Canad. Math. Bull. Vol. **41** (3), 1998 pp. 279–289

NEW CHARACTERIZATIONS OF THE REFLEXIVITY IN TERMS OF THE SET OF NORM ATTAINING FUNCTIONALS

MARÍA D. ACOSTA AND MANUEL RUIZ GALÁN

ABSTRACT. As a consequence of results due to Bourgain and Stegall, on a separable Banach space whose unit ball is not dentable, the set of norm attaining functionals has empty interior (in the norm topology). First we show that any Banach space can be renormed to fail this property. Then, our main positive result can be stated as follows: if a separable Banach space X is very smooth or its bidual satisfies the w^*-Mazur intersection property, then either X is reflexive or the set of norm attaining functionals has empty interior, hence the same result holds if X has the Mazur intersection property and so, if the norm of X is Fréchet differentiable. However, we prove that smoothness is not a sufficient condition for the same conclusion.

According to the celebrated James Theorem [15], on a non-reflexive Banach space, there is at least one (bounded and linear) functional that does not attain its norm. In some classical non-reflexive Banach spaces (for instance $c_0,\ c,\ L_1[0,1]$) the set of functionals not attaining their norms is actually dense in the dual space (for the norm topology). There are several known assertions in this line; first, as a consequence of results due to J. Bourgain and C. Stegall, for a separable Banach space with non-dentable unit ball, the set of norm attaining functionals is of first Baire category (see for instance [3, Theorem 3.5.5 and Problem 3.5.6]), hence it has empty interior. The question if the assumption of separability can be dropped appears as an open problem in [3, Problem 3.5.6]. However, for any infinite, compact and Hausdorff topological space K, Talagrand observed that the set of functionals attaining their norm on $C(K)$ is also of first Baire category in the dual space (see [3, p. 58]).

In the opposite direction, the set of norm attaining functionals can be large if the unit ball of the space is allowed to have strongly exposed points. For instance, one can easily check this kind of behaviour for the space ℓ_1. In fact, if we denote by $A(X)$ *the set of functionals attaining their norm* on a Banach space X, it is not difficult to prove that $A(X)$ has non-empty interior as soon as the unit ball of X has a "strongly vertex point", a result that will be shown later. It follows that any Banach space is almost isometric to another space for which the set of norm attaining functionals contains non-empty open balls.

Once we are convinced it may happen that the set $X^*\backslash A(X)$ is not dense in X^* (even for some non-reflexive space X), we will show a positive result for a certain class of separable Banach spaces, which contains the spaces with Fréchet differentiable norm. To prove this, we make use of a technical result known as "Simons' inequality" [23],

Received by the editors October 30, 1996; revised March 2, 1998.
Research partially supported by D.G.I.C.Y.T., project no. PB96-1906.
AMS subject classification: 46B04, 46B10, 46B20.

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

inspired by techniques of R. C. James, which has been crucial to obtain a new proof of the classical James Theorem and a number of other interesting applications (see [12]). We deduce that if X is a very smooth, separable Banach space, then either X is reflexive or $A(X)$ has empty interior. The same conclusion is obtained under the assumption that X is separable and X^{**} satisfies the w^*-Mazur intersection property, hence it also holds if X is separable with the Mazur intersection property.

Finally, we will give a procedure to construct counterexamples, which shows, in particular, that the condition "very smooth" can not be essentially weakened in the above mentioned result, since we prove that every separable Banach space X is isomorphic to a smooth space Y for which the set $A(Y)$ contains open balls. Also we exhibit an example to show that assuming that the dual space of X has no proper norming subspaces, a property shared by very smooth spaces and spaces with the Mazur intersection property, does not imply the denseness of $X^*\backslash A(X)$ in X^*.

Although all the results we will state are valid both for real and complex Banach spaces, we just consider the real case for obvious reasons. In the rest of the paper, if X is a Banach space, we will denote by B_X its closed unit ball and by S_X the unit sphere. For a subset $A \subset X$, $\mathrm{co}(A)$ and $|\mathrm{co}|(A)$ will be the convex hull and absolutely convex hull of A, respectively.

Let us begin with the first of the announced results, which is an easy generalization of the phenomenon already mentioned for ℓ_1:

PROPOSITION 1. *Let X be a Banach space such that for some subset $E \subset X$, $e_0 \in E$ and $e^* \in X^*$ it is true that*

$$B_X = \overline{|\mathrm{co}|}(E) \quad \textit{and} \quad e^*(e_0) > \sup\big\{|e^*(e)| : e \in E\backslash\{e_0\}\big\}.$$

Then e^* belongs to the norm interior of $A(X)$.

