A Determinantal Expansion for a Class of Definite Integral[1]. Part 1

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1. Let $w(x)$ be a non-negative weight function for the finite interval (a, b) such that $\int_a^b w(x)\,dx$ exists and is positive, and let $T_r(x)$, $r = 0, 1, 2, \ldots$ be the corresponding orthonormal system of polynomials. Then if $F(x)$ is continuous on (a, b) and has "Fourier" coefficients

$$c_r = \int_a^b F(x)\,T_r(x)\,w(x)\,dx, \quad r = 0, 1, 2, \ldots,$$

Parseval's formula[2] gives

$$\int_a^b w(x)\,[F(x)]^2\,dx = \sum_{r=0}^{\infty} c_r^2. \tag{1}$$

We shall show that for the weight function $w(x)\,C(x)$ and $F(x) = A(x)/C(x)$, both satisfying the conditions above, $\sum_{r=0}^{s} c_r^2$, $s = 0, 1, 2, \ldots$, of Parseval's formula takes the form of a ratio of two determinants. The successive values of this determinantal ratio will be shown to provide a sequence of convergent approximants to the value of the integral. Moreover in the case when $C(x)$ is a polynomial, an expansion of the integral is given in terms of the roots of $C(x)$. The particular case when $C(x)$ is linear indicates the relation of the present method to the expression of integrals of the type $\int \frac{w(x)}{x+z}\,dx$ as continued fractions. The case when the range of integration is infinite is to be treated in Part 2.

2. Let $\theta_r(x) = \sum_{s=0}^{r} a_{rs} x^s$, $a_{rr} \neq 0$, be a polynomial of degree r in x, and

$$\int_a^b \theta_r(x)\,\theta_s(x)\,w(x)\,C(x)\,dx = \gamma_{rs} = \gamma_{sr}, \tag{2}$$

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[1] Applications have been given in *Biometrika*, 37 (1950), 111 and 38 (1951), 58.

[2] See for example D. Jackson, *Fourier Series and Orthogonal Polynomials* (*Carus Math. Mon.*, 1941), Ch. II, and p. 228, or G. Szegö, *Orthogonal Polynomials* (New York, 1939), Ch. III.

where it is assumed that $w(x)\,C(x)$ is non-negative and $\int_a^b w(x)\,C(x)\,dx$ exists and is positive.

Further let $p_r(x) = \sum_{s=0}^{r} p_s \theta_s(x)$ be an orthonormal system associated with the weight function $w(x)\,C(x)$ on (a, b), so that[1]

$$p_r(x) = \begin{vmatrix} \theta_0(x) & \theta_1(x) & \dots & \theta_r(x) \\ \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,r} \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,r} \\ \vdots & \vdots & & \vdots \\ \gamma_{r-1,0} & \gamma_{r-1,1} & \cdots & \gamma_{r-1,r} \end{vmatrix} \div \sqrt{(\Delta_{r-1}\Delta_r)}, \tag{3}$$

where $$\Delta_r \equiv |\gamma_{0,0},\ \gamma_{1,1},\ \dots,\ \gamma_{r,r}|.$$

Hence the Fourier coefficients for $A(x)/C(x)$, assumed continuous on (a, b), are given by

$$\begin{aligned} A_s &= \int_a^b \frac{A(x)}{C(x)}\, w(x)\, C(x)\, p_s(x)\, dx \quad (s = 0, 1, 2, \dots) \\ &= \int_a^b A(x)\, w(x)\, \frac{|\theta_0(x),\ \gamma_{01},\ \gamma_{12},\ \dots,\ \gamma_{s-1,s}|}{\sqrt{(\Delta_{s-1}\Delta_s)}}\, dx \\ &= |\alpha_0,\ \gamma_{01},\ \gamma_{12},\ \dots,\ \gamma_{s-1,s}| / \sqrt{(\Delta_{s-1}\Delta_s)}, \end{aligned} \tag{4}$$

where $$\alpha_r = \int_a^b \theta_r(x)\, w(x)\, A(x)\, dx \quad (r = 0, 1, 2, \dots). \tag{5}$$

