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SHARP INTEGRAL INEQUALITIES BASED ON GENERAL TWO-POINT FORMULAE VIA AN EXTENSION OF MONTGOMERY'S IDENTITY

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Abstract

We consider families of general two-point quadrature formulae, using the extension of Montgomery's identity via Taylor's formula. The formulae obtained are used to present a number of inequalities for functions whose derivatives are from L_p spaces and Bullen-type inequalities.

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ integrable on $[a, b]$. Then the Montgomery identity

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x,t) f'(t)\,dt \tag{1.1}$$

holds [9], where $P(x, t)$ is the Peano kernel defined as

$$P(x,t) = \begin{cases} \dfrac{1}{b-a}(t-a), & a \le t \le x, \\ \dfrac{1}{b-a}(t-b), & x < t \le b. \end{cases}$$

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Suppose $w : [a, b] \to [0, \infty\rangle$ is a probability density function, that is, an integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a, b]$, the corresponding cumulative distribution function, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The identity

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x, t) f'(t)\,dt \tag{1.2}$$

(given by Pečarić in [10]) is the weighted generalization of the Montgomery identity, where the weighted Peano kernel is

$$P_w(x, t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

In a recent paper [1] the following extension of the Montgomery identity via Taylor's formula has been proved:

$$\begin{aligned} f(x) = \int_a^b w(t) f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}\,ds \\ + \frac{1}{(n-1)!} \int_a^b T_{w,n}(x, s) f^{(n)}(s)\,ds. \end{aligned} \tag{1.3}$$

Here $f : I \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, $x \in [a, b]$, $w : [a, b] \to [0, \infty\rangle$ a probability density function and

$$T_{w,n}(x, s) = \begin{cases} \displaystyle\int_a^s w(u)(u-s)^{n-1}\,du, & a \le s \le x, \\ \displaystyle -\int_s^b w(u)(u-s)^{n-1}\,du, & x < s \le b. \end{cases}$$

If we take $w(t) = 1/(b-a)$, $t \in [a, b]$, the equality (1.3) reduces to

$$\begin{aligned} f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt - \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \\ + \frac{1}{(n-1)!} \int_a^b T_n(x, s) f^{(n)}(s)\,ds, \end{aligned} \tag{1.4}$$

where $x \in [a, b]$ and

$$T_n(x, s) = \begin{cases} \dfrac{-1}{n(b-a)}(a-s)^n, & a \le s \le x, \\ \dfrac{-1}{n(b-a)}(b-s)^n, & x < s \le b. \end{cases}$$

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For $n = 1$ (1.4) reduces to Montgomery's identity (1.1) since $T_{w,1}(x, s) = P_w(x, t)$.

In this paper we study for $x \in [a, (a+b)/2]$ the general weighted two-point quadrature formula

$$\int_a^b w(t) f(t)\, dt = \frac{1}{2}\,[f(x) + f(a+b-x)] + E(f, w; x) \tag{1.5}$$

with $E(f, w; x)$ being the remainder. In the special case, for $w(t) = 1/(b-a)$, $t \in [a, b]$, (1.5) reduces to the family of two-point quadrature formulae considered by Guessab and Schmeisser in [5], where they established sharp estimates for the remainder under various regularity conditions.

The aim of this paper is to establish the general two-point formula (1.5) using the identities (1.3) and (1.4) and to give various error estimates for the quadrature rules based on such generalizations. We prove a number of inequalities which give error estimates for the general two-point formula for functions whose derivatives belong to L_p-spaces. These inequalities are generally sharp (in the case $p = 1$, the best possible). Also, we give some examples of the general two-point formula for well-known weight functions.

We recall that for a convex function f on $[a, b] \subset \mathbb{R}$, $a \neq b$, the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard's inequalities for convex functions. Inequalities

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt \geq \frac{1}{b-a} \int_a^b f(t)\, dt - f\left(\frac{a+b}{2}\right) \geq 0, \tag{1.6}$$

for any convex function f defined on $[a, b]$, were first proved by Bullen in [2]. His results were generalized for $(2n)$-convex functions ($n \in \mathbb{N}$) in [4].

