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Coordinatization Theorems For Graded Algebras

Dedicated to Robert Moody on the occasion of his 60th birthday

Bruce Allison and Oleg Smirnov

Abstract. In this paper we study simple associative algebras with finite $\mathbb{Z}$-gradings. This is done using a simple algebra F_g that has been constructed in Morita theory from a bilinear form $g\colon U \times V \to A$ over a simple algebra A. We show that finite $\mathbb{Z}$-gradings on F_g are in one to one correspondence with certain decompositions of the pair (U, V). We also show that any simple algebra R with finite $\mathbb{Z}$-grading is graded isomorphic to F_g for some bilinear from $g\colon U \times V \to A$, where the grading on F_g is determined by a decomposition of (U, V) and the coordinate algebra A is chosen as a simple ideal of the zero component R_0 of R. In order to prove these results we first prove similar results for simple algebras with Peirce gradings.

1 Introduction

Graded simple associative algebras are of considerable interest for their own sake (see for example [NvO], [S1] and [BSZ]) and because of their connection with graded Lie algebras [Z], [S2]. In this paper we study finite $\mathbb{Z}$-gradings and Peirce gradings (also known as a generalized matrix gradings) on simple associative algebras over an arbitrary base ring Φ.

Our work on simple algebras with finite $\mathbb{Z}$-gradings uses a construction of simple algebras that has arisen in earlier work of several authors on Morita theory over (possibly) nonunital rings. This construction produces a simple algebra $F_g = F_g(U, V, A)$ from a simple coordinate algebra A, idempotent torsion-free left and right A-modules U and V respectively and a nonzero nondegenerate A-bilinear form $g\colon U \times V \to A$. When A is a division algebra, F_g is the algebra of continuous finite rank A-endomorphisms of V with topology determined by g (see [J, Section 4.8]). Our main theorem, Theorem 4.7, for simple algebras with finite $\mathbb{Z}$-gradings has two parts. We show first that finite $\mathbb{Z}$-gradings on F_g of height n are in one-to-one correspondence with A-module decompositions $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ of the pair (U, V) with the properties that $U_0 \neq 0$, $U_n \neq 0$ and $g(U_i, V_i) = 0$ if $i \neq j$. We call such decompositions regular g-diagonal decompositions of (U, V). Second, given a simple algebra with finite $\mathbb{Z}$-grading $R = \bigoplus_{i=-n}^n R_i$ of height n, we show that R is graded isomorphic to $F_g(U, V, A)$, where A is any simple ideal of R_0, U, V and g are as indicated above and the grading on F_g is determined by a regular g-diagonal decomposition of g. We call Theorem 4.7 a *coordinatization* theorem since it provides a description, up

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to graded isomorphism, of arbitrary simple algebras with finite $\mathbb{Z}$-gradings in terms of coordinate structures A, g, U and V.

The proof of Theorem 4.7 exploits the connection between $\mathbb{Z}$-gradings and Peirce gradings that was studied in [S1]. Indeed, we first prove a theorem, Theorem 4.6, that describes arbitrary simple algebras with finite Peirce gradings in terms of coordinate structures. Actually, here it is convenient to weaken the hypotheses and describe idempotent torsion-free algebras with strong Peirce gradings.

In a forthcoming paper, we plan to use the methods and results of this paper to study finite $\mathbb{Z}$-gradings on simple associative algebras with involution. The information so obtained, together with the results of [Z] and [S2], will be used to study simple Lie algebras with finite $\mathbb{Z}$-gradings and their nonassociative coordinate structures.

To treat algebras and rings simultaneously we assume that all algebras and modules throughout the paper are modules over an associative commutative unital ring Φ and all maps are Φ-linear.

2 Preliminaries on Morita Contexts

In this section, we recall the basic facts that we will need about Morita contexts. Good references for most of this material are [GS] and [A].

2.1 Uni Algebras and Uni Modules

The associative algebras we are interested in arise from the study of simple Lie algebras [Z, S2]. They are simple but not necessarily unital. Therefore we need to work with an appropriate generalization of unital algebras and modules. It seems that the category of idempotent torsion-free modules is a perfect candidate for our purposes. This category includes the examples of interest to us and it allows a nice analog of classical Morita theory (see [GS] and [A] and the references therein).

Recall that an algebra A is called *idempotent* if $AA = A$. A right (resp. left) A-module M is called *idempotent* if $MA = M$ (resp. $AM = M$). An (A, B)-bimodule is called *idempotent* if $AMB = M$.

For an algebra A the sets $\operatorname{Ann}^l(A) = \{a \in A : aA = 0\}$, $\operatorname{Ann}^r(A) = \{a \in A : Aa = 0\}$, and $\operatorname{Ann}(A) = \{a \in A : AaA = 0\}$ are called the *left annihilator*, the *right annihilator*, and the *annihilator* of A respectively.

For a right (resp. left) A-module, M the set $T(M) = \{m \in M : mA = 0\}$ (resp. $T(M) = \{m \in M : Am = 0\}$) is called the *torsion submodule* of M [GS]. Similarly, if M is an (A, B)-bimodule the *torsion submodule* of M is defined as $T(M) = \{m \in M : AmB = 0\}$. A left, right or bi-module M is called *torsion-free* if $T(M) = 0$.

The category of idempotent torsion-free modules has been studied by different authors under different names. It is convenient for us to have a short name for modules in this category. Thus we say that a left, right or bi-module M is *uni* if M is idempotent and torsion-free.

We call an algebra A a *uni* algebra if ${}_AA_A$ is a uni bimodule. This means that A is an idempotent algebra and that $\operatorname{Ann}(A) = 0$ (or equivalently $\operatorname{Ann}^l(A) = \operatorname{Ann}^r(A) = 0$).

An algebra A is called *simple* if $A \neq 0$ and A has no proper nonzero ideals. (We do not require the Jacobson radical to be zero as do some authors.) Any simple algebra

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is uni.

If A is a uni algebra, the full subcategory of all uni left (resp. right) A-modules in the category A-Mod (resp. Mod-A) of all left (resp. right) A-modules is denoted by A-mod (resp. mod-A). Similarly, if A and B are uni algebras, the full subcategory of all uni (A, B)-bimodules in the category A-Mod-B of all (A, B)-bimodules is denoted by A-mod-B. For a category $\mathfrak{C}$ we write $M \in \mathfrak{C}$ to mean that M is an object in $\mathfrak{C}$.

