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BIREFLECTIONALITY IN CLASSICAL GROUPS

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1. Introduction. The motion groups of the real Euclidean plane and of the elliptic plane, the group of projectivities of a line, the projective general linear group $PGL_2(K)$, some orthogonal groups $O_3(K, Q)$ with char $K = 2$ (see [**8**]), are all bireflectional (zweispiegelig). There can be no doubt that bireflectional groups are of prime importance in any theory of groups that are generated by involutions. A brief look into F. Bachmann's book [**1**] gives convincing evidence. Already O. Veblen and J. W. Young [**10**, e.g. pp. 280, 318, 322] deliberate on this point in several instances, and before that H. Wiener [**11**] investigated bireflectional groups.

Recently, H. S. M. Coxeter [**3**, p. 3] discovered that "in the real Euclidean space of two or more dimensions, every isometry can be expressed as a product of two involutory isometries", thus proving that real orthogonal groups of any finite dimension are bireflectional. Coxeter put the question to me whether this result could be extended to symplectic groups, and F. Bachmann suggested to investigate the bireflectionality in classical groups generally.

We shall prove in § 2 that neither the symplectic (with char $K \neq 2$) nor the unitary group is bireflectional. In § 3 we shall see that the orthogonal group over any field is bireflectional if the index of the orthogonal vector space is zero.

The scope of § 3 is in fact wider. We ask whether there is a natural way to broaden the concept of bireflectionality, so that we can get results in unitary groups which are analogous to those obtained for orthogonal groups. This is indeed the case. We only have to remember that every involution is a product of commuting reflections, and that in passing from orthogonal to unitary groups in many instances we have to replace reflection by quasi-reflection. Therefore, we define a *quasi-involution* as a product of commuting quasi-reflections. Then it is possible to prove that in unitary groups of index zero, every isometry is a product of two quasi-involutions.

In his book "Regular Polytopes" [**2**, p. 215] H. S. M. Coxeter proves that every even real orthogonal transformation is a product of commuting rotations. We shall establish an analogous result for unitary groups of index zero.

For the convenience of the reader we shall now give a brief survey of the notation that we have adopted. Most definitions and results that are used in this paper can be found in [**6**].

By (V, f) we denote a metric vector space over a field K. The multiplication in K does not have to be commutative. Some authors use the words skewfield,

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sfield, division ring, or corpus instead of field. We always assume that V is a left vector space and consequently write the scalars to the left of the vectors. The metric is determined by the sesquilinear form f, together with the field element ϵ and the antiautomorphism J of K. Accordingly, we assume the following properties:

$$f(\alpha x + x', \beta y + y') = \alpha f(x, y)\beta^J + \alpha f(x, y') + f(x', y)\beta^J + f(x', y')$$
$$\text{for all } \alpha, \beta \in K \text{ and } x, x', y, y' \in V;$$

$$f(y, x) = \epsilon f(x, y)^J, \quad \epsilon\epsilon^J = 1, \quad \text{and } J^2 = I_\epsilon \text{ where } I_\epsilon \text{ is the inner automorphism of } K \text{ that maps } \xi \in K \text{ into } \epsilon^{-1}\xi\epsilon.$$

Two vectors $x, y \in V$ are called *perpendicular* to each other if $f(x, y) = 0$. The metric vector space (V, f) is *orthogonal* if $\epsilon = 1$, $J = \text{id}$, and char $K \neq 2$; it is *symplectic* if $\epsilon = -1$ and $J = \text{id}$. In general, (V, f) is called *unitary*, so that in our terminology orthogonal and symplectic vector spaces are merely special unitary spaces.

The metric vector space (V, f) is *regular*, or nonsingular as some authors say, if there is no vector in V that is perpendicular to V. The vectors in V that are perpendicular to V form a subspace R of V, the *radical* of V.