In the last section we use the obtained results to prove a generalization of Bullen-type inequalities for $(2n)$-convex functions ($n \geq 1$).

2. General weighted two-point formula and related inequalities

Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a, b]$ for some $n \geq 2$. We introduce the following notation for each $x \in [a, (a+b)/2]$:

$$D(x) = \frac{1}{2}\,[f(x) + f(a+b-x)],$$

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$$t_{w,n}(x) = \frac{1}{2}\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}\,ds\right.$$
$$\left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a+b-x)}{(i+1)!}\int_a^b w(s)(s-a-b+x)^{i+1}\,ds\right]$$

and

$$\widehat{T}_{w,n}(x,s)$$
$$= -\frac{1}{2}\left[T_{w,n}(x,s) + T_{w,n}(a+b-x,s)\right]$$
$$= \begin{cases} -\int_a^s w(u)(u-s)^{n-1}\,du, & a \le s \le x, \\ -\frac{1}{2}\left[\int_a^s w(u)(u-s)^{n-1}\,du - \int_s^b w(u)(u-s)^{n-1}\,du\right], & x < s \le a+b-x, \\ \int_s^b w(u)(u-s)^{n-1}\,du, & a+b-x < s \le b. \end{cases}$$

In the next theorem we establish a general weighted two-point formula which plays the key role in this section.

THEOREM 2.1. *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. If $w : [a,b] \to [0,\infty\rangle$ is some probability density function, then for each $x \in [a,(a+b)/2]$*

$$\int_a^b w(t)f(t)\,dt = D(x) + t_{w,n}(x) + \frac{1}{(n-1)!}\int_a^b \widehat{T}_{w,n}(x,s)f^{(n)}(s)\,ds. \tag{2.1}$$

PROOF. We put $x \equiv x$ and $x \equiv a+b-x$ in (1.3) to obtain two new formulae. After adding these two formulae and multiplying by $1/2$, we get (2.1). □

REMARK 1. Identity (2.1) holds in the case $n = 1$. It also can be obtained by taking $x \equiv x$, and $x \equiv a+b-x$ in (1.2), adding these two formulae and multiplying by $1/2$. In this special case,

$$\int_a^b w(t)f(t)\,dt = D(x) + \int_a^b \widehat{T}_{w,1}(x,s)f'(s)\,ds, \tag{2.2}$$

where

$$\widehat{T}_{w,1}(x,s) = -\frac{1}{2}\left[T_{w,1}(x,s) + T_{w,1}(a+b-x,s)\right]$$
$$= -\frac{1}{2}\left[P_w(x,s) + P_w(a+b-x,s)\right]$$

$$= \begin{cases} -W(s), & a \le s \le x, \\ \dfrac{1}{2} - W(s), & x < s \le a+b-x, \\ 1 - W(s), & a+b-x < s \le b. \end{cases}$$

DEFINITION 2.2. We say p, q with $1 \le p, q \le \infty$ are conjugate if $p^{-1} + q^{-1} = 1$.

THEOREM 2.3. *Suppose that the assumptions of Theorem 2.1 hold. Additionally assume that (p, q) is a pair of conjugate exponents. Let $f^{(n)} \in L_p[a, b]$ for some $n \ge 2$. Then for each $x \in [a, (a+b)/2]$*

$$\left| \int_a^b w(t) f(t)\, dt - D(x) - t_{w,n}(x) \right| \le \frac{1}{(n-1)!} \left\| \widehat{T}_{w,n}(x, \cdot) \right\|_q \left\| f^{(n)} \right\|_p. \quad (2.3)$$

The constant $(1/(n-1)!)\|\widehat{T}_{w,n}(x, \cdot)\|_q$ is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