2.2 Bilinear Forms and Morita Contexts

Suppose that A is an algebra, $V \in$ Mod-A, and $U \in A$-Mod. A map $g\colon U \times V \to A$ is said to be a *bilinear form* if g is biadditive and

$$g(au, v) = ag(u, v) \quad \text{and} \quad g(u, va) = g(u, v)a$$

for $u \in U$, $v \in V$, $a \in A$. The bilinear form g is called *surjective* if the set $g(U, V) = \operatorname{span}\{g(u, v) : u \in U, v \in V\}$ is equal to A. The form g is said to be *nondegenerate* if $g(u, V) = 0$ implies $u = 0$ and $g(U, v) = 0$ implies $v = 0$.

Suppose that A and B are algebras, $V \in B$-Mod-A, and $U \in A$-Mod-B. A bilinear form $g\colon U \times V \to A$ is called *balanced* provided

$$g(ub, v) = g(u, bv)$$

for $u \in U$, $v \in V$, $b \in B$. Two bilinear forms $g\colon U \times V \to A$ and $f\colon V \times U \to B$ are said to be *compatible* provided

$$f(v, u)v' = vg(u, v') \quad \text{and} \quad u' f(v, u) = g(u', v)u$$

for all $u, u' \in U$ and $v, v' \in V$.

Assume that A, B are uni algebras, $U \in A$-mod-B, $V \in B$-mod-A, and $g\colon U \times V \to A$ and $f\colon V \times U \to B$ are surjective bilinear forms that are compatible. Then the forms g and f are also nondegenerate and balanced, and we call the sextuple (A, B, U, V, g, f) a *Morita context*.

If A and B are uni algebras, it is proved in [GS] that there is a Morita context of the form (A, B, U, V, g, f) if and only if the categories mod-A and mod-B are equivalent (in which case A and B are said to be Morita equivalent). We won't need that result. In fact, the only result from Morita theory that we need is the following proposition that is proved in [GS, Proposition 3.6]. Since the proof is short and self contained we include it for the reader's convenience.

Proposition 2.1 *Suppose that (A, B, U, V, g, f) is a Morita context. Then A is simple if and only if B is simple.*

Proof We prove one direction (the other being similar). Suppose that A is simple. Let I be an ideal of B. Then $g(UI, V)$ is an ideal of A and so $g(UI, V)$ is 0 or A. If $g(UI, V) = 0$, then $UI = 0$ and so $BI = f(V, UI) = 0$ and therefore $I = 0$. On the other hand, if $g(UI, V) = A$, then $V = VA = Vg(UI, V) = f(V, UI)V = BIV \subseteq IV$ and so $B = f(V, U) \subseteq f(IV, U) \subseteq IB \subseteq I$. ■

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3 Graded Algebras

In this section, we recall the facts that we will need about graded algebras.

3.1 Peirce Gradings

Suppose that n is a nonnegative integer. A decomposition of an algebra R into the direct sum of Φ-submodules

$$R = \bigoplus_{i,j=0}^{n} R_{i,j} \tag{1}$$

is called an $(n+1)\times(n+1)$-*Peirce grading* if $R_{i,j}R_{k,l} \subseteq \delta_{jk}R_{i,l}$ for all i, j, k. If n is understood from the context, we refer to these gradings simply as *Peirce gradings*. We use the term grading because such a decomposition can be considered as a grading by the semigroup $S = \{(i, j) : 0 \le i, j \le n\} \cup \{0\}$ with multiplication $(i, j)(p, q) = \delta_{j,p}(i, q)$ and $(i, j)0 = 0(i, j) = 0$.

A Peirce grading is a generalization of the Peirce decomposition for unital algebras. Indeed, if $\{e_0, e_1, \ldots, e_n\}$ is a complete set of orthogonal idempotents of a unital algebra R then R has a Peirce grading $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ where $R_{i,j} = e_iRe_j$.

A Peirce grading can also can be written in matrix form:

$$R = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0n} \\ R_{10} & R_{11} & \cdots & R_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n0} & R_{n1} & \cdots & R_{nn} \end{bmatrix}$$

and the definition implies that the blocks of this decomposition obey the matrix multiplication rule. In [Be] Peirce graded algebras are called generalized matrix algebras. In [S1] Peirce gradings are called strict Peirce systems.

If one studies merely Peirce gradings, the nature of the indexing set $I = \{0, \ldots, n\}$ is not important. However, the assumption $I \subseteq \mathbb{Z}$ is needed to construct $\mathbb{Z}$-gradings from Peirce gradings (see Section 3.2). Also, to obtain a larger class of $\mathbb{Z}$-gradings we allow zero submodules in the decomposition (1). For this reason the set

$$P\text{-Supp}(R) = \{i \in I : R_{i,j} + R_{j,i} \neq 0 \text{ for some } j \in I\},$$

is an important numerical characteristic of the grading. We call P-Supp(R) the *support of the Peirce grading of R*.

Although the main focus of the paper is on simple algebras, the simplicity of algebras is often not needed. Instead, the next notion plays a crucial role throughout.

Borrowing terminology from group-graded algebras we say that a Peirce grading $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ is *strong* if $R \neq 0$ and $R_{i,j}R_{j,k} = R_{i,k}$ for $i, j, k \in P$-Supp(R). Observe that in that case we have

$$R_{i,i}R_{i,j} = R_{i,j}, \quad R_{i,j}R_{j,j} = R_{i,j} \quad \text{and} \quad R_{i,k}R_{k,j} = R_{i,j} \tag{2}$$

for all $i, j \in I = \{0, \ldots, n\}$ and $k \in P$-Supp(R). Obviously a strongly Peirce graded algebra is idempotent.

Lemma 3.1 *Let $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ be an algebra with a strong Peirce grading. If $i, j \in I$, then $R_{i,j} \neq 0$ if and only if $i, j \in P\text{-Supp}(R)$. In particular, $P\text{-Supp}(R) = \{i \in I : R_{i,i} \neq 0\}$.*

Proof If $R_{i,j} \neq 0$, then $i, j \in P\text{-Supp}(R)$. Conversely, suppose that $i, j \in P\text{-Supp}(R)$. If $R_{i,j} = 0$, then by (2) we have $R_{i,i} = R_{i,j}R_{j,i} = 0$, $R_{i,k} = R_{i,i}R_{i,k} = 0$, and $R_{k,i} = R_{k,i}R_{i,i} = 0$ for every $k \in I$. This contradicts the assumption that $i \in P\text{-Supp}(R)$. ■

We will need the following facts about strong Peirce gradings and simple algebras.