Hence, using Parseval's theorem, we find

$$\int_a^b \frac{[A(x)]^2 w(x)}{C(x)}\, dx = \sum_{s=0}^{\infty} \frac{|\alpha_0,\ \gamma_{01},\ \gamma_{12},\ \dots,\ \gamma_{s-1,s}|^2}{\Delta_{s-1}\Delta_s} \tag{6}$$

which, by Schweins' theorem on determinants[2], may be written as

$$-\lim_{s\to\infty} \begin{vmatrix} 0 & \alpha_0 & \alpha_1 & \dots & \alpha_s \\ \alpha_0 & \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ \alpha_1 & \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_s & \gamma_{s0} & \gamma_{s1} & \cdots & \gamma_{ss} \end{vmatrix} \div |\gamma_{00},\ \gamma_{11},\ \dots,\ \gamma_{ss}|. \tag{7}$$

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[1] See Szegö, *loc. cit.*, Ch. II.

[2] Muir, *Theory of Determinants*, Pts. I and II (London, 1906), or A C. Aitken, *Determinants and Matrices* (Edinburgh, 1946).

In general the partial sums of the series (6) and the corresponding part of the determinantal ratio (7) form non-decreasing sequences. A more general result is found by applying Parseval's formula to the functions

$$\{A(x) \pm B(x)\}/C(x),$$

assumed continuous on (a, b), whence

$$\int_a^b \frac{A(x)\,B(x)\,w(x)}{C(x)}\,dx = \sum_{s=0}^{\infty} \frac{|d_0, \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}| \cdot |\beta_0, \gamma_{01}, \gamma_{12}, \ldots, \gamma_{s-1,s}|}{\Delta_{s-1}\Delta_s} \tag{8}$$

$$= -\lim_{s\to\infty} \begin{vmatrix} 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_s \\ \beta_0 & \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ \beta_1 & \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_s & \gamma_{s0} & \gamma_{s1} & \cdots & \gamma_{ss} \end{vmatrix} \div |\gamma_{00}, \gamma_{11}, \ldots, \gamma_{ss}|, \tag{9}$$

in which β_r is given by (5) with $B(x)$ replacing $A(x)$.

It is convenient to use a matrix notation in (9) and write

$$\int_a^b \frac{A(x)\,B(x)\,w(x)}{C(x)}\,dx = -\lim \begin{vmatrix} 0 & \alpha_s \\ \beta_s' & \gamma_s \end{vmatrix} \div |\gamma_s|. \tag{10}$$

The special case $A(x) = B(x) = I$, $C(x) = x + z$ is known in the theory of continued fractions[1] in which case, with $\theta_r(x)$ suitably restricted, the determinants are of "continuant" type. The formula (10) is thus seen as an extension of this type of continued fraction. Since the convergents of a continued fraction satisfy a second order difference equation, a generalised continued fraction might be one for which the "convergents" satisfied a difference equation of the n-th order, this suggestion being given by Fürstenau in "Ueber Kettenbrüche höherer Ordnung"[2]. The determinants in (10) do not in general appear to satisfy any simple difference equation. It is of interest to note that Rogers[3], in representing certain definite integrals as continued fractions, suggested that some form of algebraic fraction might exist for cases, such as $\int_0^\infty \frac{t^n e^{-xt}\,dt}{e^t - 1}$, which were intractable by his method.

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[1] See for example H. S. Wall, *Continued Fractions* (New York, 1948), Ch. XIII onwards, or O. Perron, *Die Lehre von den Kettenbrüchen* (Leipzig, 1913), Ch. 9.

[2] Muir, *Theory of Determinants*, Vol. III (London, 1920).

[3] L. J. Rogers, "Asymptotic series as convergent continued fractions", *Proc. London Math. Soc.* (2), 4 (1905–6).