The elements π in $GL(V)$ that preserve the metric, $f(x^\pi, y^\pi) = f(x, y)$, form a subgroup $U(V)$ of $GL(V)$. $U(V)$ is the *unitary group* of (V, f). If in addition $r^\pi = r$ for all $r \in R$, then π is an element of the *weak* unitary group $U'(V)$.

With each $\pi \in U(V)$ (or more generally with each $\pi \in GL(V)$) we associate two subspaces, the *path* $B(\pi) = V^{\pi-1} = \{x^\pi - x; x \in V\}$, and the *fix* $F(\pi) = \{x \in V; x^\pi = x\}$. Clearly, if π is in the weak unitary group, then $R \subset F(\pi)$.

An element $\pi \in U(V)$ is *simple* if $\dim B(\pi) = 1$. Furthermore, a simple π is a *transvection* if $B(\pi) \subset F(\pi)$, and a *quasi-reflection* if $B(\pi) \cap F(\pi) = \{0\}$. An involutory quasi-reflection is a *reflection* (some authors call all quasi-reflections reflections).

Explicit formulas for simple transformations, transvections, and quasi-reflections are well-known [**5**]. The same is true for a number of elementary properties of $B(\pi)$ and $F(\pi)$ (see e.g. [**9**]).

2. Products of two involutory isometries. In this section we assume that (V, f) is a metric vector space over a field K. Our aim is to show that the weak group of isometries of V is not bireflectional if V is not orthogonal and, in case of char $K = 2$, not symplectic.

If V is regular with char $K \neq 2$ and ρ is an involutory isometry, then the path $B(\rho)$ is regular. Namely, $B(\rho) \cap B(\rho)^\perp = B(\rho) \cap F(\rho) = \{0\}$. As an immediate consequence we get that if V is regular and symplectic with char $K \neq 2$, and if ρ is an involutory isometry of V with finite $\dim B(\rho)$, then $\dim B(\rho)$ is even.

If V is a 2-dimensional regular symplectic vector space with char $K \neq 2$, then clearly $\rho : x \to -x$ is the only symplectic involution of V. Consequently

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in this case the identity is the only transformation that is a product of two involutions. Obviously there are other transformations, namely symplectic transvections of V. Neither the involution ρ nor the symplectic transvections of V can be written as a product of two involutions. We shall see how this result can be generalized.

THEOREM 1. *Let (V, f) be a regular metric vector space over a field K. If σ is a simple isometry that is a product of two involutions, then σ is an involution.*

Proof. Assume $\sigma = \rho_1\rho_2$, where ρ_i are involutions. We shall consider two cases.

If $F(\sigma) + F(\rho_1) = V$, then $B(\rho_2) = B(\rho_1\sigma) = B(\rho_1) + B(\sigma)$. Thus $B(\sigma)$, $B(\rho_1) \subset B(\rho_2)$ and consequently $B(\rho_1) \subset F(\sigma)$. Hence $F(\rho_1) \supset B(\sigma)$. Therefore if $x \in B(\sigma)$, then $x^\sigma = x^{\rho_1\rho_2} = x^{\rho_2} = -x$, which means that σ is an involution.

If $F(\sigma) + F(\rho_1) \neq V$, then $F(\rho_1) \subset F(\sigma)$. Thus $B(\rho_1) \supset B(\sigma)$ and therefore $B(\rho_2) = B(\rho_1\sigma) \subset B(\rho_1) + B(\sigma) = B(\rho_1)$. Hence $B(\rho_2) \subset F(\sigma)$ and consequently $F(\rho_2) \supset B(\sigma)$. Therefore if $x \in B(\sigma)$, then $x^\sigma = x^{\rho_1\rho_2} = (-x)^{\rho_2} = -x$, which means that σ is an involution.