Proposition 3.2 *Let $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ be a Peirce graded algebra. The grading is strong if and only if $R \neq 0$ and $R = RR_{i,j}R$ for any $i, j \in P\text{-Supp}(R)$.*

Proof Assume that the grading is strong and $i, j \in P\text{-Supp}(R)$. Then $R \neq 0$ and by (2) we have that $R_{k,l} = R_{k,j}R_{j,l} = R_{k,i}R_{i,j}R_{j,l} \subseteq RR_{i,j}R$ for any $k, l \in I$.

To prove the converse, assume that $i, j, k \in P\text{-Supp}(R)$. Then $R = RR_{i,j}R$ and therefore $R_{i,k} = R_{i,k} \cap (RR_{i,j}R) = R_{i,i}R_{i,j}R_{j,k} \subseteq R_{i,j}R_{j,k}$. ■

Remark 3.3 If $P\text{-Supp}(R)$ has more than one element, the statement "$R = RR_{i,j}R$ for any $i, j \in P\text{-Supp}(R)$" is equivalent to the statement "the ideal generated by $R_{i,j}$ is equal to R for any $i, j \in P\text{-Supp}(R)$".

Proposition 3.4 *Every Peirce grading on a simple algebra R is strong.*

Proof Let $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ be a simple algebra with a Peirce grading. For any subset X of R the set $\langle X \rangle = RXR$ is an ideal of R. Moreover, since R is simple, $\langle X \rangle = R$ if and only if $X \neq 0$. So, in view of Proposition 3.2, it suffices to show that $R_{i,j} \neq 0$ for any $i, j \in P\text{-Supp}(R)$.

Assume on the contrary that $R_{i,j} = 0$ for some $i, j \in P\text{-Supp}(R)$. Then $\langle R_{i,i} \rangle \langle R_{j,j} \rangle \subseteq \langle R_{i,i}R_{i,j}R_{j,j} \rangle = 0$ and therefore $R_{i,i} = 0$ or $R_{j,j} = 0$. If $R_{i,i} = 0$, then for any $l \in I$ one has $\langle R_{i,l} \rangle^2 \subseteq \langle R_{i,l}R_{l,i}R_{i,l} \rangle \subseteq \langle R_{i,i}R_{i,l} \rangle = 0$ and $\langle R_{l,i} \rangle^2 \subseteq \langle R_{l,i}R_{i,i} \rangle = 0$. This implies that $R_{i,l} = 0$ and $R_{l,i} = 0$ for any $l \in I$, which contradicts the assumption that $i \in P\text{-Supp}(R)$. Similarly, $R_{j,j} = 0$ contradicts the assumption that $j \in P\text{-Supp}(R)$. ■

For later purposes we need to find graded components of annihilators in a strongly Peirce graded algebra.

Lemma 3.5 *Assume that $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ is a strongly Peirce graded algebra. Then $\mathrm{Ann}^l(R) = \bigoplus_{i,j=0}^{n} \{r \in R_{i,j} : rR_{j,j} = 0\}$, $\mathrm{Ann}^r(R) = \bigoplus_{i,j=0}^{n} \{r \in R_{i,j} : R_{i,i}r = 0\}$ and $\mathrm{Ann}(R) = \bigoplus_{i,j=0}^{n} \{r \in R_{i,j} : R_{i,i}rR_{j,j} = 0\}$.*

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Proof We prove the first equality; the other two can be proved similarly. Assume that $r \in R_{i,j}$ and $rR_{j,j} = 0$, where $0 \le i, j \le n$. Then by (2) one has $rR = \sum_{k=0}^{n} rR_{j,k} = \sum_{k=0}^{n} rR_{j,j}R_{j,k} = 0$. So $r \in \mathrm{Ann}^l(R)$.

Conversely, suppose $r = \sum_{i,j=0}^{n} r_{i,j} \in \mathrm{Ann}^l(R)$, where $r_{i,j} \in R_{i,j}$. Then $0 = rR_{j,j} = \sum_{i=0}^{n} r_{i,j}R_{j,j}$ and therefore $r_{i,j}R_{j,j} = 0$. ■

Let $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ be a Peirce graded algebra and let V be a right (resp. left) R-module. We say that a decomposition of V into a direct sum $V = \bigoplus_{i=0}^{n} V_i$ of Φ-submodules is a *grading of* V if

(3) $$V_iR_{j,k} \subseteq \delta_{i,j}V_k \quad (\text{resp. } R_{j,k}V_i \subseteq \delta_{k,i}V_j)$$

for every $i, j, k \in I$. A module with grading is called *a graded module.*

Proposition 3.6 *Let $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ be an algebra with a strong Peirce grading. Then every right (resp. left) uni R-module V has a unique grading $V = \bigoplus_{i=0}^{n} V_i$. For this grading $V_i = VR_{i,i}$ (resp. $V_i = R_{i,i}V$).*

Proof In this proof we assume that V is a right module. The proof for left modules is similar.

First, we note that

(4) $$VR_{i,j} \subseteq VR_{j,j}$$

for every $i, j \in I$. Indeed, by (2), we have $VR_{i,j} = VR_{i,j}R_{j,j} \subseteq VR_{j,j}$.

We set $V_j = VR_{j,j}$ for $j \in I$. Then, using (4), we have $V = VR = \sum_{i,j=0}^{n} VR_{i,j} = \sum_{j=0}^{n}(\sum_{i=0}^{n} VR_{i,j}) \subseteq \sum_{j=0}^{n} VR_{j,j} = \sum_{j=0}^{n} V_j$. Also $V_kR_{i,j} = VR_{k,k}R_{i,j} \subseteq \delta_{k,i}VR_{k,j} \subseteq \delta_{k,i}VR_{j,j} = \delta_{k,i}V_j$ for $i, j, k \in I$. Besides, if $j \in I$, $\big((\sum_{k\ne j} V_k) \cap V_j\big) R_{s,t} = 0$ for every $s, t \in I$ and therefore $(\sum_{k\ne j} V_k) \cap V_j = 0$. Thus we have obtained a grading of V.

Finally, if $V = \bigoplus_{i=0}^{n} V_i'$ is a grading, we have $V_j' = V_j'R = \sum_{i=0}^{n} V_i'R_{i,j} \subseteq \sum_{i=0}^{n} VR_{i,j} \subseteq VR_{j,j} = V_j$ showing the uniqueness. ■

3.2 Finite $\mathbb{Z}$-Gradings

A decomposition of an algebra R into the direct sum of Φ-submodules $R = \bigoplus_{i\in\mathbb{Z}} R_i$ is called a $\mathbb{Z}$-*grading* of R if $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$. This grading is said to be *finite* if there exists an integer $n \ge 0$ so that $R_i = 0$ for $|i| > n$. In that case the smallest such n is called the *height* of the grading.