3. We now turn to a further use of formula (10). Consider the determinant whose elements are definite integrals

$$\Delta = \left| \int_a^b \frac{A_j(x)\,A_k(x)\,w(x)\,dx}{C(x)} \right|_n \quad (j,\ k = 1,\ 2,\ \ldots,\ n), \tag{11}$$

in which $A_j(x)$, $j = 1, 2, 3, \ldots, n$, are continuous functions over (a, b) and $w(x)\,C(x)$ satisfies the same conditions as for (10). We may approximate to Δ by replacing each integral by the corresponding ratio (10) with the same s, and using the notation

$${}_j\alpha_k = \int_a^b \theta_k(x)\,A_j(x)\,w(x)\,dx \quad (j = 1,\ 2,\ \ldots,\ n,\ \ k = 0,\ 1,\ \ldots,\ s),$$

$$[{}_j\alpha_r] \equiv [{}_j\alpha_0,\ {}_j\alpha_1,\ {}_j\alpha_2,\ \ldots,\ {}_j\alpha_r]$$

to obtain

$$\Delta = (-1)^n \lim_{s\to\infty} \left\| \begin{matrix} 0 & [{}_j\alpha_s] \\ [{}_k\alpha_s]' & [\gamma_s] \end{matrix} \right\| \div |[\gamma_s]|^n \quad (j,\ k = 1,\ 2,\ \ldots,\ n),$$

which, by an "extensional" identity in determinants[1], leads to

$$\Delta = \left| \int_a^b \frac{A_j(x)\,A_k(x)\,w(x)\,dx}{C(x)} \right|_n \tag{12}$$

$$= (-1)^n \lim_{s\to\infty} \frac{\begin{vmatrix} 0 & 0 & \cdots & 0 & {}_1\alpha_0 & {}_1\alpha_1 & \cdots & {}_1\alpha_s \\ 0 & 0 & \cdots & 0 & {}_2\alpha_0 & {}_2\alpha_1 & \cdots & {}_2\alpha_s \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & {}_n\alpha_0 & {}_n\alpha_1 & \cdots & {}_n\alpha_s \\ {}_1\alpha_0 & {}_2\alpha_0 & \cdots & {}_n\alpha_0 & \gamma_{00} & \gamma_{01} & \cdots & \gamma_{0s} \\ {}_1\alpha_1 & {}_2\alpha_1 & \cdots & {}_n\alpha_1 & \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1s} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ {}_1\alpha_s & {}_2\alpha_s & \cdots & {}_n\alpha_s & \gamma_{s0} & \gamma_{s1} & \cdots & \gamma_{ss} \end{vmatrix}}{|\gamma_{00},\ \gamma_{11},\ \cdots,\ \gamma_{ss}|}. \tag{13}$$

If the $A_j(x)$, $j = 1, 2, \ldots, n$, are linearly independent, then Δ is positive. This follows from the fact that the quadratic form

$$\sum_1^n \sum_1^n u_j u_k \int_a^b \frac{A_j(x)\,A_k(x)}{C(x)}\,w(x)\,dx = \int_a^b \frac{\left(\sum_1^n u_j A_j(x)\right)^2}{C(x)}\,w(x)\,dx$$

is positive definite. That a determinant with definite integral elements similar to Δ is positive appears to be due to Kowalewski[2]. We thus see

[1] Aitken, *loc. cit.*

[2] G. Kowalewski, *Einführung in die Determinantentheorie* (Leipzig, 1925), 224.

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that under the conditions attached to $A_j(x)$ and $w(x)\,C(x)$, the bordered determinant in (13) has the same sign as $(-1)^n$. It may be remarked that in certain cases the numerator of (13) reduces to a multiple of

$$|\gamma_{00}, \gamma_{11}, \ldots, \gamma_{ss}|$$

with a certain number of rows and corresponding columns deleted.

