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THE GROUPS OF THE REGULAR STAR-POLYTOPES

With best wishes to H. S. M. (Donald) Coxeter for his 90th birthday

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ABSTRACT. The regular star-polyhedron $\{5, \frac{5}{2}\}$ is isomorphic to the abstract polyhedron $\{5, 5 \,|\, 3\}$, where the last entry "3" in its symbol denotes the size of a hole, given by the imposition of a certain extra relation on the group of the hyperbolic honeycomb $\{5, 5\}$. Here, analogous formulations are found for the groups of the regular 4-dimensional star-polytopes, and for those of the non-discrete regular 4-dimensional honeycombs. In all cases, the extra group relations to be imposed on the corresponding Coxeter groups are those arising from "deep holes"; thus the abstract description of $\{5, 3^k, \frac{5}{2}\}$ is $\{5, 3^k, 5 \,|\, 3\}$ for $k = 1$ or 2. The non-discrete quasi-regular honeycombs in $\mathbb{E}^3$, on the other hand, are not determined in an analogous way.

1**. Introduction.** The regular 4-dimensional star-polytopes were discovered by Schläfli and Hess in the last century (we refer the reader to [2] for historical details). They all share the symmetry group of the regular convex polytope $\{3, 3, 5\}$, which is the Coxeter group $[3, 3, 5]$, and so in one sense there is nothing more to be said about them. However, Coxeter observed in [1] that the regular star-polyhedron $\{5, \frac{5}{2}\}$ is isomorphic to the abstract regular polyhedron $\{5, 5 \,|\, 3\}$. This is obtained from the regular hyperbolic honeycomb $\{5, 5\}$ by a certain identification, which forces some three of its edges to form a "hole" (we shall be more precise in Section 3).

In this paper, we investigate the regular 4-dimensional star-polytopes in the same spirit. We shall show that, regarded as abstract regular polytopes (see, for example, [9, 11, 13] for background material), they fall into two classes. Those which contain $\{5, \frac{5}{2}\}$ or its dual as a facet or vertex-figure inherit their group structure from the lower-dimensional components; for example, $\{3, 5, \frac{5}{2}\}$ is isomorphic to the universal regular polytope $\big\{\{3, 5\}, \{5, 5 \,|\, 3\}\big\}$. The two remaining examples, which are dual (and combinatorially self-dual), are obtained by identifications from the corresponding regular hyperbolic honeycomb, which again forces some three of its edges to form a "deep hole"; thus $\{5, 3, \frac{5}{2}\}$ is isomorphic to an abstract regular polytope, which will be denoted $\{5, 3, 5 \,|\, 3\}$.

A similar pattern is maintained for the non-discrete regular honeycombs in $\mathbb{E}^4$. That is, all these honeycombs are universal amalgamations (in the sense of [16]) of their facets and vertex-figures in the expected way, except that $\{5, 3, 3, \frac{5}{2}\}$ and its dual are isomorphic to an abstract regular 5-apeirotope which is denoted $\{5, 3, 3, 5 \,|\, 3\}$. The problem which

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arises here is that the groups are infinite, so that proving universality poses a certain challenge.

A contrast is provided by the closely related non-discrete quasi-regular honeycombs in $\mathbb{E}^3$. For example, the honeycomb

$$\left\{5, {3 \atop \frac{5}{2}}\right\}$$

occurs as a cut of $\{5, 3, 3, \frac{5}{2}\}$. But this honeycomb is not determined by the extra relation on

$$\left\{5, {3 \atop 5}\right\}$$

which forces $\{5, 5\} \longmapsto \{5, 5 \mid 3\} \cong \{5, \frac{5}{2}\}$, even though the same relation applied to the corresponding cut of $\{5, 3, 3, 5\}$ does collapse it onto $\{5, 3, 3, \frac{5}{2}\}$. In other words, the cut here is not universal in the sense of [12, Section 3].

2**. Abstract regular polytopes.** We shall discuss regular polytopes here largely on the geometric level, and so we shall pay little attention to the underlying theory of abstract regular polytopes. For this, we refer the reader to, for example, [10], and the forthcoming monograph [13]. Except in Section 7, we shall work exclusively in euclidean spaces; hence our polytopes will always be *realized* in the sense of [6, 8] (if we slightly extend the definition to permit realizations in hyperbolic spaces as well). These polytopes may be thought of as "classical", because for the most part they were the object of the central text [2].

For our purposes, an n-polytope $\mathcal{P}$ consists of faces of each dimension $0, 1, \ldots, n-1$; we also talk about *vertices* for 0-faces, *edges* for 1-faces, and, for a k-polytope, its *facets* and *ridges* will be faces of codimension 1 and 2, respectively. (Strictly speaking, we should use the term "rank" instead of "dimension", but here the concepts will coincide.) We may think of a polytope as built up recursively: each face is composed of its facets, and these fit two around each ridge. Two faces of $\mathcal{P}$, one of which contains the other, are called *incident*. Under inclusion, $\mathcal{P}$ is a partially ordered set; in fact, in all instances here, it will be a lattice, if we adjoin two *improper* faces, a unique minimum F_{-1} and maximum F_n.

It may happen that $\mathcal{P}$ is infinite; we then often refer to $\mathcal{P}$ as an *apeirotope*. In this case, $\mathcal{P}$ will sit naturally in an $(n-1)$-dimensional space; it may or may not be discrete. The terms *polygon* or *apeirogon* will also be used for a 2-polytope, and *polyhedron* or *apeirohedron* for a 3-polytope.

For two faces F and G of $\mathcal{P}$ with $F \subseteq G$ (here we allow improper faces), the family $G/F := \{J \in \mathcal{P} \mid F \subseteq J \subseteq G\}$ is called a *section* of $\mathcal{P}$; it is the lattice of faces of a polytope of dimension $\dim G - \dim F - 1$. The most important case is when $F = v$ $(= F_0)$ is a vertex and $G = F_n$ is the maximal improper face; in this case, F_n/v is the *vertex-figure* of $\mathcal{P}$ at v. Geometrically, we shall always be able to think of a vertex-figure as follows: the vertices of F_n/v will be the other vertices w of edges of $\mathcal{P}$ which contain v; the j-faces of F_n/v will then be the vertex-figures, in the obvious recursive sense, of the $(j+1)$-faces of $\mathcal{P}$ which contain v.

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A *flag* of an n-polytope P is a set $\Phi = \{F_0, F_1, \ldots, F_{n-1}\}$ of mutually incident faces (the convention is that F_j has dimension j). In view of the recursive formulation above, for each $j = 0, \ldots, n-1$, there is a unique flag which differs from Φ in its j-face; we denote this flag by Φ^j, and say that it is *adjacent* to Φ. The technical conditions for a poset P to be a polytope are then that every chain $G_{j(1)} < \cdots < G_{j(s)}$ of faces of P be contained in a flag, that the adjacent flag Φ^j to Φ be unique, and that P is *strongly flag-connected*, meaning that, if Φ and Ψ are any two flags, then they can be joined by some sequence $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ of flags, each containing $\Phi \cap \Psi$, and such that Φ_i is adjacent to Φ_{i-1} for each $i = 1, \ldots, k$.

The *symmetry group* $\Gamma(P)$ consists of the isometries of the ambient space which take P into itself (that is, which permute its faces of each dimension); again, we frequently confuse this with its (abstract) *automorphism group*, since the action of the symmetry group will necessarily be faithful; in any event, we often refer merely to the *group* of P. We then call P *regular* if $\Gamma(P)$ is transitive on the family $F(P)$ of flags of P.

Let $\Phi := \{F_0, \ldots, F_{n-1}\}$ be a fixed or *base* flag of P; its faces are also called the *base* faces. It is easy to show (see, for example, [9, 10, 13]) that the group $\Gamma(P)$ of a regular n-polytope P is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$ (*with respect to* Φ), where ρ_j is the unique automorphism such that $\Phi^j = \Phi\rho_j$ for $j = 0, \ldots, n-1$. These generators satisfy relations $(\rho_j \rho_k)^{p_{jk}} = \varepsilon$ for $0 \leq j \leq k \leq n-1$, where

$$p_{jk} = \begin{cases} 1, & \text{if } j = k, \\ p_k \geq 3, & \text{if } j = k-1, \\ 2, & \text{if } j \leq k-2. \end{cases} \tag{2.1}$$

Further, $\Gamma(P)$ has the *intersection property* (with respect to the distinguished generators), namely, if $I, J \subset \{0, \ldots, n-1\}$, then

$$\langle \rho_j \mid j \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_j \mid j \in I \cap J \rangle. \tag{2.2}$$

The numbers $p_k := p_{k-1,k}$ ($k = 1, \ldots, n-1$) determine the (*Schläfli*) *type* $\{p_1, \ldots, p_{n-1}\}$ of P.

Observe that, in a natural way, the group of the base facet of P is $\langle \rho_0, \ldots, \rho_{n-2} \rangle$, while that of the vertex-figure at the base vertex is $\langle \rho_1, \ldots, \rho_{n-1} \rangle$. (By the way, we shall usually refer to *the* facet or *the* vertex-figure of a regular polytope, since all the facets or vertex-figures are equivalent.) Further, the given conditions are easily seen to imply that the group $\Gamma(P)$ is *simply* transitive on the flags of P, so that there is a one-to-one correspondence $\Psi \leftrightarrow \gamma$ between flags and group elements, given by $\Psi = \Phi\gamma$.