There is an important connection between Peirce gradings and finite $\mathbb{Z}$-gradings. To describe this connection, suppose first that $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ is a Peirce graded algebra. Let

(5) $$R_i = \sum_{p-q=i} R_{p,q}$$

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for $-n \le i \le n$, and let $R_i = 0$ for $|i| > n$. Then $R = \bigoplus_{i=-n}^{n} R_i$ is a $\mathbb{Z}$-graded algebra. This finite $\mathbb{Z}$-grading is called the *$\mathbb{Z}$-grading induced from the Peirce grading*.

Note that if $V = \bigoplus_{i=0}^{n} V_i$ is a graded right (resp. left) module over a Peirce graded algebra $R = \bigoplus_{i,j=0}^{n} R_{i,j}$, then $V_i R_j \subseteq V_{i-j}$ (resp. $R_j V_i \subseteq V_{i+j}$) for the $\mathbb{Z}$-grading $R = \bigoplus_{i=-n}^{n} R_i$ defined by (5).

When studying the induced $\mathbb{Z}$-grading, it is often convenient to assume that $0, n \in P\text{-Supp}(R)$ for the Peirce grading $R = \bigoplus_{i,j=0}^{n} R_{i,j}$. An $(n+1)\times(n+1)$-Peirce grading with this property is called *regular*. For an arbitrary $(n+1)\times(n+1)$-Peirce grading $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ of a nonzero algebra, one can always consider a regular $(l-s+1)\times(l-s+1)$-Peirce grading $R = \bigoplus_{i,j=0}^{l-s} P_{i,j}$, where l is the largest number in $P\text{-Supp}(R)$, s is the smallest number in $P\text{-Supp}(R)$ and $P_{i,j} = R_{i+s,j+s}$ for any $i, j \in \{0,\dots,l-s\}$. In other words, without loss of generality one can shift the indexing set I down by s and disregard some zero rows and columns to obtain a regular Peirce grading. It is clear that these Peirce gradings induce the same $\mathbb{Z}$-grading on R.

For simple algebras we have the following fact that follows from results proved in [S1].

Proposition 3.7 *If R is simple, then (5) establishes a bijective correspondence between regular $(n+1)\times(n+1)$-Peirce gradings of R and $\mathbb{Z}$-gradings of R of height n.*

Proof If $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ is a regular Peirce grading, then it follows from Proposition 3.4 that this Peirce grading is strong and Lemma 3.1 implies that $R_{0,n} \ne 0$. Therefore the induced $\mathbb{Z}$-grading is of height n since $R_n = R_{0,n} \ne 0$.

Conversely, let $R = \bigoplus_{i=-n}^{n} R_i$ be a $\mathbb{Z}$-grading of height n. Thus, $R_n \ne 0$ or $R_{-n} \ne 0$. In fact since R is simple, the support of the $\mathbb{Z}$-grading of R is symmetric about the origin [S1, p. 180], and so both $R_n \ne 0$ and $R_{-n} \ne 0$. Hence, the ideal of R generated by R_{-n} equals R (as required to invoke Lemma 4.1 of [S1] below). Set $R_{p,q} = R_p R_{-n} R_{n-q}$ for $0 \le p, q \le n$. Then, by Lemma 4.1 and Corollary 3.5 of [S1], $R = \bigoplus_{p,q=0}^{n} R_{p,q}$ is a Peirce grading that induces the given $\mathbb{Z}$-grading. Since $R_{0,n} = R_{-n} \ne 0$, we have $0, n \in P\text{-Supp}(R)$ and hence this Peirce grading is regular.

For uniqueness, suppose that $R = \bigoplus_{p,q=0}^{n} R'_{p,q}$ is another such Peirce grading. Then $R_i = \sum_{p-q=i} R'_{p,q}$ for $-n \le i \le n$. So $R_{p,q} = R_p R_{-n} R_{n-q} = R'_{p,0} R'_{0,n} R'_{n,q} \subseteq R'_{p,q}$ and hence $R_{p,q} = R'_{p,q}$ for all p, q. ■

4 Coordinatization of Graded Algebras

The goal of this section is to prove a coordinatization theorem for uni algebras that are strongly Peirce graded. As a consequence of this we obtain a coordinatization theorem for simple algebras with finite $\mathbb{Z}$-gradings.

4.1 The Construction of $F_g(U, V, A)$

The algebra that we will use in our theorems is the algebra $F_g(U, V, A)$. It appeared in different forms in several papers on Morita theory and related topics (see for example

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[AM, A, GS, K]).

Before recalling the definition of $F_g(U,V,A)$, we need to introduce some notation. If R is an algebra, we will denote the *opposite algebra* of R by R^{op}. Recall that $R^{\mathrm{op}} = \{r^\circ : r \in A\}$ is a copy of R as an Φ-module and the product on R^{op} is defined by $r_1^\circ r_2^\circ = (r_2 r_1)^\circ$ for $r_1, r_2 \in A$. If W is a left (resp. right) R-module, then W is a right (resp. left) R^{op}-module with action defined by $wr^\circ = rw$ (resp. $r^\circ w = wr$).

To define the algebra $F_g(U,V,A)$, we assume that A is an algebra, $U \in A$-Mod, $V \in$ Mod-A and $g\colon U \times V \to A$ is a bilinear form.

Consider the algebra $E = \mathrm{End}_A(V) \oplus \mathrm{End}_A(U)^{\mathrm{op}}$. E acts on the left on V by $(x, y^\circ)v = xv$ and on the right on U by $u(x, y^\circ) = uy^\circ = yu$. Moreover, with respect to these actions U is an (A,E)-bimodule and V is a (E,A)-bimodule.

If $v \in V$ and $u \in U$, we define $x_{v,u} \in \mathrm{End}_A(V)$ and $y_{v,u} \in \mathrm{End}_A(U)$ by

$$x_{v,u}v' = vg(u,v') \quad \text{and} \quad y_{v,u}u' = g(u',v)u$$

for $v' \in V$, $u' \in U$. We let

$$e_{v,u} = \big(x_{v,u}, (y_{v,u})^\circ\big) \in E$$

for $v \in V$ and $u \in U$. Then

$$e_{v,u}v' = vg(u,v') \quad \text{and} \quad u'e_{v,u} = g(u',v)u \tag{6}$$

for $u, u' \in U$ and $v, v' \in V$. It is easy to check that

$$e_{va,u} = e_{v,au} \tag{7}$$

and that

$$e_{v,u}e_{v',u'} = e_{vg(u,v'),u'} = e_{v,g(u,v')u'} \tag{8}$$

for $v, v' \in V$, $u, u' \in U$ and $a \in A$. Finally, set

$$F_g = F_g(U,V,A) = e_{V,U} = \mathrm{span}\,\{e_{v,u} : v \in V, u \in U\}.$$

Then F_g is a subalgebra of E with product given explicitly by (8).