PROOF. Applying the Hölder inequality we have

$$\left| \frac{1}{(n-1)!} \int_a^b \widehat{T}_{w,n}(x, s) f^{(n)}(s)\, ds \right| \le \frac{1}{(n-1)!} \left\| \widehat{T}_{w,n}(x, \cdot) \right\|_q \left\| f^{(n)} \right\|_p. \quad (2.4)$$

Using inequality (2.4), from (2.1) we get estimate (2.3). Let's denote $C_n^x(s) = \widehat{T}_{w,n}(x, s)$. Now, we will prove that the constant $(1/(n-1)!)\left[\int_a^b |C_n^x(s)|^q\, ds \right]^{1/q}$ is optimal. We will find a function f such that

$$\left| \int_a^b C_n^x(s) f^{(n)}(s)\, ds \right| = \left(\int_a^b |C_n^x(s)|^q\, ds \right)^{1/q} \left(\int_a^b |f^{(n)}(s)|^p\, ds \right)^{1/p}.$$

For $1 < p < \infty$ take f to be such that $f^{(n)}(s) = \operatorname{sgn} C_n^x(s) \cdot |C_n^x(s)|^{1/(p-1)}$. For $p = \infty$ take $f^{(n)}(s) = \operatorname{sgn} C_n^x(s)$. For $p = 1$ we shall prove that

$$\left| \int_a^b C_n^x(s) f^{(n)}(s)\, ds \right| \le \sup_{s \in [a,b]} |C_n^x(s)| \left(\int_a^b |f^{(n)}(s)|\, ds \right) \quad (2.5)$$

is the best possible inequality.

The function $C_n^x(s)$ is left continuous and has finite jumps at x and $a+b-x$. Thus we have four possibilities.

(1) Suppose $|C_n^x(s)|$ attains its maximum at $s_0 \in [a, b]$ and $C_n^x(s_0) > 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \le s \le s_0 - \varepsilon, \\ \dfrac{1}{\varepsilon n!} (s - s_0 + \varepsilon)^n, & s_0 - \varepsilon \le s \le s_0, \\ \dfrac{1}{n!} (s - s_0 + \varepsilon)^{n-1}, & s_0 \le s \le b. \end{cases}$$

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Thus

$$\left|\int_a^b C_n^x(s) f_\varepsilon^{(n)}(s)\, ds\right| = \left|\int_{s_0-\varepsilon}^{s_0} C_n^x(s) \frac{1}{\varepsilon}\, ds\right| = \frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\, ds.$$

Now, from inequality (2.5),

$$\frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\, ds \le \frac{1}{\varepsilon} C_n^x(s_0)\int_{s_0-\varepsilon}^{s_0} ds = C_n^x(s_0).$$

Since

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\, ds = C_n^x(s_0)$$

the statement follows.

(2) Suppose $|C_n^x(s)|$ attains its maximum at $s_0 \in [a, b]$ and $C_n^x(s_0) < 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \dfrac{1}{n!}(s_0 - s)^{n-1}, & a \le s \le s_0 - \varepsilon, \\ -\dfrac{1}{\varepsilon n!}(s_0 - s)^n, & s_0 - \varepsilon \le s \le s_0, \\ 0, & s_0 \le s \le b, \end{cases}$$

and the rest of the proof is similar to that given in (1).

(3) Suppose $|C_n^x(s)|$ does not attain a maximum on $[a, b]$ and let $s_0 \in [a, b]$ be such that

$$\sup_{s\in[a,b]} |C_n^x(s)| = \lim_{\varepsilon\to 0^+} |f(s_0 + \varepsilon)|.$$

If $\lim_{\varepsilon\to 0^+} f(s_0 + \varepsilon) > 0$, we take

$$f_\varepsilon(s) = \begin{cases} 0, & a \le s \le s_0, \\ \dfrac{1}{\varepsilon n!}(s - s_0)^n, & s_0 \le s \le s_0 + \varepsilon, \\ \dfrac{1}{n!}(s - s_0)^{n-1}, & s_0 + \varepsilon \le s \le b, \end{cases}$$

and similarly to before we have

$$\left|\int_a^b C_n^x(s) f_\varepsilon^{(n)}(s)\, ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} C_n^x(s) \frac{1}{\varepsilon}\, ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\, ds,$$

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\, ds \le \frac{1}{\varepsilon} C_n^x(s_0)\int_{s_0}^{s_0+\varepsilon} ds = C_n^x(s_0),$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\, ds = C_n^x(s_0)$$

and the statement follows.