By a *string C-group*, we mean a group with generators ρ_j which satisfy (1) and (2). The group of a regular polytope is a string C-group. Conversely, given a string C-group, there is an associated (abstract) regular polytope of which it is the automorphism group ([10]). In verifying that a given group is a C-group, it is usually only the intersection property which causes difficulty. Note that Coxeter groups are examples of C-groups (see [10, 17]). For abstract regular polytopes, we may have $p_j = 2$ for some j; this leads to polytopes which we can think of as degenerate. In the present context, the fact that the groups will be generated by hyperplane reflexions will ensure that the intersection property must hold, and so they will automatically be C-groups.

The identification of regular polytopes with string C-groups shows that presentations of groups play an important rôle. (For the time being, we are discussing regular polytopes on the abstract level, but this context provides the appropriate language.) While in practice we do usually work with the groups, it is very convenient to have some alternative notation. Let $\mathcal{P}$ be a regular polytope, with group $\Gamma := \Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ as above, where the ρ_j are the distinguished generators associated with the base flag Φ. Now for any flag Ψ, any $j = 0, \ldots, n-1$ and any $\gamma \in \Gamma$, we clearly have $\Psi^j\gamma = (\Psi\gamma)^j$. In particular, $(\Phi\gamma)^j = \Phi^j\gamma = \Phi\rho_j\gamma$, and it follows that

$$\Phi\rho_{j(1)}\rho_{j(2)}\cdots\rho_{j(m)} = \Phi^{j(m)\cdots j(2)j(1)},$$

for any $j(1), \ldots, j(m) \in \{0, \ldots, n-1\}$. In other words, an element $\gamma = \rho_{j(1)}\rho_{j(2)}\cdots\rho_{j(m)} \in \Gamma$ corresponds to an *adjacency sequence* $w = j(m)\cdots j(2)j(1)$. Note that γ^{-1} then corresponds to the *reverse* sequence $w^{-1} := j(1)j(2)\cdots j(m)$. More particularly, a relation on Γ corresponds to an *adjacency cycle*; such a cycle may be started at any point in the sequence, which corresponds to conjugacy or a different choice of the base flag, or reversed, since each ρ_j is an involution. Thus a relation $\rho_{j(1)}\rho_{j(2)}\cdots\rho_{j(m)} = \varepsilon$ in Γ corresponds to the adjacency cycles $j(m)\cdots j(2)j(1)$, $j(1)j(2)\cdots j(m)$, $j(2)\cdots j(m)j(1)$, and so on.

There is an obvious equivalence relation on adjacency sequences, corresponding to conjugacy of group elements. In particular, equivalent sequences can be obtained by inserting or deleting terms such as $(jk)^{p_{jk}}$; we may also allow equivalence modulo other (previously) known adjacency cycles. Similarly, if w is an adjacency cycle, then so is $s^{-1}ws$ for any adjacency sequence s. However, since we work below with mixing operations (that is, changing generators), for the reader's convenience we shall keep to group elements when manipulating relations.

Now suppose that $\mathcal{Q}$ is a *quotient polytope* of $\mathcal{P}$, so that $\mathcal{Q}$ is also a regular n-polytope, whose group $\Gamma(\mathcal{Q})$ is a quotient $\Gamma(\mathcal{P})/\Sigma$ of $\Gamma(\mathcal{P})$ by some normal subgroup Σ. If Σ is the normal closure of the relators in $\Gamma(\mathcal{P})$ associated with the adjacency cycles $w_1, \ldots, w_m$, then we shall use the notation

$$\mathcal{Q} := \mathcal{P}/\langle\langle w_1, \ldots, w_m \rangle\rangle. \tag{2.3}$$

For the most part, $\mathcal{P}$ will be the universal polytope $\{p_1, \ldots, p_{n-1}\}$ with a given Schläfli symbol; we shall postpone until later examples of the notation. Observe, though, one advantage of the notation—it avoids having to use circumlocutions such as "the polytope whose group is the Coxeter group $[p_1, \ldots, p_{n-1}] = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, with the additional relations ..."; in other words, it is independent of any notation for the underlying group.

Given regular n-polytopes $\mathcal{P}$ and $\mathcal{Q}$ such that the vertex-figure of $\mathcal{P}$ is isomorphic to the facet of $\mathcal{Q}$, we denote by $\langle \mathcal{P}, \mathcal{Q} \rangle$ the *class* of all regular $(n+1)$-polytopes $\mathcal{R}$ with facet isomorphic to $\mathcal{P}$ and vertex-figure isomorphic to $\mathcal{Q}$. If $\langle \mathcal{P}, \mathcal{Q} \rangle \neq \emptyset$, then any such

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$\mathcal{R}$ is a quotient of a universal member of $\langle \mathcal{P}, \mathcal{Q} \rangle$; this *universal polytope* is denoted by $\{\mathcal{P}, \mathcal{Q}\}$ (see [10, 13, 15, 16]). In practical terms, universality means that no additional relations are imposed on the resulting automorphism group of the polytope other than those inherited from the groups of the facet and vertex-figure. The determination of whether, for given $\mathcal{P}$ and $\mathcal{Q}$, the universal $\{\mathcal{P}, \mathcal{Q}\}$ exists, and if it does whether it is finite, are central questions in the theory of abstract regular polytopes.

We end the general discussion of regular polytopes and their groups with a useful remark. Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be the group of a regular n-polytope $\mathcal{P}$, and suppose that $\gamma \in \Gamma$. Then we can express γ in the form

$$\gamma = \alpha_0 \rho_0 \alpha_1 \rho_0 \cdots \alpha_{m-1} \rho_0 \alpha_m,$$

with $\alpha_i \in \Gamma_0 := \langle \rho_1, \ldots, \rho_{n-1} \rangle$, the group of the vertex-figure of $\mathcal{P}$ at its base vertex $v := F_0$, for $i = 0, \ldots, m$. With γ, we can associate a path in $\mathcal{P}$ with m edges leading from v to $v\gamma$. If $m = 0$, the path consists of v $(= v\alpha_0)$ alone. For $m > 0$, let $(E'_1, \ldots, E'_{m-1})$ be an edge-path associated with $\alpha_0 \rho_0 \alpha_1 \rho_0 \cdots \alpha_{m-1}$. With γ is then associated the path $(E_1, \ldots, E_m)$, given by

$$E_i := \begin{cases} E\alpha_m \ (= E\rho_0\alpha_m), & \text{if } i = 1, \\ E'_{i-1}\rho_0\alpha_m & \text{if } i = 2, \ldots, m, \end{cases}$$

where $E := F_1$ is the base edge of $\mathcal{P}$. Of course, this path will not generally be unique, since it depends on the particular expression for γ.

Conversely, an edge-path $(E_1, \ldots, E_m)$ from v corresponds to such an element $\gamma \in \Gamma$, in which ρ_0 occurs m times. If $m > 0$, then there is an $\alpha_m \in \Gamma_0$ such that $E_1 = E\alpha_m$. The shorter path $(E'_1, \ldots, E'_{m-1})$, given by

$$E'_i := E_{i+1}\alpha_m^{-1}\rho_0$$

for $i = 1, \ldots, m-1$, also starts at v, and we can repeat to obtain γ as above, with a free choice of α_0.

In the context of group presentations, we deduce the following, whose condition is called the *circuit criterion*.

PROPOSITION 2.4. *Let $\mathcal{P}$ be a regular polytope. Then the group $\Gamma = \Gamma(\mathcal{P})$ of $\mathcal{P}$ is determined by the group of its vertex-figure, and the relations on the distinguished generators of Γ induced by the edge-circuits of $\mathcal{P}$ which contain the initial vertex.*

PROOF. A relation on Γ can be written in the form

$$\alpha_0 \rho_0 \alpha_1 \rho_0 \cdots \alpha_{m-1} \rho_0 = \varepsilon,$$

with $\alpha_i \in \Gamma_0$ for $i = 0, \ldots, m-1$, which corresponds to an edge-circuit starting and ending at v. Conversely, such an edge-circuit is equivalent under Γ_0 to one beginning with E, and this gives rise to a relation as above (now the element α_0 will be determined by the circuit). This is the result. ∎

3**. Deep holes.** We first consider the finite regular star-polytopes in $\mathbb{E}^4$. Our aim is to give presentations for their symmetry groups in terms of the natural generators. However, our starting point is Coxeter's observation in [1] that

$$\left\{5, \tfrac{5}{2}\right\} \cong \{5, 5 \,|\, 3\}.$$

We must begin by explaining and generalizing this notation.

For any $n \geq 2$, the *Coxeter group* $[p_1, \ldots, p_{n-1}]$ has the presentation

$$[p_1, \ldots, p_{n-1}] := \left\langle \rho_0, \ldots, \rho_{n-1} \mid (\rho_j \rho_k)^{p_{jk}} = \varepsilon (0 \leq j \leq k \leq n-1) \right\rangle,$$

where the relations of (2.1) hold, namely

$$p_{jk} = \begin{cases} 1, & \text{if } j = k, \\ p_k, & \text{if } j = k-1, \\ 2, & \text{if } j \leq k-2. \end{cases}$$

A Coxeter group has to satisfy the intersection property (2.2). We shall always use this notation; in the sense of Section 2, the ρ_j are the distinguished generators of $[p_1, \ldots, p_{n-1}]$.