Note that U is an (A, F_g)-bimodule and V is an (F_g, A)-bimodule. Also $g\colon U \times V \to A$ is balanced since $g(u'e_{v,u}, v') = g\big(g(u',v)u, v'\big) = g(u',v)g(u,v') = g\big(u', vg(u,v')\big) = g(u', e_{v,u}v')$.

We now define $f\colon V \times U \to F_g$ by

$$f(v,u) = e_{v,u} \tag{9}$$

for $v \in V$, $u \in U$. Then f is surjective, bilinear (by (6) and (8)) and balanced (by (7)). Furthermore, by (6), f and g are compatible.

The following fact is mentioned without proof in [A, Example 1.4]. For the convenience of the reader, we give the proof.

Proposition 4.1 *Suppose that A is a uni algebra, $U \in A$-mod, $V \in$ mod-A, and $g\colon U \times V \to A$ is a surjective nondegenerate bilinear form. Define $f\colon V \times U \to F_g$ by $f(v,u) = e_{v,u}$. Then (A, F_g, U, V, g, f) is a Morita context. Consequently, A is simple if and only if F_g is simple.*

Proof The last statement follows from the first by Proposition 2.1. To prove the first statement, we note first that U is a faithful left A-module. Indeed, if $aU = 0$, then $aA = ag(U,V) = g(aU,V) = 0$ and so $a = 0$. Similarly, the right A-module V is faithful.

To prove the proposition, we must show that V is a uni left F_g-module, U is a uni right F_g-module and that F_g is a uni algebra.

First of all, the left F_g-module V is idempotent since $V = VA = Vg(U,V) = f(V,U)V = F_gV$. To show that V is a torsion-free left F_g-module, suppose that $F_gv = 0$, where $v \in V$. Then $Vg(U,v) = 0$ and $v = 0$ because the A-module V is faithful and the form g is nondegenerate. Thus V is a uni left F_g-module. The proof for U is similar.

Finally, if $v \in V = F_gA$, we have $v = \sum e_{v_i,u_i} w_i$ for some $v_i, w_i \in V$ and $u_i \in U$. So if $u \in U$, we have $e_{v,u} = \sum e_{v_i g(u_i,w_i)}, u = \sum e_{v_i,u_i} e_{w_i,u}$ by (8). Thus the algebra F_g is idempotent. It remains to prove that both the left and right annihilators of the algebra F_g are zero. Indeed, if $b \in F_g$ and $bF_g = 0$, then $bV = bF_gV = 0$. Thus $Ub = 0$ because $g(Ub,V) = g(U,bV) = 0$ and the form g is nondegenerate. This implies that $b = 0$. Similarly one proves that the right annihilator of F_g is zero. ∎

Remark 4.2 Suppose that A, U, V and g are as in Proposition 4.1. In the last paragraph of the proof of Proposition 4.1, we noticed that if $b \in F_g$ and $bV = 0$ then $b = 0$. In other words, V is a faithful left F_g-module (and similarly U is a faithful right F_g-module). Thus we have $F_g \simeq x_{V,U}$, under projection onto the first factor, where $x_{V,U}$ is the subalgebra of $\mathrm{End}_A(V)$ spanned by $\{x_{v,u} : v \in V, u \in U\}$. This is simple alternate construction of the algebra F_g.

Remark 4.3 The construction of F_g contains as a special case a classical construction of Jacobson. Indeed, suppose that $\Phi = \mathbb{Z}$, A is a division ring, $U \in A$-mod, $V \in$ mod-A, and $g\colon U \times V \to A$ is a nondegenerate bilinear form. Then $F_g \simeq x_{V,U}$ (by Remark 4.2), and $x_{V,U}$ is the ring of continuous finite rank A-linear transformations of V with topology determined by g [J, Section 4.8]. If U (and hence V) is finite dimensional over the division ring A, then $F_g \simeq x_{V,U} = \mathrm{End}_A(V)$.

4.2 Coordinatization Theorems For Graded Algebras

Peirce gradings on the algebra $F_g(U,V,A)$ arise naturally from decompositions of the modules U and V.

Suppose that A is an algebra, $U \in A$-Mod, $V \in$ Mod-A and $g\colon U \times V \to A$ is a bilinear form. Assume further that $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are direct sums of A-submodules such that $g(U_i, V_j) = 0$ if $i \neq j$. Then $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ is called a *g-diagonal* $(n+1)$-decomposition of the pair (U,V). If n is understood from the context, we call these decompositions *g-diagonal decompositions*. The submodules

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$\{U_i\}_{i=0}^n$ and $\{V_i\}_{i=0}^n$ of U and V respectively are called the *g-diagonal components* of the decomposition. The subset $J = \{i : U_i \neq 0 \text{ or } V_i \neq 0\}$ of $I = \{0, \dots, n\}$ is called the *support of the g-diagonal decomposition*. Of course, if g is nondegenerate, then $J = \{i : V_i \neq 0\} = \{i : U_i \neq 0\}$. The g-diagonal decomposition of (U, V) is said to be *strong* if $g \neq 0$ and $g(U_i, V_i) = A$ for $i \in J$. Note that if A is simple and g is nonzero and nondegenerate, then any g-diagonal decomposition is strong.

If $g \colon U \times V \to A$ is nondegenerate, it is easy to see that given a g-diagonal decomposition $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ one has $U_i = \bigcap_{j \neq i} V_j^\perp$ where $V_j^\perp = \{u \in U : g(u, V_j) = 0\}$. Thus for every direct sum decomposition $V = \bigoplus_{i=0}^n V_i$ there is at most one decomposition $U = \bigoplus_{i=0}^n U_i$ so that $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ is g-diagonal. Moreover, a direct sum decomposition $V = \bigoplus_{i=0}^n V_i$ is part of a g-diagonal decomposition $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ if and only if $U = \sum_{i=0}^n U_i$ where $U_i = \bigcap_{j \neq i} V_j^\perp$.

Proposition 4.4 *Suppose that $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ is a g-diagonal decomposition relative to a bilinear form $g \colon U \times V \to A$. Then*

$$F_g = \bigoplus_{i,j=0}^{n} (F_g)_{i,j}, \quad \textit{where } (F_g)_{i,j} = e_{V_i, U_j}, \tag{10}$$

is a Pierce grading of F_g. Moreover, (10) is the unique Peirce grading of F_g relative to which $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are graded F_g-modules.