PROOF. By assumption $B_X = \overline{|\mathrm{co}|}(E)$, so $\|e^*\| = \sup\{|e^*(e)| : e \in E\}$, and in view of the hypothesis it has to be $e^*(e_0) = \|e^*\|$. Let us write

$$\rho = \sup\big\{|e^*(e)| : e \in E\backslash\{e_0\}\big\}$$

and choose $0 < r < \frac{1}{2}(\|e^*\| - \rho)$. We will observe that the ball centered at e^* with radius r is contained in the set of norm attaining functionals; in order to check this, choose $x^* \in X^*$ such that $\|x^* - e^*\| \leq r$, so, for any $e \in E\backslash\{e_0\}$ we have

$$|x^*(e)| \leq |e^*(e)| + |(x^* - e^*)(e)| \leq \rho + r < \frac{1}{2}(\|e^*\| + \rho);$$

but, on the other hand

$$|x^*(e_0)| \geq |e^*(e_0)| - |(x^* - e^*)(e_0)| \geq \|e^*\| - r \geq \frac{1}{2}(\|e^*\| + \rho).$$

Therefore, $|x^*(e_0)| = \sup\{|x^*(x)| : x \in B_X\}$, so x^* attains its norm (at e_0). ■

Note that the previous result is a partial converse to the mentioned fact that $A(X)$ has empty interior if B_X is not dentable. As a consequence of Proposition 1 we can get a general renorming result:

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

COROLLARY 2. *For any Banach space X, $e^* \in X^* \setminus \{0\}$ and $\varepsilon > 0$, there is a Banach space Y isomorphic to X such that e^* is an interior point of $A(Y)$ and the Banach-Mazur distance between X and Y is less than $1 + \varepsilon$.*

PROOF. Given $\varepsilon > 0$, choose e_0 in X such that $1 < \|e_0\| < 1 + \varepsilon$ and $e^*(e_0) > \|e^*\|$. Let us consider the set

$$B = \overline{|\operatorname{co}|}\big(B_X \cup \{e_0\}\big).$$

It is clear that B is the unit ball in X for an equivalent norm. Let Y be the linear space X endowed with this new norm. Since

$$B_X \subset B \subset \|e_0\| B_X,$$

from the choice of e_0, it follows that the Banach-Mazur distance between X and Y is less than $1 + \varepsilon$. Y and e^* also satisfy the hypothesis in the above proposition, so $e^* \in \overset{\circ}{A}(Y)$. ∎

As the above result shows, one can not expect any isomorphic condition on X to guarantee the denseness of the set $X^* \setminus A(X)$.

Next we will show some positive results, that is, our aim is to exhibit a class of Banach spaces for which a certain strengthening of James Theorem holds: the set $X^* \setminus A(X)$ is dense in X^*. For this purpose we will use the following technical result due to Simons:

LEMMA 3 [23, LEMMA 2]. *Let E be a set, $B \subset E$ and assume there is a bounded sequence $\{f_n\}$ of elements in $\ell_\infty(E)$ (real-valued bounded functions on E endowed with the usual norm) satisfying that for any sequence $\{t_n\}$ of non-negative numbers with $\sum_{n=1}^{\infty} t_n = 1$ there is $b \in B$ such that*

$$\sum_{n=1}^{\infty} t_n f_n(b) = \sup_{e \in E} \sum_{n=1}^{\infty} t_n f_n(e).$$

Then

$$\sup_{b \in B} \Big[\limsup_n f_n(b)\Big] \geq \inf_{g \in \operatorname{co}\{f_n\}} \sup\big\{g(e) : e \in E\big\}.$$

The previous result has been successfully used to get many applications (see [23, 24, 12]); for instance, it is crucial to obtain a new proof of the James Theorem for separable Banach spaces [5, Theorem I.3.2]. Even so, the above lemma has a completely elementary proof.