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(4) Suppose $|C_n^x(s)|$ does not attain a maximum on $[a, b]$ and let $s_0 \in [a, b]$ be such that

$$\sup_{s\in[a,b]} |C_n^x(s)| = \lim_{\varepsilon\to 0^+} |f(s_0+\varepsilon)|.$$

If $\lim_{\varepsilon\to 0^+} f(s_0+\varepsilon) < 0$, we take

$$f_\varepsilon(s) = \begin{cases} \dfrac{1}{n!}\,(s-s_0-\varepsilon)^{n-1}, & a \le s \le s_0, \\ -\dfrac{1}{\varepsilon n!}\,(s-s_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon, \\ 0, & s_0+\varepsilon \le s \le b, \end{cases}$$

and the rest of the proof is similar to that shown in (1). □

THEOREM 2.4. *Suppose that the assumptions of Theorem 2.3 hold. Additionally assume that $f^{(2n)}$ is a differentiable function on $\langle a, b\rangle$. Then for every $x \in [a, (a+b)/2]$ there exists $\eta \in \langle a, b\rangle$ such that*

$$\int_a^b w(t)f(t)\,dt - D(x) - t_{w,2n}(x) = \frac{f^{(2n)}(\eta)}{(2n-1)!}\int_a^b \widehat{T}_{w,2n}(x,s)\,ds. \tag{2.6}$$

PROOF. We apply (2.1) with $2n$ in place of n. Since $-\int_a^s w(u)(u-s)^{2n-1}\,du \ge 0$ for every $s \in [a, x]$, $\int_s^b w(u)(u-s)^{2n-1}\,du \ge 0$ for every $s \in \langle a+b-x, b]$ and

$$\frac{1}{2}\left[-\int_a^s w(u)(u-s)^{2n-1}\,du + \int_s^b w(u)(u-s)^{2n-1}\,du\right] \ge 0$$

for every $s \in \langle x, a+b-x]$, we have $\widehat{T}_{w,2n}(x,s) \ge 0$ for $s \in [a, b]$. By applying the integral mean value theorem to $\int_a^b \widehat{T}_{w,2n}(x,s)f^{(2n)}(s)\,ds$ we obtain (2.6). □

THEOREM 2.5. *Suppose that the assumptions of Theorem 2.1 hold for $2n$, $n \in \mathbb{N}$. If f is $(2n)$-convex, then for each $x \in [a, (a+b)/2]$ the inequality*

$$\int_a^b w(t)f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - t_{w,2n}(x) \ge 0 \tag{2.7}$$

holds. If f is $(2n)$-concave, then the inequality (2.7) is reversed.

PROOF. First note that if $f^{(k)}$ exists, then f is k-convex (k-concave) if and only if $f^{(k)} \ge 0$ ($f^{(k)} \le 0$).

From (2.1) we have that

$$\int_a^b w(t)f(t)\,dt - D(x) - t_{w,2n}(x) = \frac{1}{(n-1)!}\int_a^b \widehat{T}_{w,2n}(x,s)f^{(2n)}(s)\,ds.$$

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Let us consider the sign of the integral

$$\int_a^b \widehat{T}_{w,2n}(x,s) f^{(2n)}(s)\,ds$$

when f is $2n$-convex. We have $f^{(2n)} \geq 0$ and from the proof of Theorem 2.4, $\widehat{T}_{w,2n}(x,s) \geq 0$. Hence, $\int_a^b \widehat{T}_n(x,s) f^{(n)}(s)\,ds \geq 0$, and (2.7) follows.