4. There is an alternative form for the expansion given in (8) when $C(x)$ is a polynomial. Let $C(x) = k \prod_{\lambda\ 1}^{n} (x - x_\lambda)$. Then by Christoffel's theorem (Szegö, *loc. cit.*, 2.5), if $p_r(x)$ are orthonormal polynomials with respect to the weight function $w(x)$ on (a, b), the orthogonal set with respect to $C(x)\,w(x)$ is $q_r(x)$ where

$$q_r(x) = \begin{vmatrix} p_r(x) & p_{r+1}(x) & \ldots & p_{r+n}(x) \\ p_r(x_1) & p_{r+1}(x_1) & \ldots & p_{r+n}(x_1) \\ \vdots & \vdots & & \\ p_r(x_n) & p_{r+1}(x_n) & \ldots & p_{r+n}(x_n) \end{vmatrix} \div C(x). \tag{14}$$

If $C(x)$ has a root of multiplicity m at x_k then the corresponding rows in (14) are to be replaced by the 0, 1, 2, ..., $m-1$-th derivatives of $p(x)$ at x_k.

We further require a theorem of Darboux (Szegö, *loc. cit.*, 3.2):

$$\begin{vmatrix} p_r(x) & p_{r+1}(x) \\ p_r(y) & p_{r+1}(y) \end{vmatrix} \Big/ (x-y) = -\frac{k_{r+1}}{k_r} \sum_{s=0}^{r} p_s(x)\,p_s(y), \tag{15}$$

where the recurrence relation for the polynomials $p_r(x)$ is

$$p_r(x) = (xA_r + B_r)\,p_{r-1}(x) - C_r\,p_{r-2}(x) \tag{16}$$

with $A_r = k_r/k_{r-1}$, $C_r = A_r/A_{r-1}$, and where k_r is the highest coefficient in $p_r(x)$. An extension of this is found by using the recurrence relation in (14), namely

$$q_r(x) = (-1)^n \frac{k_{r+n}}{k_r} \sum_{s=0}^{r} |p_s(x_1),\ p_{r+1}(x_2),\ p_{r+2}(x_3),\ \ldots,\ p_{r+n-1}(x_n)|\, p_s(x) \tag{17}$$

with derivatives appearing in the rows of the determinants when $C(x) = 0$ has multiple roots. Using (14) and (17), we have

$$\int_a^b q_r^2(x)\,C(x)\,w(x)\,dx = \int_a^b \frac{|p_r(x),\ p_{r+1}(x_1),\ \ldots,\ p_{r+n}(x_n)|}{C(x)} (-1)^n \frac{k_{r+n}}{k_r}$$

$$\times \sum_{s=0}^{r} |\,p_s(x_1),\ p_{r+1}(x_2),\ \ldots,\ p_{r+n-1}(x_n)|\,C(x)\,p_s(x)\,w(x)\,dx$$

$$= (-1)^n \frac{k_{r+n}}{k_r} |p_r(x_1), p_{r+1}(x_2), \ldots, p_{r+n-1}(x_n)| . |p_{r+1}(x_1), p_{r+2}(x_2), \ldots, p_{r+n}(x_n)|$$

$$= \phi_r \text{ say.} \tag{18}$$

If now $A(x)$ and $B(x)$ are polynomials of degree L and M respectively, we may write

$$A(x)=\sum_{\lambda=0}^{L} a_\lambda p_\lambda(x), \quad B(x)=\sum_{\lambda=0}^{M} b_\lambda p_\lambda(x).$$

Hence the Fourier coefficients of $A(x)/C(x)$ with respect to the orthogonal set $q_r(x)$ and the weight function $C(x)\,w(x)$ are given by

$$\begin{aligned}\alpha_r\phi_r &= \int_a^b A(x)\,q_r(x)\,w(x)\,dx \\ &= (-)^n \frac{k_{r+n}}{k_r}\left| \sum_{\lambda=0}^{r} a_\lambda p_\lambda(x_1),\ p_{r+1}(x_2),\ p_{r+2}(x_3),\ \ldots,\ p_{r+n-1}(x_n)\right| \qquad (19)\end{aligned}$$