If $A(X)$ has non-empty interior we can easily fulfill the requirements in Simons' inequality to get some non-trivial but still rather technical information. This will be done in our next lemma. We need some notation: given a Banach space X and $x \in S_X$, we denote by $D(x, X)$ the set of *support functionals* for the unit ball at x, *i.e.*,

$$D(x, X) = \big\{x^* \in S_{X^*} : x^*(x) = 1\big\};$$

if $x^{**} \in X^{**}$, $V(x, x^{**})$ will be the *numerical range* of x^{**} relative to x, that is,

$$V(x, x^{**}) = \big\{x^{***}(x^{**}) : x^{***} \in D(x, X^{**})\big\}.$$

LEMMA 4. *Let X be a separable Banach space and assume that there exist $r > 0$ and $x_0^* \in S_{X^*}$ such that $x_0^* + rB_{X^*}$ is contained in $A(X)$, then:*

$$x^{**}(x_0^*) + r\|x^{**} + X\| \leq \max V(x_0, x^{**}), \quad \forall x^{**} \in X^{**},$$

where x_0 is any element in S_X satisfying $x_0^*(x_0) = 1$.

PROOF. First we will show that

$$(*) \qquad x^{**}(x_0^*) + r\|x^{**} + X\| \leq \|x^{**}\|, \quad \forall x^{**} \in X^{**}.$$

For this purpose, we essentially use the same argument as in the proof of [5, Theorem I.3.2]. Since the inequality $(*)$ is clearly satisfied for any element $x \in X$, we will show it holds for elements in $X^{**} \backslash X$. So, if we fix $x^{**} \in X^{**} \backslash X$, the Hahn-Banach Theorem provides us with an element x^{***} in the unit sphere of X^{***} also satisfying

$$x^{***}(x^{**}) = \|x^{**} + X\|, \quad x^{***}(x) = 0, \quad \forall x \in X.$$

We are assuming X is separable, so the restriction of the $\sigma(X^{***}, X \cup \{x^{**}\})$-topology to bounded sets is metrizable. Hence, in view of the w^*-denseness of B_{X^*} in $B_{X^{***}}$, for any $\varepsilon > 0$ we can find a sequence $\{x_n^*\}$ in B_{X^*} satisfying

$$\left\{x^{**}(x_n^*)\right\} \longrightarrow x^{***}(x^{**}) = \|x^{**} + X\|$$

and also the following conditions:

$$(1) \qquad \left\{x_n^*(x)\right\} \longrightarrow 0, \quad \forall x \in X$$

and

$$(2) \qquad x^{**}(x_n^*) \geq \|x^{**} + X\| - \varepsilon, \quad \forall n \in \mathbb{N}.$$

Now, for any natural number n, the element $x_0^* + rx_n^*$ is in the closed ball of radius r centered at x_0^*, and, by the assumptions of the lemma, this ball is contained in the set of norm attaining functionals on X. Clearly we can apply Lemma 3 taking $E = B_{X^{**}}$, $B = B_X$ and $\{x_0^* + rx_n^*\}$ playing the role of the sequence of bounded functions on E, and we get

$$(3) \qquad \sup_{x \in B_X} \left\{\limsup_n \,(x_0^* + rx_n^*)(x)\right\} \geq \inf\left\{\|x^*\| : x^* \in \mathrm{co}\{x_0^* + rx_n^*\}\right\}.$$

Let us compute the left hand term in the previous inequality; since we know by condition (1) that $\{x_n^*(x)\}$ converges to 0, for every $x \in X$, then $\limsup (x_0^* + rx_n^*)(x) = x_0^*(x)$, so the left hand term is just $\|x_0^*\| = 1$. Now the inequality (3) provides us an element $x^* \in X^*$ that can be expressed as $x^* = \sum_{i=1}^m t_i(x_0^* + rx_i^*)$ for some $t_i \geq 0$ with $\sum_{i=1}^m t_i = 1$ and $\|x^*\| < 1 + \varepsilon$. So, we clearly deduce

$$\begin{aligned} \|x^{**}\|(1 + \varepsilon) &\geq \|x^{**}\| \, \|x^*\| \geq x^{**}(x^*) = x^{**}(x_0^*) + r\sum_{i=1}^m t_i x^{**}(x_i^*) \quad \text{(by (2))} \\ &\geq x^{**}(x_0^*) + r\|x^{**} + X\| - r\varepsilon \end{aligned}$$

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

and $(*)$ follows letting $\varepsilon \longrightarrow 0$.