We now define a new group $[p_1, \ldots, p_{n-1} \,|\, h]$ by imposing on $[p_1, \ldots, p_{n-1}]$ the single extra relation

$$(\rho_0 \rho_1 \cdots \rho_{n-1} \rho_{n-2} \cdots \rho_1)^h = \varepsilon.$$

If this new group is indeed a C-group in the sense of Section 2 (and in our applications, we may take this for granted), then we denote the associated regular polytope by $\{p_1, \ldots, p_{n-1} \,|\, h\}$. In other words, in terms of the notation we introduced in Section 2,

$$\{p_1, \ldots, p_{n-1} \,|\, h\} := \{p_1, \ldots, p_{n-1}\} / \left\langle\left\langle \left(01 \cdots (n-2)(n-1)(n-2) \cdots 1\right)^h \right\rangle\right\rangle.$$

We observe that the extra relation is preserved under duality, so that the dual polytope is $\{p_{n-1}, \ldots, p_1 \,|\, h\}$. Henceforth, we shall only consider one out of each dual pair of polytopes.

If $n = 3$, the extra group relation has an appealing geometric interpretation. Edge-paths in the polyhedron $P := \{p, q \,|\, h\}$ which leave vertices by the next edge from which they entered obviously trace out faces $\{p\}$ of P. Edge-paths which leave by the second edge similarly trace out a polygon $\{h\}$, which is called a *hole*. The polyhedron P is completely determined by its Schläfli type $\{p, q\}$, and its hole $\{h\}$.

For larger n, the geometry is usually less intuitive, though by Proposition 2 the relation still gives the length h of a certain edge-circuit in the polytope, which we shall call a *deep hole*. (In this case, the associated edge-circuit is fairly clear, since the relation gives the period of the product of ρ_0, which interchanges the two vertices of the base edge E, with a certain conjugate of ρ_{n-1}, which leaves fixed the base vertex v of E.)

The following result is one to which we shall frequently appeal. If β and γ are elements of a group, we shall write $\beta \rightleftharpoons \gamma$ to mean that β and γ commute (that is, $\beta\gamma = \gamma\beta$). We also denote by $\sim$ conjugacy in a group.

LEMMA 3.1. *Let Γ be a group, and let ρ, σ and $\tau \in \Gamma$ be involutions such that $(\rho\sigma)^3 = \varepsilon$ and $\rho \rightleftharpoons \tau$. Then*

$$\rho\sigma\tau\sigma \sim \sigma\tau.$$

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PROOF. The proof is easy. We have

$$\begin{aligned}\rho\sigma\tau\sigma &\sim \sigma\rho\sigma\tau\\ &= \rho\sigma\rho\tau\\ &= \rho\sigma\tau\rho\\ &\sim \sigma\tau,\end{aligned}$$

as claimed. ∎

There is an important special case of holes, which is a corollary of Lemma (3.1), to whose proof we shall need to refer later. Here and elsewhere, we denote by r^k a string $r, \ldots, r$ of length k.

COROLLARY 3.2. *The deep hole of an n-polytope of type* $\{3^{n-2}, q\}$ *is a q-gon* $\{q\}$.

PROOF. This result follows from applying Lemma (3.1) $n-2$ times. We obtain

$$\rho_0\rho_1\cdots\rho_{n-2}\rho_{n-1}\rho_{n-2}\cdots\rho_1 \sim \rho_{n-2}\rho_{n-1},$$

which proves the corollary.

More generally, when $p_2 = \cdots = p_{n-2} = 3$, so that the facet $\{p_1, 3, \ldots, 3\}$ is simple when it is finite, then we have the following picture of the deep hole. When two facets F, F' of the polytope P (say) meet on a common $(n-2)$-face G containing a vertex v, there are two edges of P through v which do not lie in G, one in F, and the other in F'. At the other end of the edge in F' is another $(n-2)$-face G', and then a further edge in a facet F'' which meets F' in G'. We can continue in this way, and the deep hole is then the resulting polygon. The general picture is only a little more complicated than this, but we shall not encounter it.

We shall see below how deep holes occur in the regular star-polytopes and honeycombs. However, let us first give one example in a different context. For $s \geq 2$, the regular toroidal $(n+1)$-polytope $\{4, 3^{n-2}, 4\}_{(s,0^{n-1})}$ is obtained from the cubic honeycomb $\{4, 3^{n-2}, 4\}$, whose vertices are all vectors with integer cartesian coordinates in $\mathbb{E}^n$, by identification under the lattice generated by $(s, 0^{n-1})$ and its images under permutation of coordinates. Comparison with [12, Theorem 3.2] shows that an alternative notation for this polytope is $\{4, 3^{n-2}, 4 \mid s\}$.

4. **Regular star-polytopes.** As we said in Section 3, we need only consider one of each pair of dual polytopes. Further, the regular (or quasi-regular) star-polytopes and honeycombs in $\mathbb{E}^3$ and $\mathbb{E}^4$ occur in isomorphic pairs, obtained by interchanging 5 and $\frac{5}{2}$ in their Schläfli symbols, and so we may also confine our attention to one of each such pair. (In fact, we shall implicitly verify this isomorphism in the section.) As a result, the only (finite) 4-dimensional polytopes we need look at are $\{3, 5, \frac{5}{2}\}$, $\{5, \frac{5}{2}, 5\}$ and $\{5, 3, \frac{5}{2}\}$.

The regular star-polytopes are abstractly described by

THEOREM 4.1. *The regular* 4*-dimensional star-polytopes satisfy the following isomorphisms:*

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(a) $\{3,5,\frac{5}{2}\} \cong \big\{\{3,5\},\{5,5\,|\,3\}\big\} = \{3,5,5\}/\langle\langle(1232)^3\rangle\rangle$;
(b) $\{5,\frac{5}{2},5\} \cong \big\{\{5,5\,|\,3\},\{5,5\,|\,3\}\big\} = \{5,5,5\}/\langle\langle(0121)^3,(1232)^3\rangle\rangle$;
(c) $\{5,3,\frac{5}{2}\} \cong \{5,3,5\,|\,3\}$.

PROOF. All these polytopes have groups isomorphic to $[3,3,5] = \langle\rho_0,\rho_1,\rho_2,\rho_3\rangle$, with the convention introduced in Section 3. We therefore choose new generators for $[3,3,5]$, guided by the process of systematic vertex-figure replacement described in [5]; the dissection theorems of [4] are also relevant in this context. In each case, we shall have an invertible (mixing) operation $(\rho_0,\rho_1,\rho_2,\rho_3) \longmapsto (\sigma_0,\sigma_1,\sigma_2,\sigma_3)$ in the sense of [10].

(a) The first operation is

$$(\rho_0,\rho_1,\rho_2,\rho_3) \longmapsto (\rho_0,\rho_1,\rho_2\rho_3\rho_2,\rho_3) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3).$$

Most of the relations satisfied by the new generators are obvious (for example, they are all involutions), and so we concentrate on those which are not. We first observe that $\sigma_2\sigma_3 = (\rho_2\rho_3)^2$, reflecting the change from 5 to $\frac{5}{2}$ in the Schläfli symbol; thus $(\sigma_2\sigma_3)^5 = \varepsilon$. Further,

$$\sigma_1\sigma_2 = \rho_1\rho_2\rho_3\rho_2 \sim \rho_2\rho_3,$$

by Lemma (3.1), since $(\rho_1\rho_2)^3 = \varepsilon$.

The inverse operation is

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \longmapsto (\sigma_0,\sigma_1,\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2,\sigma_3) = (\rho_0,\rho_1,\rho_2,\rho_3).$$

It follows that the automorphism group $\Gamma(\{3,5,\frac{5}{2}\})$ of $\{3,5,\frac{5}{2}\}$ is obtained from the Coxeter group $[3,5,5]$ by imposing the single extra relation arising from $(\rho_1\rho_2)^3 = \varepsilon$. Now

$$\begin{aligned}\rho_1\rho_2 &= \sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\\ &= \sigma_1\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\\ &= \sigma_3\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\\ &\sim \sigma_1\sigma_2\sigma_3\sigma_2.\end{aligned}$$

Here we have used $(\sigma_2\sigma_3)^5 = \varepsilon$. That is,

$$\big\{3,5,\tfrac{5}{2}\big\} \cong \{3,5,5\}/\langle\langle(1232)^3\rangle\rangle = \big\{\{3,5\},\{5,5\,|\,3\}\big\},$$

as claimed.