with a similar expression for $B(x)$. In the expression $\sum_{\lambda=0}^{r} a_\lambda p_\lambda(x_1)$ it is to be understood that $a_\lambda=0$, $\lambda>L$, and similarly, in $\Sigma b_\lambda p_\lambda(x_1)$, $b_\lambda=0$, $\lambda>M$. Hence if $A(x)/C(x)$, $B(x)/C(x)$, $w(x)\,C(x)$ satisfy the conditions of Parseval's theorem, we have from (18) and (19)

$$\int_a^b \frac{A(x)\,B(x)\,w(x)\,dx}{C(x)}$$

$$= (-)^n \sum_{r=0}^{\infty} \frac{k_{r+n}\left|\sum_{\lambda=0}^{r} a_\lambda p_\lambda(x_1),\ p_{r+1}(x_2),\ p_{r+2}(x_3),\ \ldots,\ p_{r+n-1}(x_n)\right| \cdot \left|\sum_{0}^{r} b_\lambda p_\lambda(x_1),\ p_{r+1}(x_2),\ \ldots,\ p_{r+n-1}(x_n)\right|}{k_r\,|p_r(x_1),\ p_{r+1}(x_2),\ \ldots,\ p_{r+n-1}(x_n)| \cdot |p_{r+1}(x_1),\ p_{r+2}(x_2),\ \ldots,\ p_{r+n}(x_n)|}. \qquad (20)$$

If (20) is compared with (8) and (9) it will be observed that (*a*) the determinants in (20) are all of order n whereas in (8) the order increases with the term, (*b*) whereas the partial sums of (8) may be expressed as (9) by Schweins' theorem, this does not appear to be the case with the partial sums of (20). It is however clear that there will be determinantal identities between the denominators Δ_{s-1} of (8) and $|p_s(x_1),\ p_{s+1}(x_2),\ \ldots,\ p_{s+n-1}(x_n)|$ of (20). The special case $A(x)=B(x)=1$, $C(x)=x+z$ (z real) gives the expansion

$$\int_a^b \frac{w(x)\,dx}{x+z} = (-1)\sum_{s=0}^{\infty} \frac{k_{s+1}}{k_s\,p_s(-z)\,p_{s+1}(-z)}. \qquad (21)$$

5. The relation of (21) to the corresponding continued fraction expansion appears from (7). For (21) is an example of (7) with $A(x)=1$, $C(x)=x+z$ and $\theta_r(x)=p_r(x)\left\{\dfrac{-k_{r+1}}{k_r p_r(-z)\,p_{r+1}(-z)}\right\}^{\frac{1}{2}}$, $p_r(x)$ being the orthonormal set with respect to $w(x)$. If however we take $\theta_r(x)=p_r(x)$ and use the recur-

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rence relation (16), then $\alpha_0 = 1$ and $\alpha_r = 0$, $r \neq 0$. Moreover, writing $x+z = k_1^{-1} p_1(x) + \gamma + z$, where $k_1 \neq 0$, we find

$$\gamma_{r,s} = (z - B_{r+1} A_{r+1}^{-1}) \delta_{r,s} + A_{r+1}^{-1} \delta_{r+1,s} + A_r^{-1} \delta_{r-1,s} \quad (r, s = 1, 2, \ldots)$$

and $\gamma_{0,0} = z - B_1 A_1^{-1}$, $A_s > 0$.

From (7),
$$\int_a^b \frac{w(x)}{x+z}\,dx = -\lim_{s\to\infty} \frac{N_s(z)}{D_s(z)}, \tag{22}$$
where

$$N_s(z) = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot \\ 1 & z-B_1A_1^{-1} & A_1^{-1} & 0 & \cdot & \cdot \\ 0 & A_1^{-1} & z-B_2A_2^{-1} & A_2^{-1} & \cdot & \cdot \\ 0 & 0 & A_2^{-1} & z-B_3A_3^{-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{s+1}$$

and $D_s(z)$ is $N_s(z)$ with the first row and column deleted. In other words we have the continued fraction expansion

$$\int_a^b \frac{w(x)}{x+z}\,dx = \frac{1}{z-B_1A_1^{-1}-}\,\frac{A_1^{-2}}{z-B_2A_2^{-1}-}\,\frac{A_2^{-2}}{z-B_3A_3^{-1}-\ldots}. \tag{23}$$