Now we will show the inequality announced in the lemma. So let us fix an element $x_0 \in S_X$ with $x_0^*(x_0) = 1$ and choose $x^{**} \in X^{**}$ and $t > 0$; by using $(*)$ for the element $x_0 + tx^{**}$ in X^{**} we have

$$1 + tx^{**}(x_0^*) + rt\|x^{**} + X\| \leq \|x_0 + tx^{**}\|,$$

so, we also have, for any $t > 0$

$$x^{**}(x_0^*) + r\|x^{**} + X\| \leq \frac{\|x_0 + tx^{**}\| - 1}{t}.$$

It is well-known that

$$\lim_{t \longrightarrow 0^+} \frac{\|x_0 + tx^{**}\| - 1}{t} = \max V(x_0, x^{**})$$

(see for instance [19] or [7, Theorem V.9.5]), so

$$x^{**}(x_0^*) + r\|x^{**} + X\| \leq \max V(x_0, x^{**})$$

holds for any $x^{**} \in X^{**}$. ∎

Before proving the main result, let us recall that a Banach space X is said to be *very smooth* if it is smooth (*i.e.*, $D(x, X)$ is a singleton for any $x \in S_X$) and the duality mapping $x \longmapsto D(x, X)$ is norm to weak continuous. This notion, which is due to J. Diestel and B. Faires [6, Section 4], has also received attention from some other authors (see for instance [9]).

Let us note that assuming $D(x_0, X^{**})$ is contained in a small ball, we can get a sharp control of the right hand term in the inequality of Lemma 4. We will use this idea in case the Banach space X satisfies the so called *Mazur intersection property*, that is, any bounded, closed and convex set in X can be expressed as an intersection of closed balls. This notion was introduced in [20] and afterwards, it was used by several authors (see for instance [21, 25]). J. Giles, D. Gregory and B. Sims characterized the Mazur intersection property in terms of the extremal structure of the dual unit ball [10] and they also gave the corresponding definition for dual spaces: it is said that X^* has the *w^*-Mazur intersection property* if, and only if, any w^*-compact convex set can be represented as an intersection of closed balls.

There are spaces very smooth not satisfying the Mazur intersection property. For instance, let us take $X = \ell_2$ endowed with the norm given by

$$|x| = \max\left\{\|x\|_2, 2|x(1)|\right\} + \|\alpha x\|_2 \quad (x \in X),$$

where α is a fixed sequence in ℓ_2 of positive real numbers. Since the norm $|\cdot|$ is strictly convex, then the corresponding norm on X^* is smooth, so X^* is very smooth. However, the open set of the unit ball given by

$$\left\{x \in B_X : \|x\|_2 < 2|x(1)|\right\}$$

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

does not contain denting points and so, X^* does not have the Mazur intersection property (see [9, Theorem 2.1]). On the other hand, there are even finite-dimensional examples satisfying the Mazur intersection property which are not smooth: any norm in $\mathbb{R}^3$ whose dual unit sphere contains a proper line segment but has a dense set of extreme points [21, Theorem 4.4]. In fact, there are spaces that can be renormed to have the Mazur intersection property and do not admit even a Gateaux differentiable norm (see [16] and [13]).

THEOREM 5. *Let X be a separable Banach space satisfying at least one of the following properties:*

(i) X is very smooth.

(ii) X^{**} has the w^*-Mazur intersection property.

Then X is reflexive if, and only if, the set of norm attaining functionals has non-empty interior.

PROOF. Assume X is separable, very smooth and there are $\rho > 0$ and $x_0^* \in X^*$ such that $x_0^* + \rho B_{X^*} \subset A(X)$; the set $A(X)$ is a cone, hence we can suppose $\|x_0^*\| = 1$. Let us choose $x_0 \in S_X$ with $x_0^*(x_0) = 1$; since X is very smooth, $D(x_0, X^{**})$ is a singleton [9, Theorem 3.1], so it is just $\{x_0^*\}$ and the inequality appearing in Lemma 4 gives us

$$x^{**}(x_0^*) + \rho\|x^{**} + X\| \leq x^{**}(x_0^*), \quad \forall x^{**} \in X^{**},$$

hence $\|x^{**} + X\| = 0$, for any x^{**} in X^{**}, that is, X is reflexive.