(b) The next operation is

$$(\rho_0,\rho_1,\rho_2,\rho_3) \longmapsto (\rho_0,\rho_1\rho_2\rho_3\rho_2\rho_1,\rho_3,\rho_2) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3).$$

We observe that

$$\sigma_0\sigma_1 = \rho_0\rho_1\rho_2\rho_3\rho_2\rho_1 \sim \rho_2\rho_3,$$

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by two applications of Lemma (3.1) (compare Corollary (3.2)). Next,

$$\begin{aligned}\sigma_1\sigma_2 &= \rho_1\rho_2\rho_3\rho_2\rho_1\rho_3\\ &= \rho_1\rho_2\rho_3\rho_2\rho_3\rho_1\\ &\sim (\rho_2\rho_3)^2.\end{aligned}$$

As before, this reflects the change from 5 to $\frac{5}{2}$ in the Schläfli symbol. Finally

$$\begin{aligned}\sigma_1\sigma_3 &= \rho_1\rho_2\rho_3\rho_2\rho_1\rho_2\\ &= \rho_1\rho_2\rho_3\rho_1\rho_2\rho_1\\ &= \rho_1\rho_2\rho_1\rho_3\rho_2\rho_1\\ &= \rho_2\rho_1\rho_2\rho_3\rho_2\rho_1\\ &= \sigma_3\sigma_1.\end{aligned}$$

The inverse operation is

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \longmapsto (\sigma_0, \sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3, \sigma_3, \sigma_2) = (\rho_0, \rho_1, \rho_2, \rho_3).$$

It thus follows that $\Gamma(\{5, \frac{5}{2}, 5\})$ is obtained from $[5, 5, 5]$ by imposing the two extra relations arising from $(\rho_0\rho_1)^3 = \varepsilon = (\rho_1\rho_2)^3$. Now

$$\begin{aligned}\rho_0\rho_1 &= \sigma_0\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\\ &= \sigma_3\sigma_2\sigma_0\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\\ &\sim \sigma_0\sigma_1\sigma_2\sigma_1,\end{aligned}$$

while

$$\begin{aligned}\rho_1\rho_2 &= \sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^2\\ &= \sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\\ &= \sigma_3\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\\ &= \sigma_1\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\\ &\sim \sigma_3\sigma_2\sigma_1\sigma_2\\ &\sim \sigma_1\sigma_2\sigma_3\sigma_2.\end{aligned}$$

That is, $\{5, \frac{5}{2}, 5\} \cong \{5, 5, 5\}/\langle\langle(0121)^3, (1232)^3\rangle\rangle = \big\{\{5, 5\,|\,3\}, \{5, 5\,|\,3\}\big\}$, as claimed. Notice that we have used $(\sigma_1\sigma_2)^5 = \varepsilon$ here.

(c) The final operation is

$$(\rho_0, \rho_1, \rho_2, \rho_3) \longmapsto (\rho_0, \rho_1\rho_2\rho_3\rho_2\rho_1, \rho_3\rho_2\rho_3, \rho_2) =: (\sigma_0, \sigma_1, \sigma_2, \sigma_3).$$

We see that

$$\begin{aligned}\sigma_0\sigma_1 &= \rho_0\rho_1\rho_2\rho_3\rho_2\rho_1\\ &\sim \rho_2\rho_3,\end{aligned}$$

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as in case (b). Next,

$$\begin{aligned}\sigma_0\sigma_2 &= \rho_0\rho_3\rho_2\rho_3\\ &= \rho_3\rho_2\rho_3\rho_0\\ &= \sigma_2\sigma_0.\end{aligned}$$

Then we have

$$\begin{aligned}\sigma_1\sigma_2 &= \rho_1\rho_2\rho_3\rho_2\rho_1\rho_3\rho_2\rho_3\\ &= \rho_1\rho_2\rho_3\rho_2\rho_3\rho_1\rho_2\rho_3\\ &\sim \rho_1\rho_2\rho_3\rho_1\rho_2\rho_3\rho_2\rho_3\\ &= \rho_1\rho_2\rho_1\rho_3\rho_2\rho_3\rho_2\rho_3\\ &= \rho_2\rho_1\rho_2\rho_3\rho_2\rho_3\rho_2\rho_3\\ &\sim \rho_1\rho_2\rho_3\rho_2\rho_3\rho_2\rho_3\rho_2\\ &= \rho_1\rho_3\rho_2\rho_3\\ &= \rho_3\rho_1\rho_2\rho_3\\ &\sim \rho_1\rho_2.\end{aligned}$$

We also have $\sigma_1\sigma_3 = \sigma_3\sigma_1$ as in case (b), and $\sigma_2\sigma_3 = (\rho_3\rho_2)^2 \sim (\rho_2\rho_3)^2$, so that $(\sigma_2\sigma_3)^5 = \varepsilon$.

The inverse operation is

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \longmapsto (\sigma_0,\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1,\sigma_3,\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2) = (\rho_0,\rho_1,\rho_2,\rho_3).$$

It follows that $\Gamma(\{5,3,\frac{5}{2}\})$ is obtained from $[5,3,5]$ by imposing the extra relations arising from $(\rho_0\rho_1)^3 = \varepsilon = (\rho_1\rho_2)^3$. Now the second is given by $(\sigma_1\sigma_2)^3 = \varepsilon$; for the first,

$$\begin{aligned}\rho_0\rho_1 &= \sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\\ &= \sigma_0\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\\ &= \sigma_0\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\\ &= \sigma_0\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\\ &= \sigma_2\sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\\ &\sim \sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1\\ &= \sigma_0\sigma_1\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\\ &= \sigma_3\sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_3\\ &\sim \sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1.\end{aligned}$$

That is, $\{5,3,\frac{5}{2}\} \cong \{5,3,5\}/\langle\langle(012321)^3\rangle\rangle = \{5,3,5\,|\,3\}$, as claimed. ■

Let us emphasize that we have shown that $\{3,5,\frac{5}{2}\}$ and $\{5,\frac{5}{2},5\}$ are the universal polytopes of their respective types; in particular, these universal polytopes exist, and

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are finite. Of course, $\{5, 3, \frac{5}{2}\}$ cannot be universal of type $\{5, 3, 5\}$, since the latter is infinite. Finally, we observe that $\{5, 3, 5 \,|\, 3\}$ is (combinatorially) self-dual, so that the dual polytopes $\{5, 3, \frac{5}{2}\}$ and $\{\frac{5}{2}, 3, 5\}$ are isomorphic; this justifies our earlier assertion that interchange of 5 and $\frac{5}{2}$ in Schläfli symbols leads to isomorphic polytopes.

5**. Three-dimensional honeycombs.** In preparation for dealing with the non-discrete regular honeycombs in Section 6, we now consider the non-discrete quasi-regular honeycomb

$$Q := \left\{\tfrac{5}{2}, {3 \atop 5}\right\}$$

in $\mathbb{E}^3$, and its two related honeycombs

$$Q' := \left\{3, {5 \atop \frac{5}{2}}\right\}, \qquad Q'' := \left\{5, {3 \atop \frac{5}{2}}\right\},$$

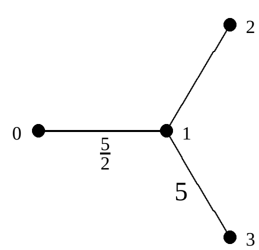
with the same group. (Combinatorially, Q' is actually regular; it is of type $\big\{\{3, 5\}, \{5, 4\}_6\big\} = \{3, 5, 4\}/\langle\langle(123)^6\rangle\rangle$, though not, as we shall see, isomorphic to it. As in, for example, [3], the notation $\{p, q\}_r \cong \{p, q\}/\langle\langle(012)^r\rangle\rangle$ refers to a polyhedron of type $\{p, q\}$ determined solely by the lengths of its Petrie polygons.)

We merely remark here on the rôle played by these honeycombs in the construction of quasi-periodic tilings of $\mathbb{E}^3$; compare [7].

The generating reflexions $\rho_0, \dots, \rho_3$ of the symmetry group of Q are as follows. Let $\tau := \frac{1}{2}(1 + \sqrt{5})$ be the golden section. For $j = 0, \dots, 3$, the mirror of the reflexion ρ_j is the plane $H_j := \{x \in \mathbb{E}^3 \mid \langle x, u_j\rangle = \alpha_j\}$, with u_j a unit vector and $\alpha_j \in \mathbb{R}$, so that $x\rho_j = x + 2(\alpha_j - \langle x, u_j\rangle)u_j$ for $x \in \mathbb{E}^3$. Here we have $\alpha_0 = 1$, $\alpha_j = 0$ for $j = 1, 2, 3$, and

$$\begin{aligned} u_0 &= (1, 0, 0), \\ u_1 &= \tfrac{1}{2}(-\tau^{-1}, -\tau, -1), \\ u_2 &= (0, 0, 1), \\ u_3 &= (0, 1, 0). \end{aligned}$$

In other words, the group is that with Coxeter diagram



The label "j" against a node denotes the reflexion ρ_j.

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The three quasi-regular honeycombs have the same vertices and edges, and are related to each other by replacement of (quasi-regular) vertex-figures with the same vertices and groups. In fact, the following operations yield the groups of the other two. First,

$$(\rho_0,\dots,\rho_3) \longmapsto (\rho_0, \rho_1\rho_3\rho_1, \rho_3, \rho_2) =: (\sigma_0,\dots,\sigma_3)$$

gives the group of $\mathcal{Q}'$; the inverse operation is

$$(\sigma_0,\dots,\sigma_3) \longmapsto (\sigma_0, \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \sigma_3, \sigma_2) = (\rho_0,\dots,\rho_3).$$

Second,

$$(\rho_0,\dots,\rho_3) \longmapsto (\rho_0, \rho_1\rho_3\rho_1\rho_2\rho_1\rho_3\rho_1, \rho_3, \rho_2) =: (\sigma_0,\dots,\sigma_3)$$

gives the group of $\mathcal{Q}''$; the inverse operation is

$$(\sigma_0,\dots,\sigma_3) \longmapsto (\sigma_0, \sigma_1\sigma_3\sigma_1\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_1\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_1\sigma_3\sigma_1, \sigma_3, \sigma_2) = (\rho_0,\dots,\rho_3).$$

We shall see in the course of proving Theorems (6.3) and (6.6) that the 4-dimensional honeycomb $\{\frac{5}{2}, 3, 3, 5\}$ has $\mathcal{Q}$ as a section; similarly, $\{3, 5, \frac{5}{2}, 3\}$ has $\mathcal{Q}'$ as a section, and $\{5, 3, 3, \frac{5}{2}\}$ has $\mathcal{Q}''$ as a section.