Proof First of all, we obtain a Pierce grading for the unital algebra $E = \big(\mathrm{End}_A(V), \mathrm{End}_A(U)^{\mathrm{op}}\big)$. For $i = 0, \dots, n$, let p_i be the projection of V onto V_i, let q_i to be the projection of U onto U_i, and let $e_i = \big(p_i, (q_i)^\circ\big)$ in E. Then the set $\{e_i : i = 0, \dots, n\}$ is a complete set of orthogonal idempotents in E and $E = \bigoplus_{i,j=0}^n e_i E e_j$ is a Peirce grading of E.

Second, we obtain the Pierce grading for F_g. We certainly have $F_g = e_{V,U} = \sum_{i,j=0}^n e_{V_i, U_j}$. Also, one easily checks that $e_i e_{v_i, u_j} = e_{v_i, u_j}$ and $e_{v_i, u_j} e_j = e_{v_i, u_j}$ all $v_i \in V_i$ and $u_j \in U_j$. Thus, we have $e_{V_i, U_j} = e_i e_{V_i, U_j} e_j \subseteq e_i E e_j$ for all i, j. Hence the sum $F_g = \sum_{i,j=0}^n e_{V_i, U_j}$ is direct and

$$e_{V_i, U_j} = e_i F_g e_j$$

for all i, j. It follows from this that $F_g = \bigoplus_{i,j=0}^n e_{V_i, U_j}$ is a Peirce grading, and that $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are graded F_g-modules relative to this Peirce grading. Finally, if $F_g = \bigoplus_{i,j=0}^n (F_g)'_{i,j}$ is any Peirce grading relative to which $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are graded modules, we have $(F_g)'_{i,j} = e_i (F_g)'_{i,j} e_j \subseteq (F_g)_{i,j}$ for all i, j. ■

Remark 4.5 If the idempotents $e_i = \big(p_i, (q_i)^\circ\big)$ are constructed as in the above proof, then q_i is an adjoint of p_i, that is $g(q_i u, v) = g(u, p_i v)$ for every $u \in U$ and $v \in V$. In particular, it follows that if $V = \bigoplus_{i=0}^n V_i$ is a part of g-diagonal decomposition of (U, V) then every projection $p_i \colon V \to V_i$ has an adjoint. If g is non-degenerate, it is easy to show that the converse is also true. Namely, if $V = \bigoplus_{i=0}^n V_i$ is a direct sum

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of A-modules such that every projection $p_i\colon V \to V_i$ has an adjoint $q_i \in \operatorname{End}_A(U)$ relative to g, then $\{q_i : i = 0, \dots, n\}$ is a complete set of orthogonal idempotents in $\operatorname{End}_A(U)$ and $(\bigoplus_{i=0}^n q_i U, \bigoplus_{i=0}^n V_i)$ is a g-diagonal decomposition of (U, V).

Suppose that g is as in Proposition 4.4. The Peirce grading defined by (10) will be called the *Peirce grading of F_g determined by the g-diagonal decomposition* $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$. The $\mathbb{Z}$-grading induced by this Peirce grading is given by

$$(11) \qquad F_g = \bigoplus_{i=-n}^{n} (F_g)_i, \quad \text{where } (F_g)_i = \sum_{p-q=i} e_{V_p, U_q},$$

and called the *$\mathbb{Z}$-grading of F_g determined by the g-diagonal decomposition* $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$.

When one considers the $\mathbb{Z}$-gradings (11) it is often convenient to assume that the g-diagonal decomposition $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ is *regular*, which means that 0 and n are in the support of the decomposition. For an arbitrary g-diagonal $(n+1)$-decomposition $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ of a nontrivial pair (U, V), one can consider the regular $(l - s + 1)$-decomposition $(\bigoplus_{i=0}^{l-s} U_i', \bigoplus_{i=0}^{l-s} V_i')$, where l is the largest number and s is the smallest number in the support of the decomposition and where $U_i' = U_{i+s}$ and $V_i' = V_{i+s}$ for $i, j \in \{0, \dots, l - s\}$. This change in enumeration of components does not affect the $\mathbb{Z}$-grading (11).

We are now ready to prove a coordinatization theorem for uni algebras with strong Peirce gradings.

Theorem 4.6

(i) *Let A be a uni algebra, $U \in A\text{-mod}$, $V \in \text{mod-}A$, and let $g\colon U \times V \to A$ be a nonzero surjective nondegenerate bilinear form. Then $F_g(U, V, A)$ is a uni algebra and (10) establishes a bijective correspondence between strong g-diagonal $(n+1)$-decompositions of (U, V) and strong $(n+1)\times(n+1)$-Peirce gradings of $F_g(U, V, A)$. Under this correspondence, the support of the decomposition is equal to the Peirce support of the corresponding Peirce grading.*

(ii) *Conversely, let $R = \bigoplus_{i,j=0}^n R_{i,j}$ be a uni algebra with a strong $(n+1) \times (n+1)$-Peirce grading. Then there exists a uni algebra A, $U \in A\text{-mod}$, $V \in \text{mod-}A$, and a nonzero surjective nondegenerate bilinear form $g\colon U \times V \to A$ so that R is graded isomorphic to the Peirce graded algebra $F_g(U, V, A)$, where the Peirce grading on $F_g(U, V, A)$ is determined by a strong g-diagonal $(n+1)$-decomposition of (U, V). The components A, U, V and g and the g-diagonal components $\{U_i\}_{i=0}^n$ and $\{V_i\}_{i=0}^n$ can be chosen as follows: Fix i with $R_{i,i} \neq 0$ (such an i exists by Lemma 3.1). Put $A = R_{i,i}$, $U = \bigoplus_{j=0}^n U_j$ where $U_j = R_{i,j}$, and $V = \bigoplus_{j=0}^n V_j$ where $V_j = R_{j,i}$, and define $g\colon U \times V \to A$ by $g(u, v) = uv$ (multiplication in R).*

Proof (i): By Proposition 4.1, F_g is a uni algebra, $V \in F_g\text{-mod}$, and $U \in \text{mod-}F_g$. By Proposition 4.4, (10) defines a mapping from the set of g-diagonal $(n+1)$-decompositions of (U, V) to the set of $(n+1) \times (n+1)$-Peirce gradings of F_g. Let $(\bigoplus_{i=0}^n U_i$,

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$\bigoplus_{i=0}^n V_i)$ be a strong g-diagonal decomposition of (U,V). To see that the corresponding Peirce grading on F_g is strong, note first that $F_g \neq 0$ (by Proposition 4.1 since $g \neq 0$). Also, if J is the support of this decomposition and $f\colon V \times U \to F_g$ is defined by (9), we have $f(V_i,U_k) = f(V_iA,U_k) = f\big(V_i g(U_j,V_j),U_k\big) = f\big(f(V_i,U_j)V_j,U_k\big) = f(V_i,U_j)f(V_j,U_k)$ for $i,j,k \in J$. On the other hand if either $i \notin J$ or $j \notin J$ we trivially have $f(V_i,U_j) = 0$. Thus, the Peirce grading on F_g is strong. Furthermore, uniqueness in Proposition 3.6 implies that a g-diagonal decomposition that induces a given strong Peirce grading is unique.