Now let us assume that X^{**} satisfies the w^*-Mazur intersection property and $A(X)$ has non-empty interior, so there are $x_0^* \in S_{X^*}$ and $r > 0$ such that the ball centered at x_0^* with radius $2r$ is contained in the set of norm attaining functionals on X. Since X^{**} has the w^*-Mazur intersection property, the set of denting points of B_{X^*} is dense in S_{X^*} [10, Theorem 3.1], hence we can take x_0^* as a denting point of B_{X^*}. We fix $0 < \varepsilon < r$, and now the definition of denting point provides us an element $z_0^{**} \in S_{X^{**}}$ verifying that for some $\alpha > 0$ the slice $S(B_{X^*}, z_0^{**}, \alpha)$ given by

$$S(B_{X^*}, z_0^{**}, \alpha) = \left\{z^* \in B_{X^*} : z_0^{**}(z^*) > 1 - \alpha\right\}$$

contains x_0^* and has diameter less than ε. Now, choose z_0^* with $z_0^{**}(z_0^*) > 1 - \delta^2/4$, for some $\delta < \min\{\alpha/2, r - \varepsilon, 1\}$, which obviously implies that $z_0^* \in S(B_{X^*}, z_0^{**}, \alpha)$. By using Bishop-Phelps-Bollobás Theorem [2, Theorem 16.1] we can take $y_0^* \in S_{X^*}$ and $y_0^{**} \in D(y_0^*, X^*)$ with

$$\max\{\|y_0^* - z_0^*\|, \|y_0^{**} - z_0^{**}\|\} < \delta. \tag{1}$$

Therefore, by using that $x_0^*, z_0^* \in S(B_{X^*}, z_0^{**}, \alpha)$ and the choice of δ, we get

$$\|y_0^* - x_0^*\| \leq \|y_0^* - z_0^*\| + \|z_0^* - x_0^*\| < \delta + \varepsilon < r$$

and so, since x_0^* belongs to the interior of $A(X)$, we know that

$$y_0^* + rB_{X^*} \subset x_0^* + 2rB_{X^*} \subset A(X). \tag{2}$$

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

If we apply Lemma 4 for y_0^* we get

$$1 + r\|y_0^{**} + X\| \leq \|y_0^{**}\| = 1,$$

that is, $y_0^{**} \in X$, so we will write $y_0 = y_0^{**}$. Let us check that

$$S(B_{X^*}, y_0, \delta) \subset S(B_{X^*}, z_0^{**}, \alpha);$$

if $x^* \in B_{X^*}$ and $x^*(y_0) > 1 - \delta$ then

$$\begin{aligned} z_0^{**}(x^*) &> 1 - \delta - \|y_0 - z_0^{**}\| \quad \text{(by (1))} \\ &> 1 - 2\delta > 1 - \alpha, \end{aligned}$$

because of the choice of δ. So, also the diameter of $S(B_{X^*}, y_0, \delta)$ is less than ε and, by using that $y_0^* \in S_{X^*}$ and $y_0^*(y_0) = 1$ it follows that $S(B_{X^*}, y_0, \delta) \subset y_0^* + \varepsilon B_{X^*}$. Making use of the w^*-denseness of B_{X^*} in $B_{X^{***}}$ we clearly deduce

$$D(y_0, X^{**}) \subset S(B_{X^{***}}, y_0, \delta) \subset \overline{S(B_{X^*}, y_0, \delta)}^{w^*} \subset y_0^* + \varepsilon B_{X^{***}}. \tag{3}$$

By (2) y_0^* satisfies the hypothesis in Lemma 4, so we get

$$x^{**}(y_0^*) + r\|x^{**} + X\| \leq \max V(y_0, x^{**}), \quad \forall x^{**} \in X^{**}.$$

But, in view of (3), we can estimate

$$\max V(y_0, x^{**}) \leq x^{**}(y_0^*) + \varepsilon, \quad \forall x^{**} \in S_{X^{**}},$$

and linking the last two inequalities we conclude $r\|x^{**} + X\| \leq \varepsilon$, for any $0 < \varepsilon < r$ and any $x^{**} \in S_{X^{**}}$, and this clearly implies X to be reflexive. ∎

As a direct consequence of the previous result we obtain:

COROLLARY 6. *If a non-reflexive and separable Banach space X has a Fréchet differentiable norm, then the set $A(X)$ has empty interior.*

PROOF. A Banach space whose norm is Fréchet differentiable has the Mazur intersection property. Under this assumption the set of w^*-denting points of B_{X^*} is dense in S_{X^*} [10, Theorem 2.1], hence X^{**} satisfies the w^*-Mazur intersection property [10, Theorem 3.1] and this time condition (ii) in Theorem 5 is the appropriate hypothesis to be used. ∎

The result in the above corollary was obtained by M. Jiménez Sevilla and J. P. Moreno for any Banach space with the Mazur intersection property, by using James Theorem after renorming the original space [17].

