A REMARK ON SEPARABLE ORDERS

Klaus W. Roggenkamp

(received November 10, 1968)

Notation. K = algebraic number field,
R = algebraic integers in K,
A = finite dimensional semi-simple K-algebra,
$A = \sum_{i=1}^{n} \oplus A_i$; A_i = simple K-algebra, $i = 1, \ldots, n$,
K_i = center of A_i, $i = 1, \cdots, n$,
G = R-order in A,
$R_i = G \cap K_i$.

All modules under consideration are finitely generated left modules. A G-lattice is a G-module which is R-torsion-free.

In this note we shall prove the following result.

THEOREM. G is a separable R-order in A if and only if

(i) A_i is unramified at all the finite primes of K_i, $i = 1, \cdots, n$,

(ii) every prime ideal of R_i is unramified over R, $i = 1, \cdots, n$,

(iii) G is a maximal R-order in A.

Remark. This result can also be obtained from Auslander-Goldman [1]. However, we would like to sketch an independent proof, which, from the viewpoint of the theory of orders, uses little machinery.

Proof. By G^{op} we denote the opposite order to G, and $G^e = G \otimes_R G^{op}$ is the enveloping algebra of G. We may view G in a natural way as left G^e-module. G is called a separable R-order in A if G is a projective G^e-module. Let us recall (cf. [4; 2]) that the Higman ideal $i(G)$ of the R-order G is defined as

https://doi.org/10.4153/CMB-1969-055-4 Published online by Cambridge University Press

$$i(G) = \{ r \in R : \operatorname{Ext}^1_{G^e}(G,V)r = 0, \text{ for every left } G^e\text{-module } V \}.$$

Thus, G is separable if and only if $i(G) = R$. Since the Higman ideal localizes properly, G is separable if and only if $R^*_p \otimes_R G$ is separable for every finite prime p of R, where R^*_p denotes the p-adic completion of R. Thus it suffices to prove the theorem under the assumption that R is a complete discrete rank one valuation ring with quotient field K, where K is a complete number field.

1) Let G be a separable R-algebra in A. Since G is separable, it follows immediately that $G' = G/pG$ is a separable $R' = R/pR$-algebra, where p is a prime element in R. Thus G is hereditary and $pG = \operatorname{rad}(G)$, where $\operatorname{rad}(G)$ denotes the Jacobson radical of G; and so we may assume that A is a simple K-algebra. Moreover, since $i(G) = R$, where $i(G)$ is the Higman ideal of G, every irreducible G-lattice remains irreducible if reduced modulo p (cf. Maranda [5]). If M, N are two irreducible G-lattices, then $KM \cong KN$, since A is simple, and hence M/pM and N/pN have the same composition factors; i.e., $M/pM \cong N/pN$. This implies $M \cong N$, and G is maximal. Since $\operatorname{rad}(G) = pG$, the center R^* of G is unramified over R.

It remains to show that A is a full matrix ring over its center K^*. Since p is also a prime element of R^*, we may view A as a central simple K^*-algebra and G as an R^*-order in A; moreover, G' is a separable $R^{*\prime} = R^*/pR^*$-algebra. It should be observed that G' is a simple $R^{*\prime}$-algebra. Now, let M be a fixed irreducible G-lattice. Then we have a Morita equivalence between the category of G-lattices and the category of E(M)-lattices, where $E(M) = \operatorname{End}_G(M)$. $E(M)$ is the maximal R^*-order in the skew-field $D = \operatorname{End}_A(KM)$. We intend to show that $D = K^*$; this would finish one part of the proof. The above Morita equivalence induces a Morita equivalence between the finitely generated G'-modules and the category of the finitely generated $E(M)'$-modules, where

$$E(M)' = \operatorname{End}_{G'}(M/pM) \cong \operatorname{End}_G(M)/p\operatorname{End}_G(M).$$

This shows in particular that $E(M)'$ is simple; i.e., $p\operatorname{End}_G(M) = \operatorname{rad}(E(M))$. But then $D = K^*$ (cf. Hasse [3]), since $(D:K^*) = n^2$ implies that the ramification index of $\operatorname{rad}(E(M))$ over $pE(M)$ is n. This proves one direction of the theorem.

2) Assume that G satisfies the conditions of the theorem. Since a direct sum of orders is separable if and only if each summand is separable, we may assume that A is simple. Since all maximal R-orders in A are isomorphic under an inner automorphism of A, it suffices to show, that one maximal R-order in A is separable. Hence we may assume G is the form $G = (R^*)_n$, where R^* is the integral closure of R in the center of A. Since R^* is unramified over R, $rad(G) = (pR^*)_n$, where $pR = rad(R)$. Consequently $G' = G/pG = (R^*/pR^*)_n$ is a semisimple $R' = R/pR$-algebra, hence G' is a separable R'-algebra; i.e., G' is a projective G'^e-module. Thus G is a projective G^e-module; i.e., G is separable. This proves the theorem.

REFERENCES

1. M. Auslander and O. Goldman, The Brauer group of a commutative ring. Trans. Am. Math. Soc. 97 (1960) 367-409.

2. C.W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras. (Interscience, New York, 1962)

3. H. Hasse, Über p-adische Schiefkörper und ihre Bedeurung für die Arithmetik hyperkomplexer Zahlsysteme. Math. Ann. 104 (1931) 495-534.

4. D.G. Higman, Representations of orders over Dedekind domains. Canad. J. Math. 12 (1960) 107-125.

5. J.M. Maranda, On the equivalence of representations of finite groups by groups of automorphisms of modules over Dedekind rings. Canad. J. Math. 7 (1955) 516-526.

Université de Montréal

https://doi.org/10.4153/CMB-1969-055-4 Published online by Cambridge University Press