THEOREM 2. *Let (V, f) be a metric vector space which is not orthogonal and which, in case of* char $K = 2$, *contains a trace-valued nonisotropic vector. Then the weak group of isometries of V is not bireflectional.*

Proof. Let σ be a simple isometry with $B(\sigma) \cap R = \{0\}$ and $\sigma = \rho_1\rho_2$ where ρ_i are involutions. The quotient space $\bar{V} = V/R$ is a regular metric but not orthogonal vector space which satisfies the assumptions of our Theorem 2. The induced mappings $\bar{\sigma}$, $\bar{\rho}_i$ are isometries, $\bar{\sigma}$ is simple, $\bar{\sigma} = \bar{\rho}_1\bar{\rho}_2$, and $\bar{\rho}_i$ are involutions or, in case char $K = 2$, one of the $\bar{\rho}_i$ could be the identity. In both cases we get that $\bar{\sigma}$ itself is an involution, using Theorem 1 if $\bar{\sigma}$ is a product of two involutions. Since the mapping $\pi \to \bar{\pi}$ for all isometries π of (V, f) is a surjection, we can assume that V is regular.

If V is symplectic, then by our assumption char $K \neq 2$, consequently clearly no simple isometry is an involution. Hence no simple isometry can be a product of two reflections.

If V is not symplectic, then V contains a trace-valued nonisotropic vector v. Then there are simple isometries with path Kv that are not involutions.

Therefore, the weak group of isometries of (V, f) is not bireflectional.

Clearly, all trace-valued nonsymplectic spaces with char $K = 2$ satisfy the assumptions of Theorem 2. Other examples can be found in [**4**].

In order to prepare for the introduction of quasi-involutions, we shall prove the following lemma, which states that every involution is a product of commuting reflections.

LEMMA 3. *Let (V, f) be a regular metric but not symplectic vector space over a field K with* char $K \neq 2$. *If ρ is an involutory isometry and* $\dim B(\rho) = d$ *is finite,*

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then $\rho = \sigma_1 \ldots \sigma_d$, *where* σ_i *are involutory dilatations with* $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $i, j = 1, \ldots, d$.

Proof. Since V is not symplectic and char $K \neq 2$, there is some nonisotropic vector $v \in B(\rho)$. We define $\sigma\colon x \to x + f(x, v^\rho - v)\alpha^{-1}(v^\rho - v)$ where $\alpha = f(v, v^\rho - v) = f(v, -2v) = f(v, v)(-2) \neq 0$. Then $\sigma\colon x \to x - 2f(x, v)f(v, v)^{-1}v$ is an involutory dilatation, $B(\rho\sigma) = B(\rho) \cap v^\perp$, and dim $B(\rho\sigma) <$ dim $B(\rho)$. Also $\rho\sigma$ is an involution; namely, if $w \in B(\rho\sigma) = B(\rho) \cap v^\perp$, then $w^{\rho\sigma} = (-w)^\sigma = -w$. Now we can use induction.

3. Hyperrotations and quasi-involutions. As we have seen, most of the classical groups are not bireflectional. In this section we shall investigate whether at least some of the desirable properties of bireflectionality still hold in a more general context. Therefore we shall define *quasi-involutions* as the natural substitute for involutions. Then we shall show that in unitary groups of index zero, every element is a product of two quasi-involutions. Our result includes orthogonal groups of index zero, and the theorem specializes to the statement that these orthogonal groups are bireflectional. In fact we shall prove a stronger result, namely that under the assumption already mentioned, every unitary isometry is a hyperrotation. Our investigations include all metric vector spaces, and we mention especially those whose fields of coordinates have characteristic 2. The assumption about the index excludes symplectic vector spaces, though.

We start with a somewhat technical looking lemma.