Let us note that this conclusion may be a little bit surprising since for some situations the set of norm attaining functionals on a Banach space X is residual, for instance, this happens in case X has the Mazur intersection property and X^* also satisfies the w^*-Mazur intersection property [18, Theorem 2.8].

By using techniques which give equivalent norms satisfying some smoothness condition we obtain:

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

COROLLARY 7. *Every separable Banach space X which is not weakly sequentially complete admits an equivalent norm such that $A(X)$ does not contain balls.*

PROOF. Let us choose $z^{**} \in X^{**} \backslash X$ such that z^{**} is the ω^*-limit of a sequence $\{x_n\}$ of elements in X, and now, proceeding as in the proof of [11, Theorem 1.2] one can construct an equivalent norm in X which is differentiable at every $x \in X \backslash \{0\}$ in the direction of z^{**}. By [11, Lemma 1.3], this implies that

$$\inf_{\alpha > 0} \operatorname{Osc}\big(z^{**}, S(B_{X^*}, x, \alpha)\big) = 0, \quad \forall x \in X,$$

where

$$\operatorname{Osc}\big(z^{**}, S(B_{X^*}, x, \alpha)\big) = \sup z^{**}\big(S(B_{X^*}, x, \alpha)\big) - \inf z^{**}\big(S(B_{X^*}, x, \alpha)\big)$$

and, as a consequence, we have

$$\inf_{\alpha > 0} \operatorname{Osc}\big(z^{**}, S(B_{X^{***}}, x, \alpha)\big) = 0.$$

Since, given $x_0 \in S_X$ we have $D(x_0, X^{**}) \subseteq S(B_{X^{***}}, x_0, \alpha)$ for any $0 < \alpha < 1$, then if $x_0^* \in S_{X^*}$ attains its norm at x_0, $V(x_0, z^{**}) = \{z^{**}(x_0^*)\}$, and so, from Lemma 4, we deduce that x_0^* is not an interior point of $A(X)$. ∎

We do not know if the above result holds if the condition of weakly sequentially completeness is omitted.

Note that Theorem 5 cannot be deduced from the result by J. Bourgain and C. Stegall that was mentioned in the introduction, since a Banach space with dentable unit ball may be very smooth. Actually, a very smooth space can be renormed in such a way that the unit ball has at least a denting point and the condition "very smooth" still holds. To this purpose, if $X = Y \oplus \mathbb{R}x$ ($x \neq 0$) is such space, it suffices to define an equivalent norm in X by

$$|y + \lambda x|^2 := \|y\|^2 + |\lambda|^2 \quad (y \in Y,\ \lambda \in \mathbb{R}),$$

where we denote by $\| \ \|$ the original norm in X. Since $(X, \| \ \|)$ is very smooth, it is straightforward to check that $(X, | \ |)$ is also very smooth and the element x in the unit sphere is a denting point of the unit ball. So, Theorem 5 can be applied at least to one norm (with dentable unit ball) in any separable space which admits a very smooth norm.

Lemma 4 can also be useful in situations not covered by Theorem 5, for example:

PROPOSITION 8. *Let X be a separable Banach space such that X^{**}/X is infinite dimensional and for every x_0 in S_X the set $D(x_0, X^{**})$ is contained in a finite-dimensional space, then $A(X)$ has empty interior.*

PROOF. Let us fix x_0 in S_X and denote by Y the linear subspace of X^{***} generated by $D(x_0, X^{**})$. Since Y is finite dimensional, its annihilator Y° ($\subseteq X^{**}$) has finite codimension in X^{**}, so, by using that X^{**}/X is infinite dimensional, Y° cannot be contained in X.

https://doi.org/10.4153/CMB-1998-040-x Published online by Cambridge University Press

If we assume now that $x_0^* + rB_{X^*} \subset A(X)$ for some $x_0^* \in S_{X^*}$, $r > 0$, let us fix $x_0 \in S_X$ such that $x_0^*(x_0) = 1$ and Lemma 4 gives us

$$x^{**}(x_0^*) + r\|x^{**} + X\| \leq \max V(x_0, x^{**}), \quad \forall x^{**} \in X^{**}.$$

Now, choose $x^{**} \in X^{**} \backslash X$ such that x^{**} belongs to the annihilator of $D(x_0, X^{**})$, so we have $V(x_0, x^{**}) = \{0\}$ and also $x^{**}(x_0^*) = 0$, hence, in view of the previous inequality x^{**} belongs to X, a contradiction.