Using this description of the group, we see that we may take the vertices of the icosidodecahedral vertex-figure of $\mathcal{Q}$ at its initial vertex $o = (0, 0, 0)$ to be the cyclic permutations of $(\pm 2, 0, 0)$ and $(\pm\tau, \pm 1, \pm\tau^{-1})$, giving $6 + 24 = 30$ points in all. Since this vertex-figure is centrally symmetric, it easily follows that the complete vertex-set of $\mathcal{Q}$ is the set Λ_3 of the integer linear combinations of these 30 points.

LEMMA 5.1. *The vertex-set Λ_3 of $\mathcal{Q}$ is an additive subset of $\mathbb{E}^3$ of rank* 6.

PROOF. This can be seen directly. However, the following approach is more transparent. Consider the double-size integer lattice $2\mathbb{Z}^6$ in $\mathbb{E}^6$, generated by the points $2e_j$ for $j = 1,\dots,6$, where $\{e_1,\dots,e_6\}$ is the standard basis of $\mathbb{E}^6$. The 12 points $\pm 2e_j$, for $j = 1,\dots,6$, are thus the vertices of a regular 6-crosspolytope $\mathcal{C}$. There is obviously a (scaled) orthogonal projection, mapping these 12 vectors onto the 12 vertices of the regular icosahedron $\mathcal{J}$ of edge-length $2\tau^{-1}$, with vertices all cyclic permutations of $(\pm 2, 0, \pm 2\tau^{-1})$ (the scaling factor is actually $2^{1/2}\tau^{-1/2}5^{1/4}$; compare [14]). The complementary projection take the vectors onto another icosahedron, and we then deduce that the additive group generated by the 12 vertices of $\mathcal{J}$ has rank 6. (If it had smaller rank, then it would lift to a lattice of rank less than 6.)

Now the mid-points of the edges of $\mathcal{J}$ are the 30 points which generate Λ_3. Since these mid-points come from among the 60 mid-points $\pm e_i \pm e_j$ of the edges of $\mathcal{C}$, which generate the lattice usually known as D_6 (its points are those of $\mathbb{Z}^6$ with an even sum), we see that Λ_3 has rank at most 6. When we observe that the other 30 images of the mid-points of the edges of $\mathcal{C}$, namely the cyclic permutations of $(\pm 2\tau^{-1}, 0, 0)$ and $(\pm 1, \pm\tau^{-1}, \pm\tau^{-2})$, also lie in Λ_3 (for instance, $(2\tau^{-1}, 0, 0) = (\tau^{-1}, \tau, 1) - (-\tau^{-1}, \tau, 1)$), we conclude that the rank is exactly 6, as claimed. ∎

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The circuit criterion of Proposition (2.4) says that we can find a presentation for the group of $\mathcal{Q}$ from one of its vertex-figure, together with relations arising from its edge-circuits. (The fact that the vertex-figure is quasi-regular, rather than regular, requires only a trivial extension of the proposition.) Naturally, we look for *basic* circuits, corresponding to the simplest relations which determine the group. A general relation in the group will arise from *concatenating* basic circuits, by means of symmetric differences. In the present case, since the edges are just the projections of those of the semi-regular honeycomb $h\delta_7$ (in the notation of [2, p. 155]), it will suffice to determine the relations arising from the equilateral triangular circuits, and those coming from the rhombs given by pairs of radii (to vertices) of the icosidodecahedron, with angles $\pi/5$, $\pi/3$, $2\pi/5$ and $\pi/2$.

A rhomb with angle $\pi/3$ is clearly formed by concatenating two triangles. We must thus consider the other three kinds.

LEMMA 5.2. *The relations in $\Gamma(\mathcal{Q})$ which arise from the rhombs with angles $\pi/5$, $2\pi/5$ and $\pi/2$ are equivalent.*

PROOF. What we show first is that two rhombic circuits with angles $2\pi/5$ and $\pi/2$ are equivalent, modulo triangles (the latter rhomb is, of course, a square). Consider the following 9 points $o, a_\pm, b, c_\pm, d_\pm, e$ of Λ_3, where $o = (0, 0, 0)$ as usual, and

$$a_\pm = (\pm\tau, 1, \tau^{-1}), \quad b = (0, 2, 2\tau^{-1}), \quad c_\pm = (\pm 1, \tau^2, \tau),$$
$$d_\pm = (\pm 1, \tau^2, -\tau^{-2}), \quad e = (2, 0, 0).$$

Listing polygons by their vertex-sets, the quadrilateral circuits given by $\{o, a_+, b, a_-\}$ and $\{c_+, c_- d_- d_+\}$ are rhombs with angles $2\pi/5$ and $\pi/2$, respectively, while we have the 10 triangles $\{a_+, c_+, d_+\}$, $\{a_-, c_-, d_-\}$, $\{a_+, b, d_+\}$, $\{a_-, b, d_-\}$, $\{b, d_\pm\}$, $\{o, a_+, e\}$, $\{o, a_-, e\}$, $\{a_+, c_+, e\}$, $\{a_-, c_-, e\}$ and $\{c_\pm, e\}$. The equivalence is most easily seen by noticing that concatenating the two rhombs and the first five triangles yields the regular pentagon with vertices o, a_+, c_+, c_-, a_- (in cyclic order), while the last five triangles concatenate to the same pentagon.

The equivalence of the square circuit with the rhomb of angle $\pi/5$ can be seen similarly; the easiest way is to observe that following the automorphism $\tau \leftrightarrow -\tau^{-1}$ of the ring $\mathbb{Z}[\tau]$ by interchange of the second and third coordinates in $\mathbb{E}^3$ induces an involutory automorphism of Λ_3. This gives the lemma. ∎

Let $\Gamma(\mathcal{Q}) = \langle \rho_0, \ldots, \rho_3 \rangle$ be the group of $\mathcal{Q}$ as above, so that the generators ρ_j satisfy (among others) the relations $(\rho_j\rho_k)^{m_{jk}} = \varepsilon$, with

$$m_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 5, & \text{if } (j,k) = (0,1) \text{ or } (1,3), \\ 3, & \text{if } (j,k) = (1,2), \\ 2, & \text{if } (j,k) = (0,2), (0,3) \text{ or } (2,3), \end{cases} \tag{5.3}$$

together with $(\rho_0\rho_1\rho_3\rho_2)^3 = \varepsilon$, the last representing the triangular holes of the facets $\{\frac{5}{2}, 5\} \cong \{5, 5 \,|\, 3\}$. The translation γ which takes the initial vertex o into the other vertex $(2, 0, 0)$ (say) of the initial edge may be taken to be

$$\gamma := (\rho_1\rho_2\rho_3)^5\rho_2\rho_3 \cdot \rho_0 = (\rho_1\rho_2\rho_3)^4\rho_1\rho_0.$$

(The first part of the product is the reflexion in the plane through o parallel to ρ_0, and is itself the product of the central reflexion $(\rho_1\rho_2\rho_3)^5$ in o and the half-turn $\rho_2\rho_3$ about the initial edge.)

The relation corresponding to a rhomb with angle $\pi/5$ expresses the fact that the translations γ and $\rho_1\gamma\rho_1$ commute, that is, that

$$(\gamma\rho_1)^2 = (\rho_1\gamma)^2.$$

Simplifying the resulting relation (which we leave to the reader) then yields

$$\big((\rho_1\rho_2\rho_3)^3(\rho_1\rho_0)^2\big)^2 = \varepsilon. \tag{5.4}$$

In conclusion, putting Lemma (5.2) together with (5.4), we see that we have proved

THEOREM 5.5. *The group $\Gamma(\mathcal{Q})$ of the non-discrete quasi-regular honeycomb*

$$\mathcal{Q} = \left\{\tfrac{5}{2}, {3 \atop 5}\right\}$$

is the Coxeter group

$$\langle \rho_0, \ldots, \rho_3 \rangle = \left[5, {3 \atop 5}\right],$$

which satisfies $(\rho_j\rho_k)^{m_{jk}} = \varepsilon$ with the numbers m_{jk} given by (5.3), together with the relations

$$(\rho_0\rho_1\rho_3\rho_1)^3 = \big((\rho_1\rho_2\rho_3)^3(\rho_1\rho_0)^2\big)^2 = \varepsilon.$$

6**. The four-dimensional honeycombs.** In this section, we shall treat the non-discrete regular honeycombs with fivefold symmetries in $\mathbb{E}^4$, and we shall show that, somewhat surprisingly perhaps, these are all either universal of their type, or are again determined by a deep hole.

Before we embark on the formalities, it is instructive to look at the problem from an heuristic viewpoint. Consider a honeycomb formed from cells $\{5, 3, 3\}$ with a fixed edge-length, embedded in a hyperbolic space $\mathbb{H}^4$ of given negative curvature. That is, we just glue copies of the 120-cell facet against facet, ignoring the fact that the resulting vertex-figure will usually be infinite, and the honeycomb non-discrete. For the universal $\{5, 3, 3, 5\}$, the cells have dihedral angle $2\pi/5$. As the curvature tends to 0, so the dihedral angle of the facets $\{5, 3, 3\}$ increases. Thus the honeycomb, while remaining hyperbolic, will pass through $\{5, 3, 3, 4\}$ (with dihedral angle $\pi/2$) and then $\{5, 3, 3, 3\}$ (with dihedral angle $2\pi/3$); for the latter, as we have seen, its deep holes will be pentagons $\{5\}$. At the limit, we obtain $\{5, 3, 3, \frac{5}{2}\}$ in $\mathbb{E}^4$; it is easy to check geometrically (and we shall do this algebraically below) that the deep holes are now triangles $\{3\}$.