It is left to show that a strong Peirce grading $F_g = \bigoplus_{i,j=0}^n (F_g)_{i,j}$ is determined by a strong g-diagonal decomposition. Set $U_i = U(F_g)_{i,i}$ and $V_i = (F_g)_{i,i}V$ for $0 \le i \le n$. By Proposition 3.6 the modules $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are graded F_g-modules relative to $F_g = \bigoplus_{i,j=0}^n (F_g)_{i,j}$. These gradings constitute a g-diagonal decomposition of (U,V) because $g(U_i,V_j) = g\big(U(F_g)_{i,i},(F_g)_{j,j}V\big) = g\big(U,(F_g)_{i,i}(F_g)_{j,j}V\big) = 0$ if $i \neq j$.

Note that U and V are faithful F_g-modules (see Remark 4.2), and therefore, by Lemma 3.1, the support of this decomposition equals P-$\mathrm{Supp}(F_g)$. Moreover, for $i,j \in P$-$\mathrm{Supp}(F_g)$, we have

$$\begin{aligned} g(U_i,V_i) &= g\big(U(F_g)_{i,i},(F_g)_{i,i}V\big) = g\big(U,(F_g)_{i,i}V\big) = g\big(U,(F_g)_{i,j}(F_g)_{j,i}V\big) \\ &= g\big(U(F_g)_{i,j},(F_g)_{j,i}V\big) \subseteq g\big(U(F_g)_{j,j},(F_g)_{j,j}V\big) = g(U_j,V_j). \end{aligned}$$

Therefore $g(U_i,V_i) = g(U_j,V_j)$ and $A = g(U,V) = \sum_{k=0}^n g(U_k,V_k) = g(U_i,V_i)$. That is, this g-diagonal decomposition is strong.

Finally, the given Peirce grading of F_g and the Peirce grading of F_g determined by the g-diagonal decomposition $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ are both Peirce gradings relative to which $U = \bigoplus_{i=0}^n U_i$ and $V = \bigoplus_{i=0}^n V_i$ are graded modules. Hence, by uniqueness in Proposition 4.4, these two Peirce grading are the same.

(ii): Suppose that $R = \bigoplus_{i,j=0}^n R_{i,j}$ be a uni algebra with a strong Peirce grading. Let $I = \{0,\dots,n\}$ and let J be the Peirce support of the grading of R. Fix $i \in J$, and define A, U, V and g as in the last sentence of (ii).

It follows from the definition of strong Peirce grading, (2) and Lemma 3.5 that A is a uni algebra, $U \in A$-mod, $V \in$ mod-A, g is a nonzero surjective bilinear form, and $(\bigoplus_{i=0}^n U_i, \bigoplus_{i=0}^n V_i)$ is a strong g-diagonal decomposition of (U,V) with support J. In fact the support of this decomposition equals the Peirce support of R.

To show that g is nondegenerate, we verify that the radical of g on the left is zero (the radical on the right is handled in the same way). For this it is enough to show that if $u_j \in U_j$ and $g(u_j,V_j) = 0$ then $u_j = 0$. So $u_j \in R_{i,j}$ and $u_jR_{j,i} = 0$. We can assume that $j \in J$. Then, for any $k \in J$, we have $uR_{j,k} = uR_{j,i}R_{i,k} = 0$. Hence, for any $k \in I$, $uR_{j,k} = 0$. Thus $uR = 0$ and so $u = 0$. Therefore g is nondegenerate.

To prove that $R \simeq F_g(U,V,A)$ we define $\varphi\colon R \to \big(\mathrm{End}_A(V),\mathrm{End}_A(U)^{\mathrm{op}}\big)$ by $\varphi(x) = \big(L_x|_V,(R_x|_U)^\circ\big)$. It is clear that φ is an algebra homomorphism.

We now look at the kernel of φ. Since $R_{p,q} = R_{p,i}R_{i,q}$ for every $p,q \in J$, we have $R = VU$. So, $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ann}^l(R) = 0$.

Next, we consider the image of φ. For any $v,v' \in V$ and $u,u' \in U$ one has $L_{vu}v' =$

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$vuv' = vg(u, v') = x_{v,u}v'$ and $R_{vu}u' = u'vu = g(u', v)u = y_{v,u}u'$. So, $\varphi(vu) = \big(L_{vu}|_V, (R_{vu}|_U)^\circ\big) = \big(x_{v,u}, (y_{v,u})^\circ\big) = e_{v,u}$ and $\varphi(R) = \varphi(VU) = F_g(U, V, A)$.

Finally, if $p, q \in J$, we have $\varphi(R_{p,q}) = \varphi(R_{p,i}R_{i,q}) = \varphi(V_pU_q) = f(V_p, U_q) = (F_g)_{p,q}$, where f is defined as in (9). Since $R_{p,q} = 0$ if p or q is not in J, it follows that φ preserves the Peirce grading. The proof is complete. ■

We now apply Theorem 4.6 to prove a coordinatization theorem for simple algebras with finite $\mathbb{Z}$-gradings.

Theorem 4.7

(i) *If A is a simple algebra, $U \in A$-mod, $V \in$ mod-A, and $g\colon U\times V \to A$ is a nonzero nondegenerate bilinear form, then $F_g(U, V, A)$ is a simple algebra and (11) establishes a bijective correspondence between regular g-diagonal $(n+1)$-decompositions of (U, V) and finite $\mathbb{Z}$-gradings of $F_g(U, V, A)$ of height n.*

(ii) *Conversely, suppose that $R = \bigoplus_{i=-n}^{n} R_i$ is a simple algebra with finite $\mathbb{Z}$-grading of height n. Then there is a simple ideal A of R_0, $U \in A$-mod, $V \in$ mod-A, and a nonzero nondegenerate bilinear form $g\colon U \times V \to A$ such that R is graded isomorphic to the algebra $F_g(U, V, A)$, where the $\mathbb{Z}$-grading on $F_g(U, V, A)$ is determined by a regular g-diagonal $(n + 1)$-decomposition of (U, V). In fact, R_0 is the nonzero direct sum of finitely many simple ideals and A can be taken to be any one of these ideals.*

Proof (i): Our assumptions imply that g is surjective, so F_g is simple by Propositions 4.1. Furthermore, every Peirce grading on F_g is strong by Proposition 3.4 and every g-diagonal decomposition is strong because A is simple. Hence (10) describes a bijective correspondence between g-diagonal $(n+1)$-decompositions of (U, V) and $(n+1)\times(n+1)$-Peirce gradings of F_g by Theorem 4.6(i). Under this correspondence regular decompositions correspond to regular Peirce gradings (by the last statement in Theorem 4.6(i)). Now an application of Proposition 3.7 completes the proof.