LEMMA 4. *Let* (V, f) $[(V, Q)]$ *be a regular metric vector space whose index is zero. Assume* π *is an isometry of* V *with* dim $B(\pi) = d$ *finite.*

Then there are simple isometries σ_i, $i = 1, \ldots, k$ *where* $k = d/2$ *or* $(d+1)/2$, *and* ρ_j, $j = 1, \ldots, d - k$, *with the following properties* (i) *to* (vi). *We put* $\pi_0 = \pi$, $\pi_i' = \pi_i\sigma_{i+1}^{-1}$ *for* $i \geqq 0$, $\pi_i = \rho_i^{-1}\pi_{i-1}'$ *for* $i \geqq 1$.

$$\left.\begin{array}{l}
\text{(i) } B(\sigma_{i+1}) \subset B(\pi_i)\\
\text{(ii) } B(\pi_{i+1}) \subset B(\pi_i') \cap F(\sigma_{i+1})\\
\text{(iii) } B(\rho_{i+1}) \subset B(\pi_i')\\
\text{(iv) } B(\pi_{i+1}') \subset B(\pi_{i+1}) \cap F(\rho_{i+1})\\
\text{(v) } \dim B(\pi_i') = \dim B(\pi_i) - 1\\
\text{(vi) } \dim B(\pi_{i+1}) = \dim B(\pi_i') - 1
\end{array}\right\} \textit{for } i \geqq 0.$$

Proof. We can assume that dim $B(\pi) = d \geqq 2$. According to the nature of the statement we need two steps.

I. Put $B(\rho_i) = Kv$ for some $v \in V$ and $i \geqq 1$. If $v \in F(\pi_i)$, then $F(\rho_i) = B(\rho_i)^\perp = v^\perp \supset B(\pi_i)$. For $i \geqq 0$ we define

$$\sigma_{i+1} : x \to x + f(x, y^{\pi_i} - y)\alpha^{-1}(y^{\pi_i} - y) \quad \text{for all } x \in V$$
$$\text{and some } y \in V \backslash F(\pi_i) \text{ and } \alpha = f(y, y^{\pi_i} - y).$$

Then $B(\sigma_{i+1}) \subset B(\pi_i)$ and $B(\pi_i') = B(\pi_i\sigma_{i+1}^{-1}) = B(\pi_i) \cap y^\perp \subset B(\pi_i)$;

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thus $B(\pi_i') \subset B(\pi_i) = B(\pi_i) \cap F(\rho_i)$ for $i \geqq 1$, and $\dim B(\pi_i') = \dim B(\pi_i) - 1$.

If $v \notin F(\pi_i)$, then $v^{\pi_i} - v \neq 0$. We define

$$\sigma_{i+1} : x \to x + f(x, v^{\pi_i} - v)\alpha^{-1}(v^{\pi_i} - v) \quad \text{for all } x \in V \text{ and } \alpha = f(v, v^{\pi_i} - v).$$

Then $B(\sigma_{i+1}) \subset B(\pi_i)$ and $B(\pi_i') = B(\pi_i\sigma_{i+1}^{-1}) = B(\pi_i) \cap v^\perp$; thus $B(\pi_i') = B(\pi_i) \cap v^\perp = B(\pi_i) \cap F(\rho_i)$ for $i \geqq 1$, and $\dim B(\pi_i') = \dim B(\pi_i) - 1$.

II. Put $B(\sigma_{i+1}) = Kw$ for some $w \in V$. If $w \in F(\pi_i')$, then $F(\sigma_{i+1}) = B(\sigma_{i+1})^\perp = w^\perp \supset B(\pi_i')$. Define

$$\rho_{i+1}^{-1} : x \to x + f(x, y^{\pi_i'^{-1}} - y)\alpha^{-1}(y^{\pi_i'^{-1}} - y) \quad \text{for all } x \in V$$
$$\text{and some } y \in V\backslash F(\pi_i') \text{ and } \alpha = f(y, y^{\pi_i'^{-1}} - y).$$

Then $B(\rho_{i+1}) \subset B(\pi_i')$, and $B(\pi_{i+1}) = B(\rho_{i+1}^{-1}\pi_i') = B(\pi_i') \cap y^\perp \subset B(\pi_i') \cap F(\sigma_{i+1})$, and $\dim B(\pi_{i+1}) = \dim B(\pi_i') - 1$.