Moreover, in the last two cases, the holes in their turn determine the dihedral angles of the facets, and so, in a sense, their type also. In fact, there will always be edge-paths corresponding to deep holes, although they will generally not close. Nevertheless, the

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curvature of the space can be calculated from the angle of this polygonal deep hole (which is planar).

Our task in this section is to show that this heuristic argument can be validated. Among other things, this will imply that $\{5, 3, 3, \frac{5}{2}\} \cong \{5, 3, 3, 5 \,|\, 3\}$. Our proof will comprise three stages. First, we show that $\{5, 3, 3, \frac{5}{2}\}$ is a quotient of $\{5, 3, 3, 5 \,|\, 3\}$. Next, we establish presentations of the groups of related honeycombs which are equivalent to $\{5, 3, 3, \frac{5}{2}\} \cong \{5, 3, 3, 5 \,|\, 3\}$. Finally, we prove that one of these honeycombs does indeed have the group presentation required.

In fact, we shall find it more convenient to work with the dual honeycomb $\{\frac{5}{2}, 3, 3, 5\}$. This will enable us to use the previous results about vertex-figure replacement in the family derived from $\{3, 3, 5\}$, at least to some extent. Of course, the deep hole relation is symmetric between a polytope and its dual.

Our first result is then

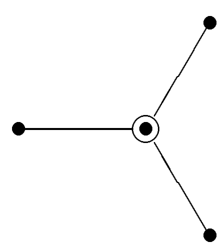
LEMMA 6.1. *The honeycomb* $\{\frac{5}{2}, 3, 3, 5\}$ *is of type* $\{5, 3, 3, 5 \,|\, 3\}$.

PROOF. To prove this, all we have to do is choose appropriate generators for the group, and check that the hole relation holds. Each generator ρ_j of the (geometric) group $\langle \rho_0, \ldots, \rho_4 \rangle$ of $\{\frac{5}{2}, 3, 3, 5\}$ is the reflexion in some hyperplane $H_j := \{x \in \mathbb{E}^4 \mid \langle x, u_j \rangle = \alpha_j\}$, where $\alpha_0 = 1$, $\alpha_j = 0$ for $j = 1, \ldots, 4$, and

$$\begin{aligned} u_0 &= (1, 0, 0, 0), \\ u_1 &= \tfrac{1}{2}(-\tau^{-1}, -1, 0, -\tau), \\ u_2 &= (0, 1, 0, 0), \\ u_3 &= \tfrac{1}{2}(0, -1, -\tau, \tau^{-1}), \\ u_4 &= (0, 0, 1, 0), \end{aligned}$$

with $\tau := \frac{1}{2}(1 + \sqrt{5})$ as before. These u_j are chosen so that, with initial vertex $o = (0, 0, 0, 0)$, the vertex-figure has vertices all 120 points derived from $(2, 0, 0, 0)$, $(1, 1, 1, 1)$ and $(\tau, 1, \tau^{-1}, 0)$ by applying even permutations and arbitrary changes of sign to their coordinates.

An important feature to note for the future is that the 24 points so derived from $(2, 0, 0, 0)$ and $(1, 1, 1, 1)$ form the vertices of the 24-cell $\{3, 4, 3\}$. Actually, this occurs in the guise of the polytope denoted by

$\left\{\begin{matrix}3\\3\\3\end{matrix}\right\} =$ 

(compare [2, Section 11.6]), whose group is a subgroup of $[3, 3, 5]$.

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Since $\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1$ is the conjugate of ρ_4 by $\rho_3\rho_2\rho_1$, it follows that it is the reflexion in the hyperplane $\{x \in \mathbb{E}^4 \mid \langle x, u\rangle = 0\}$, with

$$u := u_4\rho_3\rho_2\rho_1 = \tfrac{1}{2}(-1, 0, -\tau^{-1}, \tau).$$

Thus $\langle u_0, u\rangle = -\frac{1}{2}$, and it follows at once that $\rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1$ has period 3. ∎

Let Λ_4 denote the vertex-set of $\{\frac{5}{2}, 3, 3, 5\}$, that is, Λ_4 is the set of integer linear combinations of the 120 vertices of $\{3, 3, 5\}$ described above. Our analysis will depend crucially on a well-known important fact about Λ_4.

LEMMA 6.2. *The vertex-set* Λ_4 *of* $\{\frac{5}{2}, 3, 3, 5\}$ *is an additive set of rank* 8.

PROOF. The idea of the proof is similar to that of Lemma (5.1), and so we shall not give many details. This time, the basic observation is that the 240 vertices of the semi-regular Gosset polytope $4_{2,1}$ (see [2, 11.8]) project orthogonally onto two copies of the vertices of $\{3, 3, 5\}$, one set being τ^{-1} as large as the other (compare [7, 14]). Moreover, just as in the proof of Lemma (5.1), each vertex in the "small" copy is an integer combination of vertices in the "big" one. The complementary projection has the same property; naturally, the 120 vertices which go into the "big" $\{3, 3, 5\}$ in one projection go into the "small" $\{3, 3, 5\}$ in the other. These 240 points are all permutations of $(\pm 2, \pm 2, 0^6)$, and all points $\big((\pm 1)^8\big)$ with an even number of minus signs. They generate the lattice often called E_8; it has the vertices of the semi-regular honeycomb $5_{2,1}$. The assertion about the rank follows immediately. ∎

Before we proceed further, we shall list the ten (non-discrete) regular honeycombs derived from $\{\frac{5}{2}, 3, 3, 5\}$. In fact, they all have the same vertex-set (and, indeed, the same edges), and among the relationships between them are those obtained from the process of *vertex-figure replacement* (see [5]); that is, we change the vertex-figure of a given regular polytope (whose symmetry group is generated by reflexions in hyperplanes) for another regular polytope with the same vertices, while keeping the same edges.

We arrange the new vertex-figures after the pattern of [5, Table 2]. This gives

$\{\frac{5}{2}, 3, 3, 5\}$		
$\{\frac{5}{2}, 3, 5, \frac{5}{2}\}$	$\{3, \frac{5}{2}, 5, 3\}$	
$\{\frac{5}{2}, 5, \frac{5}{2}, 5\}$		
$\{\frac{5}{2}, 5, 3, \frac{5}{2}\}$		$\{5, \frac{5}{2}, 3, 5\}$
		$\{5, \frac{5}{2}, 5, \frac{5}{2}\}$
	$\{3, 5, \frac{5}{2}, 3\}$	$\{5, 3, \frac{5}{2}, 5\}$
		$\{5, 3, 3, \frac{5}{2}\}$

The honeycombs in the same column have the same 2-faces as well; in addition, in each of the two columns of four, the first two honeycombs have the same 3-faces, as have the last two.

To determine the groups of these honeycombs, it will be convenient to base our analysis on $\{3, \frac{5}{2}, 5, 3\}$ instead. Our next step therefore is to find presentations of the groups of certain of the other honeycombs, which are equivalent to $\{\frac{5}{2}, 3, 3, 5\} \cong$

$\{5,3,3,5\,|\,3\}$. The notation of (2.3) which we introduced in Section 2 will prove usful here.

THEOREM 6.3. *The following isomorphisms are equivalent:*
(a) $\{\frac{5}{2},3,3,5\} \cong \{5,3,3,5\,|\,3\}$;
(b) $\{\frac{5}{2},3,5,\frac{5}{2}\} \cong \{5,3,5,5\}/\langle\langle(012321)^3,(2343)^3\rangle\rangle$;
(c) $\{\frac{5}{2},5,\frac{5}{2},5\} \cong \{5,5,5,5\}/\langle\langle(0121)^3,(1232)^3,(2343)^3\rangle\rangle$.
(d) $\{3,\frac{5}{2},5,3\} \cong \{3,5,5,3\}/\langle\langle(1232)^3\rangle\rangle$;

PROOF. Before we start the details of the proof, let us observe the advantage of our notation over the standard one for universal polytopes. Theorem (6.3)(c) is equivalent to

$$\left\{\tfrac{5}{2},5,\tfrac{5}{2},5\right\} \cong \Big\{\big\{\{5,5\,|\,3\},\{5,5\,|\,3\}\big\},\big\{\{5,5\,|\,3\},\{5,5\,|\,3\}\big\}\Big\},$$

while Theorem (6.3)(d) is equivalent to

$$\left\{3,\tfrac{5}{2},5,3\right\} \cong \Big\{\big\{\{3,5\},\{5,5\,|\,3\}\big\},\big\{\{5,5\,|\,3\},\{5,3\}\big\}\Big\},$$

which is nearly as bad. These expressions are clumsy and difficult immediately to comprehend. (Of course, we deliberately chose the worst two examples, although (b) is not much better.)

What we must do is trace the operations which yield the group generators for one of these honeycombs in terms of another, and the corresponding presentations. (For practical reasons, we work with the groups, since we are employing mixing operations.) The only case which involves anything going beyond what we did in Section 4 is $\{3,\frac{5}{2},5,3\}$—naturally, since this is the one in which we are most interested.

Let us treat first the easier cases of the equivalence of (a), (b) and (c). We have already seen the appropriate operations; they are those in the proof of Theorem (4.1), with the indices increased by 1, and a new generator ρ_0 which remains unchanged. In each case, we begin with (a).