Therefore, under these conditions $A(X)$ has empty interior, as we wanted to show. ∎

It is not difficult to check that Proposition 8 can be applied, at least, in case we take as X the space c_0 or some "canonical" preduals of Lorentz spaces $d(w, 1)$ considered in [8, 22], since in both cases X is *M-ideal* in its bidual (see [14, Examples III.1.4] and [26, Proposition 2.2]), that is,

$$X^{***} = X^* \oplus X^\circ,$$

where X° denotes the annihilator of X (in X^{***}) and

$$\|x^* + x^\circ\| = \|x^*\| + \|x^\circ\|, \quad \forall x^* \in X^*, \quad \forall x^\circ \in X^\circ.$$

From the previous definition it follows that $D(x, X) = D(x, X^{**})$ for any $x \in S_X$ and the serious lack of extreme points of the unit ball in both cases (see [1, Lemma 1.3]) allows one to check that $D(x, X)$ generates a finite dimensional subspace. However, for non-reflexive M-ideals, the unit ball is not dentable, so Bourgain-Stegall's result applies.

Now we will show that, in a certain sense, the hypothesis of very smooth is not too far from being sharp in Theorem 5. Taking into account that the assumption of smoothness on X implies that the set $A(X)$ is not too large (there is just one normalized functional attaining its norm at each point in the unit sphere of X), it is easy to believe that the complement of $A(X)$ is dense for a non-reflexive Banach space satisfying this kind of condition. But we will prove that assuming X is just smooth one cannot expect the same result.

PROPOSITION 9. *Let X be a separable Banach space, then there is a smooth space Y isomorphic to X such that $A(Y)$ has non-empty interior.*

PROOF. In view of Corollary 2 we may assume that $X^* \backslash A(X)$ is not dense in X^*. Now the proof of [4, Theorem 9.(4)] provides us an equivalent norm $|\ |$ on X such that $Y = (X, |\ |)$ is smooth and also $A(Y) = A(X)$, so the condition $A(Y)$ has non-empty interior still holds. ∎

On the other hand, one can consider a class of Banach spaces including spaces which are either very smooth or satisfy the Mazur intersection property, which is those X whose dual space X^* does not contain norming proper (closed) subspaces. Again this last hypothesis is not sufficient to get the same conclusion as in Theorem 5 or Corollary 6. To show this, let us consider the following case:

EXAMPLE 10. There is a non-reflexive Banach space X whose dual space has no norming (proper) subspaces and $X^*\backslash A(X)$ fails to be dense in X^*.

It is enough to consider as X the space c_0 endowed with the norm $|\ |$ whose unit ball is given by the set

$$B = \overline{|\operatorname{co}|}\big(B_{c_0} \cup \{2e_1\}\big),$$

where we denote by $\{e_n\}$ the usual basis of c_0. By Proposition 1 we know that for this space the set of norm attaining functionals has non-empty interior. We need just to check that X^* does not contain proper norming subspaces. Under the usual identification $X^* \equiv \ell_1$, we clearly have in X^*

$$|y| = \max\Big\{\sum_{k=1}^{\infty} |y(k)|, 2|y(1)|\Big\} \quad (y \in X^*)$$

where we also denote by $|\ |$ the dual norm of $(X, |\ |)$. Now, if $M \subset X^*$ is a norming subspace, then we clearly have for every $n \geq 2$

$$|e_n| = \sup\big\{|y(n)| : y \in M, |y| \leq 1\big\} = 1; \tag{1}$$

since $|y| \geq \sum_{k=1}^{\infty} |y(k)| \geq |y(n)|$ for any $y \in X^*$ and M is closed, (1) clearly forces that $e_n^* \in M$, for any $n \geq 2$ ($\{e_n^*\}$ is the sequence of biorthogonal functionals of the basis $\{e_n\}$). But using again that M is norming, there has to be an element $y \in M$ with $y(1) \neq 0$; from this condition and the fact that M contains the subset $\{e_n^* : n \geq 2\}$ it follows $M = \ell_1 \equiv X^*$, as we wanted to show.