If $w \notin F(\pi_i')$, then $w^{\pi_i'^{-1}} - w \neq 0$. Define

$$\rho_{i+1}^{-1} : x \to x + f(x, w^{\pi_i'^{-1}} - w)\alpha^{-1}(w^{\pi_i'^{-1}} - w) \text{ for all } x \in V \text{ and}$$
$$\alpha = f(w, w^{\pi_i'^{-1}} - w).$$

Then $B(\rho_{i+1}) \subset B(\pi_i')$, and $B(\pi_{i+1}) = B(\rho_{i+1}^{-1}\pi_i') = B(\pi_i') \cap w^\perp = B(\pi_i') \cap F(\sigma_{i+1})$ and $\dim B(\pi_{i+1}) = \dim B(\pi_i') - 1$.

We shall draw more information from Lemma 4. Namely, we get $B(\sigma_{i+1}) \subset B(\pi_i) \subset B(\pi_{i-1}')$, $F(\sigma_i)$, and further $B(\sigma_{i+1}) \subset F(\sigma_i)$, $F(\rho_{i-1})$, $B(\pi_{i-1})$. Therefore, $B(\sigma_{i+1}) \subset F(\sigma_j)$ for $j \leqq i$ and $B(\sigma_{i+1}) \subset F(\rho_j)$ for $j \leqq i - 1$. Similarly, we obtain $B(\rho_{i+1}) \subset F(\sigma_j)$, $F(\rho_j)$ for $j \leqq i$. Furthermore, $\pi = \rho_1 \ldots \rho_r\sigma_s \ldots \sigma_1$ where $r + s = d$ and either $r = s$ or $r + 1 = s$.

The preceding observations show that

$$(*) \quad \left.\begin{aligned} \rho_i\rho_j &= \rho_j\rho_i \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \end{aligned}\right\} \text{for all } i, j,$$
$$\sigma_i\rho_j = \rho_j\sigma_i \quad \text{for } i < j,$$
$$\sigma_i\rho_j = \rho_j\sigma_i \quad \text{for } j < i - 1.$$

Before we formulate our results we shall introduce a few concepts.

An isometry π with $\dim B(\pi) = d$ is called a *quasi-involution* if there are simple isometries ω_i such that $\pi = \omega_1 \ldots \omega_d$ and $\omega_i\omega_j = \omega_j\omega_i$.

A product of two simple isometries is called a *rotation.*

An isometry π with $\dim B(\pi) = d$ is called a *hyperrotation* if there are rotations η_i for $i \leqq d/2$ and a simple isometry $\eta_{(d+1)/2}$ in case d is odd, with $\eta_i\eta_j = \eta_j\eta_i$ for $j \neq i \pm 1$ and $\pi = \eta_1 \ldots \eta_t$. Here $t = d/2$ if d is even, and $t = (d + 1)/2$ if d is odd.

Now we are ready to state our main results. For the proof we use Lemma 4 and $(*)$.

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THEOREM 5. *Let* (V, f) $[(V, Q)]$ *be a regular metric vector space. Assume that the index of* V *is zero.*

Then every isometry π *of* V *with finite* $\dim B(\pi)$ *is a hyperrotation, and every isometry* π *of* V *with finite* $\dim B(\pi)$ *is a product of two quasi-involutions.*

For orthogonal vector spaces (V, Q) every simple isometry is an involution and consequently every quasi-involution is an involution. Therefore, in this case we get an especially nice result.

COROLLARY 6. *Every regular orthogonal vector space* V *is bireflectional if* $2 \leqq \dim V < \infty$ *and if* $\operatorname{index} V = 0$.

Proof. We should only mention that one of the quasi-involutions of Theorem 5 could be the identity and the other just one reflection σ. Then there is a second reflection σ' whose path is perpendicular to the path of σ and $\sigma = \sigma\sigma' \cdot \sigma'$.

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