(b) The operation here is

$$(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4) \longmapsto (\rho_0,\rho_1,\rho_2,\rho_3\rho_4\rho_3,\rho_4) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4).$$

It is inverted by

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4) \longmapsto (\sigma_0,\sigma_1,\sigma_2,\sigma_3\sigma_4\sigma_3\sigma_4\sigma_3,\sigma_4) = (\rho_0,\rho_1,\rho_2,\rho_3,\rho_4).$$

The verification of the equivalence largely follows Theorem (4.1)(a). Indeed, we may just add 1 to the appropriate indices, for the relations which do not involve $\rho_0 = \sigma_0$. Moreover, $\rho_j = \sigma_j$ for $j = 1,2$ also.

The only relation not so far explored is that for the deep hole. Here, we have

$$\rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1 = \sigma_0\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1.$$

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Thus the deep hole in the facet of $\{5,3,5,5\}/\langle\langle(012321)^3,(2343)^3\rangle\rangle$ is equivalent to that of $\{5,3,3,5\,|\,3\}$.

(c) Now the operation is

$$(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4)\longmapsto(\rho_0,\rho_1,\rho_2\rho_3\rho_4\rho_3\rho_2,\rho_4,\rho_3) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4).$$

Its inverse is

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4)\longmapsto(\sigma_0,\sigma_1,\sigma_4\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4,\sigma_4,\sigma_3) = (\rho_0,\rho_1,\rho_2,\rho_3,\rho_4).$$

Again, the equivalence of the various relations which do not involve $\rho_0=\sigma_0$ follow as in Theorem (4.1)(b). We also have $\rho_1=\sigma_1$, and so once again we are left with the deep hole relation. Finally, then,

$$\begin{aligned}\rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1 &= \sigma_0\sigma_1\cdot\sigma_4\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4\cdot\sigma_4\sigma_3\sigma_4\cdot\sigma_4\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4\cdot\sigma_1\\ &= \sigma_0\sigma_1\sigma_4\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_4\sigma_1\\ &= \sigma_0\sigma_1\sigma_4\sigma_2\sigma_4\sigma_1\\ &\sim \sigma_0\sigma_1\sigma_2\sigma_1,\end{aligned}$$

which is the hole relation for the 3-face of $\{5,5,5,5\}/\langle\langle(0121)^3,(1232)^3,(2343)^3\rangle\rangle$.

(d) For the equivalence of (a) and (d), we actually have the same operation on $(\rho_1,\ldots,\rho_4)$ as in case (b), except that we must then follow it by duality, and conjugate to make the initial vertex of the new vertex-figure the same as that of the original. The intermediate operation (case (b) followed by duality) gives

$$(\rho_1,\rho_2,\rho_3,\rho_4)\longmapsto(\rho_4,\rho_3\rho_4\rho_3,\rho_2,\rho_1) =: (\tau_1,\tau_2,\tau_3,\tau_4).$$

Conjugating these elements τ_j by $\rho_3\rho_2\rho_1$ and adjoining $\rho_0=\sigma_0$ finally gives the operation

$$(\rho_0,\rho_1,\rho_2,\rho_3,\rho_4)\longmapsto(\rho_0,\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1,\rho_4,\rho_3,\rho_2) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4).$$

(We could have produced the operation like a rabbit out of a hat, but this approach seems more instructive.)

The inverse operation is now

$$(\sigma_0,\sigma_1,\sigma_2,\sigma_3,\sigma_4)\longmapsto(\sigma_0,\sigma_4\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_4,\sigma_4,\sigma_3,\sigma_2) = (\rho_0,\rho_1,\rho_2,\rho_3,\rho_4).$$

The equivalence between the relations for $\langle\sigma_0,\ldots,\sigma_4\rangle$ and those for $\langle\rho_0,\ldots,\rho_4\rangle$ is easily checked. Note that $(\sigma_0\sigma_1)^3=\varepsilon$ is equivalent to the deep hole relation for $\langle\rho_0,\ldots,\rho_4\rangle$. Further, observe that $\sigma_1\sigma_2\sim(\rho_3\rho_4)^2=(\sigma_3\sigma_2)^2$, reflecting the change from 5 to $\frac{5}{2}$ in the Schläfli symbol. ∎

Now we must return to $\{\frac{5}{2},3,3,5\}$.

LEMMA 6.4. *The mixing operation on the group* $[5,3,3,5]=\langle\rho_0,\ldots,\rho_4\rangle$ *given by*

$$(\rho_0,\ldots,\rho_4)\longmapsto\big(\rho_0,\rho_1,\rho_2,(\rho_3\rho_2\rho_4)^4\rho_3\big) =: (\sigma_0,\sigma_1,\sigma_2,\sigma_3)$$

yields the group

$$\left[5,\begin{matrix}3\\5\end{matrix}\right].$$

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PROOF. Geometrically, this is clear; we are taking the cut

$$\left\{5, \begin{matrix}3\\5\end{matrix}\right\} \subset \{5,3,3,5\}$$

determined by the central section

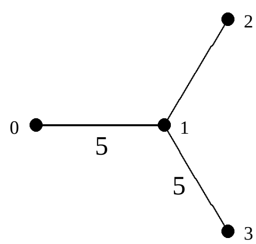
$$\left\{\begin{matrix}3\\5\end{matrix}\right\} \subset \{3,3,5\},$$

which lies in the mirror ρ_4. To see what is happening algebraically, we begin by noting that we can write σ_3 in the form

$$\sigma_3 = (\rho_2\rho_3\rho_4)^5 \cdot \rho_2\rho_4,$$

the first term being the central involution in the group $\langle \rho_2, \rho_3, \rho_4 \rangle = [3,5]$, so that, first, the order of the elements ρ_2, ρ_3 and ρ_4 in the bracket is immaterial, and, second, $\rho_j \rightleftharpoons (\rho_2\rho_3\rho_4)^5$ for $j = 2,3,4$, where as before we use "$\rightleftharpoons$" to mean "commutes with". (Of course, we also have $\rho_2 \rightleftharpoons \rho_4$.)

The subgroup of the statement of the lemma is that with diagram



As before, the label "j" against a node denotes the involution σ_j. It is straightforward to verify that $\langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle$ satisfies most of the implied relations; indeed, only those involving σ_3 need to be checked. First, σ_3 is an involution, from the above remarks about it. Second, $\sigma_3 \rightleftharpoons \sigma_0 = \rho_0$, since $\sigma_3 \in \langle \rho_2, \rho_3, \rho_4 \rangle$. Next, $\sigma_3 \rightleftharpoons \sigma_2$, again an easy deduction from the remarks. Finally, freely using $\rho_i \rightleftharpoons \rho_j$ if $|i-j| > 1$ and $\rho_{j-1}\rho_j\rho_{j-1} = \rho_j\rho_{j-1}\rho_j$ for $j = 2,3$, we have

$$\begin{aligned}
\sigma_1\sigma_3 &= \rho_1\rho_3\rho_4\rho_2\rho_3\rho_2\rho_4\rho_3\rho_4\rho_2\rho_3\rho_2\rho_4\rho_3 \\
&\sim \rho_1\rho_3\rho_2\rho_3\rho_4\rho_3\rho_4\rho_3\rho_2\rho_3 \\
&\sim \rho_1\rho_2\rho_4\rho_3\rho_4\rho_3\rho_4\rho_2 \\
&\sim \rho_1\rho_2\rho_3\rho_4\rho_3\rho_2 \\
&\sim \rho_3\rho_4,
\end{aligned}$$

as in Lemma (3.2). Observe that we had to use $(\rho_3\rho_4)^5 = \varepsilon$ in the course of the proof. ∎

We now wish to consider the effect of imposing hole relations.

LEMMA 6.5. *In the subgroup* $\langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle \leq \langle \rho_0, \ldots, \rho_4 \rangle$ *of Lemma (6.4),*

$$\sigma_0\sigma_1\sigma_3\sigma_1 \sim \rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1.$$

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PROOF. We have

$$\begin{aligned}\sigma_0\sigma_1\sigma_3\sigma_1 &= \rho_0\rho_1\rho_3\rho_4\rho_2\rho_3\rho_2\rho_4\rho_3\rho_4\rho_2\rho_3\rho_2\rho_4\rho_3\rho_1\\ &\sim \rho_0\rho_1\rho_2\rho_3\rho_2\rho_4\rho_3\rho_4\rho_2\rho_3\rho_2\rho_1\\ &= \rho_0\rho_1\rho_3\rho_2\rho_3\rho_4\rho_3\rho_4\rho_3\rho_2\rho_3\rho_1\\ &\sim \rho_0\rho_1\rho_2\rho_4\rho_3\rho_4\rho_3\rho_4\rho_2\rho_1\\ &\sim \rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1,\end{aligned}$$

as required. Again, we have used $(\rho_3\rho_4)^5 = \varepsilon$ in the course of the proof. ∎

From Lemma (6.5), it follows that the effect of imposing the relation $(\sigma_0\sigma_1\sigma_3\sigma_1)^3 = \varepsilon$ on the subgroup $\langle\sigma_0,\ldots,\sigma_3\rangle$, is that of imposing $(\rho_0\rho_1\rho_2\rho_3\rho_4\rho_3\rho_2\rho_1)^3 = \varepsilon$ on $\langle\rho_0,\ldots,\rho_4\rangle$. In turn, the effect on the corresponding apeirotopes is to obtain the cut

$$\left\{\tfrac{5}{2}, {3 \atop 5}\right\} \subset \left\{\tfrac{5}{2}, 3, 3, 5\right\},$$

with which we began this discussion. It should be observed that our choices of generating reflexions of the two groups are such that the cut is given by the section by the hyperplane $\{(\xi_1,\ldots,\xi_4) \in \mathbb{E}^4 \mid \xi_4 = 0\}$, with the last coordinate then dropped.