It is of some interest to notice another form for the expansion. Take $\theta_r(x) = (x+z)^r$ so that $\alpha_r = \int_a^b (x+z)^r w(x)\,dx = m_r$ say, and is an Appell polynomial of degree r in z. (These are treated by J. Geronimus, *Journal London Math. Soc.*, 6 (1931), 55.) Similarly for $\gamma_{r,s} = m_{r+s+1}$. From (7), with the usual notation for persymmetric determinants,

$$\int_a^b \frac{w(x)}{x+z}\,dx = -\lim_{s\to\infty} \frac{P(0, m_0, m_1, \ldots, m_{2s-1})}{P(m_1, m_2, \ldots, m_{2s-1})}. \tag{24}$$

6. The expressions (21)-(24) indicate that there are relations between the various forms of the approximants to the definite integral. Consider $\phi_r = \int_a^b (x+z)\,w(x)\,q_r^2(x)\,dx$, where $\{q_r(x)\}$ is an orthogonal system with respect to $(x+z)\,w(x)$, the coefficient of x^r in $q_r(x)$ being unity. Then

$$q_r(x) = (-)^r \theta_r^{-1} |\,\theta_0(x), q_{01}, q_{12}, \ldots, q_{r-1,r}| / |q_{00}, q_{11}, \ldots, q_{r-1,r-1}|,$$

where $\theta_s(x)$ is an *arbitrary* polynomial of precise degree s with highest coefficient θ_s, and $q_{\alpha,\beta} = \int_a^b (x+z)\,\theta_\alpha(x)\,\theta_\beta(x)\,w(x)\,dx$. But ϕ_r is invariant with respect to the choice of $\theta(x)$. Hence taking $\theta_s(x) = (x+z)^s$, we find $\phi_r = P(m_1, m_2, \ldots, m_{2r+1})/P(m_1, m_2, \ldots, m_{2r-1})$. Again take $\{\theta_s(x)\}$ to be the orthonormal set with respect to $w(x)$, and use (14) and (15), so that $\phi_r = -p_{r+1}(-z)/\{k_r k_{r+1} p_r(-z)\}$. But from the recurrence relation (16) and the continued fraction expansion (23), $p_s(-z) = (-)^s k_s D_s(z)$ Hence

$$P(m_1, m_2, \ldots, m_{2r+1}) = \frac{D_{r+1}(z)}{(k_0 k_1 \ldots k_r)^2}. \tag{25a}$$

It therefore follows from (22) and (24) that

$$P(0, m_0, m_1, \ldots, m_{2r+1}) = \frac{N_{r+1}(z)}{(k_0 k_1 \ldots k_r)^2}. \tag{25b}$$

The relations (25) have been derived by Geronimus (*loc. cit.*) by another method.

7. As an illustration consider the hypergeometric function

$$F(1, a;\ b;\ t) = \sum_{s=0}^{\infty} \frac{a(a+1)\ldots(a+s-1)}{b(b+1)\ldots(b+s-1)} t^s = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \frac{w(x)}{1-xt}\, dx,$$

where $\quad w(x) = x^{a-1}(1-x)^{b-a-1}, \quad 0 \leqslant x \leqslant 1, \quad a > 0, \quad b-a > 0,$

and [1] $\quad p_s(x) = \sqrt{r_s}\, F(s+b-1, -s;\ a;\ x),$

$$r_s = \frac{(2s+b-1)\,\Gamma(s+a)\,\Gamma(s+b-1)}{\Gamma(s+1)\,\Gamma(s+b-a)\,\Gamma(a)^2}, \quad k_s = (-)^s \frac{\Gamma(2s+b-1)\,\Gamma(a)\sqrt{r_s}}{\Gamma(s+a)\,\Gamma(s+b-1)}.$$