(ii): By Proposition 3.7, there is a regular Peirce grading $R = \bigoplus_{i,j=0}^{n} R_{i,j}$ on R that induces the given $\mathbb{Z}$-grading. Furthermore, by Proposition 3.4, this Peirce grading is strong. Also, $R_0 = \bigoplus_{i=0}^{n} R_{i,i}$, and since R is simple the summands are either simple or 0 [S1, Lemma 3.7]. The theorem now follows from Theorem 4.6(ii). ■

In order to make use of some classical facts about simple rings, we assume from now on that Φ is the ring of integers.

Remark 4.8 Suppose that R is an artinian simple ring. Then, by the Wedderburn-Artin theorem, we may identify $R = \mathrm{End}_A(V) \simeq F_g(U, V, A)$, where A is a division ring, V is a finite dimensional right vector space over A, U is the dual space of V and $g\colon U \times V \to A$ is the natural pairing (see Remarks 4.2 and 4.3). It is clear in this setting that any A-module decomposition $V = \bigoplus_{i=0}^{n} V_i$ determines a unique g-diagonal decomposition $(\bigoplus_{i=0}^{n} U_i, \bigoplus_{i=0}^{n} V_i)$ of the pair (U, V). Hence, Theorem 4.7(i) provides a bijective correspondence from the set of A-module decompositions $V = \bigoplus_{i=0}^{n} V_i$ with $V_0 \neq 0$ and $V_n \neq 0$ onto the set of finite $\mathbb{Z}$-gradings of R of height n. This correspondence is already known. Indeed, it can be deduced from the results of [NvO, Chapter I] (see I.2.3, I.5.8, I.4.3 and I.5.4). It also follows from Theorem 1 in [ZS].

In conclusion, we describe applications of Theorem 4.7 to simple rings with nonzero socle.

First, we show that the classical description of a simple ring R with nonzero socle that is due to Jacobson follows from Theorem 4.7. By [J, Propositions 3.9.1 and 4.3.1], R contains a nonzero idempotent e so that eRe is a division ring. Thus R has a 2×2 Peirce grading $R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$. (Although R may not be unital, the components here have obvious interpretations.) The $\mathbb{Z}$-grading of R induced by this Peirce grading has height 0 if $R = eRe$ and height 1 otherwise. By Theorem 4.7(ii), R is isomorphic to $F_g(U, V, A)$, where A is a division ring (namely $A = eRe$), $U \in A$-mod, $V \in$ mod-A, and $g\colon U \times V \to A$ is a nonzero nondegenerate bilinear form. (See also [J, Section 4.9] where the larger class of primitive rings with nonzero socle is described.)

Second, Theorem 4.7(i) gives the following description of finite $\mathbb{Z}$-gradings on the ring F_g. Recall that since A is a division ring, V is a topological vector space with subbase of neighborhoods of zero $\{\{v : g(u, v) = 0\} : u \in U\}$, and an element of $\mathrm{End}_A(V)$ is continuous if and only if it has an adjoint relative to g ([J, Theorem 1, p. 72]). Thus, Theorem 4.7(i) and Remark 4.5 imply

Corollary 4.9 *Suppose that A is a division ring, $U \in A$-mod, $V \in$ mod-A, and $g\colon U \times V \to A$ is a nonzero nondegenerate bilinear form. Then there is a bijective correspondence between the set of finite $\mathbb{Z}$-gradings of F_g of height n and the set of vector space decompositions $V = \bigoplus_{i=0}^{n} V_i$ with $V_0 \neq 0$ and $V_n \neq 0$ such that every projection $p_i\colon V \to V$, defined by $p_i(\sum_{j=0}^{n} v_j) = v_i$, is continuous.*

References

[AM] P. N. Ánh and L. Márki, *Rees matrix rings.* J. Algebra **81**(1983), 340–369.

[A] P. Ara, *Morita equivalence for rings with involutions.* Algebr. Represent. Theory **2**(1999), 227–247.

[BSZ] Yu. A. Bahturin, S. K. Sehgal and M. V. Zaicev, *Group gradings on associative algebras.* J. Algebra **241**(2001), 677–698.

[Be] G. M. Bergman, *Radicals, tensor products and algebraicity.* In: Ring Theory in Honor of S. A. Amitsur, Weizmann Science Press, Jerusalem, 1989, 150–192.

[BD] C. Boboc and S. Dăscălescu *Gradings of matrix algebras by cyclic groups.* Comm. Algebra **29**(2001), 5013–5021.

[DKvW] S. Dăscălescu, A. V. Kelarev and L. van Wyk, *Semigroup gradings of full matrix rings.* Comm. Algebra **29**(2001), 5023–5031.

[GS] J. L. Garcia and J. J. Simon, *Morita Equivalence for idempotent rings.* J. Pure Appl. Algebra **76**(1991), 39–56.

[J] N. Jacobson, *Structure of rings.* Amer. Math. Soc. Colloq. Publ. **37**, Providence, R.I., 1956.

[K] S. Kyono, *Equivalence of module categories.* Math. J. Okayama Univ. **28**(1986), 147–150.

[NvO] C. Năstăsescu and F. Van Oystaen, *Graded ring theory.* North-Holland Publ. Comp. **29**, 1982.

[S1] O. Smirnov, *Simple associative algebras with finite $\mathbb{Z}$-grading.* J. Algebra **196**(1997), 171–184.

[S2] ———, *Finite $\mathbb{Z}$-gradings of Lie algebras and symplectic involutions.* J. Algebra **218**(1999), 246–275.

https://doi.org/10.4153/CMB-2002-048-4 Published online by Cambridge University Press

[ZS] M. V. Zaicev and S. K. Sehgal, *Finite gradings of simple Artinian rings*. Vestnik Moskov. Univ. Ser. I Mat. Mech., (to appear).

[Z] E. I. Zelmanov, *Lie algebras with finite $\mathbb{Z}$-grading*. Math. USSR-Sb. **52**(1985), 347–385.

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