show that $\Lambda_1 \times \Lambda_2$ is a Rosenthal set, it suffices to show that $C_{\Lambda_1 \times \Lambda_2}(G_1 \times G_2)$ has the Radon-Nikodym property [5]. $C_{\Lambda_1 \times \Lambda_2}(G_1 \times G_2)$ is isomorphic to $C_{\Lambda_1}(G_1, C_{\Lambda_2}(G_2))$. Since $L^\infty_{\Lambda_2}(G_2)$ has the Schur property,

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Λ_2 is a Rosenthal set, by the preceding Remark, so $C_{\Lambda_2}(G_2)$ has the Radon-Nikodym property. An application of Corollary 2 completes the proof.

REMARK. Corollary 3 was proved independently by F. Lust-Piquard in [8, Chapter 4, Theorem 5].

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