It follows from (21) with $z = -t^{-1}$, $|t| < 1$, that

$$F(1, a;\ b;\ t) = \sum_{s=1}^{\infty} \frac{\Gamma(b)\,\Gamma(s+a-1)\,\Gamma(s+b-2)\,\Gamma(s+b-a-1)\,\Gamma(s)\,t^{2s-2}}{\Gamma(a)\,\Gamma(b-a)\,\Gamma(2s+b-2)\,\Gamma(2s+b-3)\,F(-s, 1-a-s;\ 2-b-2s;\ t) \times F(1-s, 2-a-s;\ 4-b-2s;\ t)}. \tag{26}$$

Moreover, using the recurrence relation for $p_s(x)$ (see Szegö, *loc. cit.*; a slight change of notation is required in Szegö, 4.5.1) it will be found from (23) that $F(1, a;\ b;\ t)$ is the even part of the continued fraction

$$\frac{1}{1-}\,\frac{b_1 t}{1-}\,\frac{b_2 t}{1-}\,\frac{b_3 t}{1-}\ldots, \tag{27}$$

where

$$b_{2s+1} = \frac{(s+a)(s+b-1)}{(2s+b-1)(2s+b)}, \quad b_{2s+2} = \frac{(s+1)(s+b-a)}{(2s+b)(2s+b+1)}. \tag{27a}$$

[1] H. Bateman, *Partial Differential Equations* (New York, 1944), 392.

In a similar way the odd part of the continued fraction (27) arises from a consideration of the relation

$$F(1,\, a;\; b;\; t) = 1 + \frac{t\,\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \int_0^1 \frac{xw(x)}{r - xt}\, dx,$$

and approximating to the integral by using $xw(x)$ for $w(x)$ in (23). There is also a corresponding series expansion similar to (26). If we call the s-th convergent of (27) n_s/d_s, then the series expansion (26) may be derived from the identity

$$\frac{n_s}{d_s} - \frac{n_{s-2}}{d_{s-2}} = \frac{b_1 b_2 \dots b_{s-2} t^{s-2}}{d_{s-2}\, d_s}, \quad s = 2,\, 4,\, 6,\, \dots .$$

8. Finally, consider the relations between the persymmetric determinants and continued fraction convergents given by (25), in connection with the hypergeometric function. We find

$$(k_0 k_1 \dots k_r)^2 = \prod_{s=0}^{r} \frac{\Gamma(r+b+s)}{\Gamma(s+1)\,\Gamma(s+a)\,\Gamma(s+b-a)},$$

$$m_s = z^s \frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(b)} F(-s,\, a;\; b;\; -z^{-1}).$$

Inserting these in (25a) and (25b), we find that

$$z^{r^2} P(F_1,\, F_2,\, \dots,\, F_{2r-1}) = F(r+b-1,\, -r;\; a;\; -z) \prod_{s=0}^{r-1} \frac{s!\,(b-a)_s\, a_{s+1}}{b_{r+s}},$$

$$z^{r^2-1} P(0,\, F_0,\, F_1,\, \dots,\, F_{2r-1}) = -\bar{N}_r(z) \prod_{s=0}^{r-1} \frac{s!\,(b-a)_s\, a_s}{b_{r+s-1}}, \quad r = 1,\, 2,\, 3,\, \dots, \tag{28}$$

where $F_\alpha \equiv F(-\alpha,\, a;\; b;\; -z^{-1})$,

$\lambda_s \equiv \lambda(\lambda+1) \dots (\lambda+s-1)$ under the continued product sign,

and $\bar{N}_r(z)$ is the numerator of the r-th convergent of

$$\frac{1}{z+b_1-}\;\frac{b_1 b_2}{z+b_2+b_3-}\;\frac{b_3 b_4}{z+b_4+b_5-}\dots,$$

the b's being given in (27a). A similar pair of relations would also be found by considering the weight function $xw(x)$ in place of $w(x)$. Burchnall[1] has recently given similar expressions for $P(F_0,\, F_1,\, \dots,\, F_{2r-1})$ and its minors.

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[1] J. L. Burchnall, *Quart. Journ. Math., Oxford*, 2nd Series 3, 10 (1952), 151–157.

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