However, it is important to recall Theorem (5.5), which says that the group of the cut is not obtained merely by imposing the given relation; in other words, the cut

$$\left\{5, {3 \atop 5}\right\} \subset \{5, 3, 3, 5\}$$

is not universal with respect to the relation $(\sigma_0\sigma_1\sigma_3\sigma_1)^3 = \varepsilon$.

The discussion is now completed by proving Theorem (6.3)(d), namely

THEOREM 6.6. $\{3, \frac{5}{2}, 5, 3\} \cong \{3, 5, 5, 3\}/\langle\langle(1232)^3\rangle\rangle$.

PROOF. We already know from Lemma (6.1) and Theorem (6.3) that $\{3, \frac{5}{2}, 5, 3\}$ is of type $Q := \{3, 5, 5, 3\}/\langle\langle(1232)^3\rangle\rangle$ (that is, it is a quotient of the latter). First, we have $\{\frac{5}{2}, 5, 3\} \cong \{5, 5, 3\}/\langle\langle(0121)^3\rangle\rangle$, and hence (with a shift of 1 in the indices) the vertex-figure of Q will have the same vertices and edges as $\{\frac{5}{2}, 5, 3\}$ (there is an ambiguity about the 2-faces, since there is no combinatorial way of distinguishing between $\{\frac{5}{2}, 5, 3\}$ and $\{5, \frac{5}{2}, 3\}$). However, the central feature is that the triangular holes determined by the adjacency cycle $(1232)^3$ (or by $(\rho_1\rho_2\rho_3\rho_2)^3 = \varepsilon$ in its group $\langle\rho_0,\ldots,\rho_4\rangle$) are the same for both polytopes.

We now come to the crux of the argument: *every edge-circuit in the graph of* $\{3, \frac{5}{2}, 5, 3\}$ *can be built up from triangles, which are either* 2*-faces or holes in its vertex-figure.* In fact, we already noted that among the 120 vertices of $\{3, 3, 5\}$ are the 24 of $\{3, 4, 3\}$, and actually its edges are among those of $\{\frac{5}{2}, 5, 3\} \cong \{5, 5, 3\}/\langle\langle(0121)^3\rangle\rangle$. Thus the vertices and edges of the honeycomb $\{3, 3, 4, 3\}$ occur among those of $\{3, \frac{5}{2}, 5, 3\}$. More specifically, of the four 2-faces of the initial 3-face $\{3, 3\}$, those

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which contain the initial vertex o are faces of $\{3, \frac{5}{2}, 5, 3\}$, while the fourth is a hole of the vertex-figure.

We saw in Lemma (6.2) that the vertex-set of $\{3, \frac{5}{2}, 5, 3\}$ forms an additive set of rank 8. From this, it follows that the basic edge-circuits in the edge-graph of $\{3, \frac{5}{2}, 5, 3\}$ are all obtained by concatenating those in triangles, or rhombs with angle $\pi/5$, $\pi/3$, $2\pi/5$ or $\pi/2$, these being the angles between diameters of $\{3, \frac{5}{2}, 5, 3\}$.

Now each of these rhombs is represented in the section

$$\left\{3, \begin{matrix} 5 \\ \frac{5}{2} \end{matrix}\right\}.$$

The key observation is that we already know from Theorem (5.5) what the group of the section looks like; it is just obtained from that of Q by permuting ρ_0, ρ_2 and ρ_3 cyclically. More to the point, Lemma (5.2) tells us that rhombs of angle $\pi/3$ come from triangles, while the other three kinds of rhomb are equivalent modulo triangles.

In view of this, the proof is easily completed. A rhomb of angle $\pi/2$ is a diametral square of a facet $\{3, 3, 4\}$ of the honeycomb $\{3, 3, 4, 3\}$ whose vertices, edges and 2-faces occur among those of $\{3, \frac{5}{2}, 5, 3\}$. It is therefore formed by concatenating four triangles in a diametral octahedral section $\{3, 4\}$ of $\{3, 3, 4\}$. ∎

7. **Hyperbolic honeycombs.** For completeness, we briefly discuss the discrete regular star-honeycombs in the hyperbolic spaces $\mathbb{H}^3$ and $\mathbb{H}^4$. A natural restriction is that such honeycombs should have finite vertices; however, we should not insist that the facets also be finite.

If the vertex-figure of a regular hyperbolic honeycomb is a star-polytope (in which case it belongs to one of the families of 3- or 4-polytopes considered in Section 4), then we may apply the process of vertex-figure replacement of [5] in exactly the same way, to obtain a new regular honeycomb whose vertex-figures are convex. Indeed, the only difference in the proof is that now the Schläfli determinant (which is negative) will *increase* strictly.

With a suitable regular honeycomb as starting point, the process can also be reversed; a convex regular vertex-figure is replaced by any regular star-polytope with the same vertices. Moreover, if the replacement has the same edges as the original vertex-figure, then the new regular honeycomb will have the same vertices, edges and 2-faces as the starting honeycomb. Where the argument departs from that of [5] is that the new facets may not be finite, and so we cannot dualize and repeat the process.

Indeed, this last is the only procedure which will actually yield any new regular honeycombs. If the new vertex-figure does not have the same edges as the old one, then in most instances, the new honeycomb will have apeirogonal (infinite) 2-faces. Except for a couple of examples for illustration, we shall ignore such cases.

In $\mathbb{H}^3$, we can begin to apply the process of vertex-figure replacement to $\{p, 3, 5\}$ for $p = 4, 5, 6$. We obtain $\{p, 5, \frac{5}{2}\} \cong \big\{\{p, 5\}, \{5, 5 \mid 3\}\big\} \cong \{p, 5, 5\}/\langle\langle(1232)^3\rangle\rangle$, which is universal of its type, with a universal facet. The proof of this assertion exactly follows

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that of Theorem (4.1)(a), which never involved ρ_0. However, if we go further, and attempt to replace $\{3,5\}$ by $\{\frac{5}{2},5\}$ or $\{3,\frac{5}{2}\}$, then we obtain apeirogonal 2-faces. Indeed, the mixing operations which yield these apeirotopes are just as in Theorem (4.1)(b,c). They show that the facets are of type $\{\infty,5\,|\,p\}$ for $\{\infty,\frac{5}{2},5\}$. For the other case, the same discussion will show that the resulting relation again gives a deep hole corresponding to the original 2-face, so that the apeirotope is $\{\infty,3,5\,|\,p\}$. (Note that the argument never mentioned the period of $\sigma_0\sigma_1$; by the way, this indicates that the first entry "5" in $\{5,3,5\,|\,3\}$ is redundant!)

The only other possible example in $\mathbb{H}^3$ to which we might attempt to apply vertex-figure replacement is $\{3,5,3\}$. Comparison with the discussion of Theorem (6.3)(d) shows that, once again, the deep hole determines the apeirotope, and we obtain $\{\infty,\frac{5}{2},3\} \cong \{\infty,5,3\,|\,3\}$.

We now move on to $\mathbb{H}^4$, where there are many more examples. Here, we start with $\{p,3,3,5\}$ for $p = 3,4,5$. We first obtain $\{p,3,5,\frac{5}{2}\}$, $\{p,5,\frac{5}{2},5\}$ and $\{p,5,3,\frac{5}{2}\}$, all of whose group presentations are determined purely by the vertex-figures. For $p = 4$ or 5, the facets are always infinite; $\{p,3,5\}$ is the universal hyperbolic honeycomb, and $\{p,5,\frac{5}{2}\} \cong \{p,5,5\}/\langle\langle(1232)^3\rangle\rangle$ as in the examples in $\mathbb{H}^3$. In addition, the facet of $\{3,5,3,\frac{5}{2}\} \cong \{3,5,3,5\}/\langle\langle(123432)^3\rangle\rangle$ is the universal hyperbolic honeycomb $\{3,5,3\}$.

Except when $p = 3$, if we replace the vertex-figure $\{3,3,5\}$ by any of the other six regular star-polytopes with the same vertices, then the resulting apeirotopes have apeirogonal 2-faces; we shall therefore not consider them further. When $p = 3$, replacing $\{3,3,5\}$ by $\{\frac{5}{2},5,3\}$ gives $\{5,\frac{5}{2},5,3\}$, the dual of $\{3,5,\frac{5}{2},5\}$, and replacing it by $\{5,\frac{5}{2},3\}$ (with the same edges as $\{\frac{5}{2},5,3\}$) gives an apeirotope $\{5,5,\frac{5}{2},3\}$, with facets of type $\{5,5,5\}/\langle\langle(1232)^3\rangle\rangle$. Employing any of the other four polytopes with the same vertices as $\{3,3,5\}$ gives apeirogonal 2-faces.

Only in case $p = 3$ do we get apeirotopes with finite facets, where we can dualize, and try to interate vertex-figure replacement; these are $\{3,3,5,\frac{5}{2}\}$ and $\{3,5,\frac{5}{2},5\}$. In $\{\frac{5}{2},5,3,3\}$, we may only replace the vertex-figure by $\{\frac{5}{2},3,3\}$, and the 2-faces become infinite. In $\{5,\frac{5}{2},5,3\}$, the vertex-figure can be replaced by any of the other nine regular star-polytopes with the same vertices as $\{3,3,5\}$; however, we have just seen that this polytope is already obtainable directly from $\{3,3,3,5\}$, and so this family is now complete.

Since there are no more regular honeycombs in $\mathbb{H}^4$ to which the method of replacing vertex-figures can be applied, this completes the discussion.

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