Finally, we would like to point out that if one considers other topologies in the dual space the assertion in Corollary 6 may be satisfied without assuming any additional hypothesis; for instance, G. Debs, G. Godefroy and J. Saint Raymond prove in [4, Lemma 11] that for any separable non-reflexive Banach space X the set $X^*\backslash A(X)$ is always w^*-dense in the dual space X^*.

ACKNOWLEDGEMENT. The authors would like to thank Rafael Payá for bringing the question dealt with in this paper to their attention. Also they are grateful to Gilles Godefroy for his suggestions.

REFERENCES

1. M. D. Acosta, F. J. Aguirre and R. Payá, *A space by W. Gowers and new results on norm and numerical radius attaining operators*. Acta Univ. Carolin. Math. Phys. **33**(1992), 5–14.
2. F. F. Bonsall and J. Duncan, *Numerical Ranges II*. London Math. Soc. Lecture Note Ser. **10**, Cambridge University Press, 1973.
3. R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*. Lecture Notes in Math. **993**, Springer-Verlag, Berlin, 1983.
4. G. Debs, G. Godefroy and J. Saint Raymond, *Topological properties of the set of norm-attaining linear functionals*. Canad. J. Math. **47**(1995), 318–329.
5. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*. Pitman Monographs Surveys Pure Appl. Math. **64**, Longman Sci. Tech., New York, 1993.
6. J. Diestel and B. Faires, *On vector measures*. Trans. Amer. Math. Soc. **198**(1974), 253–271.

7. N. Dunford and J. T. Schwarz, *Linear Operators Part I: General theory*. Interscience Publishers, New York, 1958.
8. D. J. H. Garling, *On symmetric sequence spaces*. Proc. London Math. Soc. **16**(1966), 85–106.
9. J. R. Giles, D. A. Gregory and B. Sims, *Geometrical implications of upper semi-continuity of the duality mapping on a Banach space*. Pacific J. Math. **79**(1978), 99–109.
10. ______, *Characterisation of normed linear spaces with Mazur's intersection property*. Bull. Austral. Math. Soc. **18**(1978), 105–123.
11. G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms*. Studia Math. **95**(1989), 1–15.
12. ______, *Some applications of Simons' inequality*. Murcia Seminar on Functional Analysis II (to appear).
13. R. Haydon, *A counterexample to several questions about scattered compact spaces*. Bull. London Math. Soc. **22**(1990), 261–268.
14. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*. Lecture Notes in Math. **1547**, Springer-Verlag, Berlin, 1993.
15. R. C. James, *Weak compactness and reflexivity*. Israel J. Math. **2**(1964), 101–119.
16. M. Jiménez Sevilla and J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*. J. Funct. Anal. **144**(1997), 486–504.
17. ______, *A note on norm attaining functionals*. Proc. Amer. Math. Soc. (to appear).
18. P. S. Kenderov and J. R. Giles, *On the structure of Banach spaces with Mazur's intersection property*. Math. Ann. **291**(1991), 463–473.
19. S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*. Studia Math. **4**(1933), 70–84.
20. ______, *Über schwache Konvergentz in den Räumen* ℓ^p. Studia Math. **4**(1933), 128–133.
21. R. R. Phelps, *A representation theorem for bounded convex sets*. Proc. Amer. Math. Soc. **11**(1960), 976–983.
22. W. L. C. Sargent, *Some sequence spaces related to the* ℓ^p *spaces*. J. London Math. Soc. **35**(1960), 161–171.
23. S. Simons, *A convergence theorem with boundary*. Pacific J. Math. **40**(1972), 703–708.
24. ______, *Maximinimax, minimax, and antiminimax theorems and a result of R. C. James*. Pacific J. Math. **40**(1972), 709–718.
25. F. Sullivan, *Dentability, smoothability and stronger properties in Banach spaces*. Indiana Math. J. **26**(1977), 545–553.
26. D. Werner, *New classes of Banach spaces which are M-ideals in their biduals*. Math. Proc. Cambridge Philos. Soc. **111**(1992), 337–354.

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain
email: dacosta@goliat.ugr.es

Departamento de Matemática Aplicada
E.U. Arquitectura Técnica
Universidad de Granada
18071 Granada
Spain
email: mruizg@goliat.ugr.es