The reversed (2.7) can be obtained analogously. □

REMARK 2. If in Theorem 2.3 we set $x = (a+b)/2$ we get the generalized midpoint inequality (see [1])

$$\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(t) f(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{(i+1)!} \int_a^b w(s)\left(s - \frac{a+b}{2}\right)^{i+1} ds \right|$$
$$\leq \frac{1}{(n-1)!} \left(\int_a^b \left| T_{w,n}\left(\frac{a+b}{2}, s\right)\right|^q ds \right)^{1/q} \left\| f^{(n)} \right\|_p .$$

For the generalized trapezoid inequality we apply (2.3) with $x = a$ or $x = b$:

$$\left| \frac{f(a)+f(b)}{2} - \int_a^b w(t) f(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{2(i+1)!} \int_a^b w(s)(s-a)^{i+1}\,ds \right.$$
$$\left. + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{2(i+1)!} \int_a^b w(s)(s-b)^{i+1}\,ds \right|$$
$$\leq \frac{1}{2(n-1)!} \left(\int_a^b \left| T_{w,n}(a,s) + T_{w,n}(b,s)\right|^q ds \right)^{1/q} \left\| f^{(n)} \right\|_p$$

where

$$T_{w,n}(a,s) + T_{w,n}(b,s) = \int_a^s w(u)(u-s)^{n-1}\,du - \int_s^b w(u)(u-s)^{n-1}\,du.$$

For the applications to follow we introduce the notation

$$f_k^*(x) = \sum_{\substack{0 \leq i \leq k \\ j=1,2}} (-1)^{i(j+1)} x^i f^{(i)}\left((-1)^j x\right) \quad k = 0, 1, 2.$$

3. Application to Gaussian quadrature formulae

Gaussian quadrature formulae are formulae of the type

$$\int_a^b \varrho(t) f(t)\,dt \approx \sum_{i=1}^{k} A_i f(x_i).$$

Without loss of generality, we shall restrict ourselves to $[a,b] = [-1,1]$.

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3.1. The case $\varrho(t) = (1/\sqrt{1-t^2})$, $t \in [-1, 1]$ In this case we have a *Gauss–Chebyshev formula*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, dt = \sum_{i=1}^{k} A_i f(x_i) + E_k(f), \tag{3.1}$$

where $A_i = \pi/k$, $i = 1, \ldots, k$ and the x_i $i = 1, \ldots, k$ are zeros of the *Chebyshev polynomials of the first kind* defined as

$$T_k(x) = \cos(k \arccos(x)).$$

The polynomial $T_k(x)$ has exactly k distinct zeros, all of which lie in the interval $[-1, 1]$ (see for instance [13]) and are given by

$$x_i = \cos\left(\frac{(2i-1)\,\pi}{2k}\right).$$

The error of the approximation formula (3.1) is given by

$$E_k(f) = \frac{\pi}{2^{2k-1}(2k)!} f^{(2k)}(\xi), \quad \xi \in \langle -1, 1\rangle.$$

In the case $k = 2$ (3.1) reduces to

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, dt = \frac{\pi}{2} f_0^*\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{192} f^{(4)}(\xi), \quad \xi \in \langle -1, 1\rangle.$$

REMARK 3. If we apply (2.2) with $a = -1$, $b = 1$, $x = -\sqrt{2}/2$ and $w(t) = 1/(\pi\sqrt{1-t^2})$, $t \in [-1, 1]$, we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, dt = \frac{\pi}{2} f_0^*\left(\frac{\sqrt{2}}{2}\right) + \pi \int_{-1}^{1} R_1(s) f'(s)\, ds,$$

where

$$R_1(s) = \begin{cases} -\dfrac{1}{2} - \dfrac{1}{\pi} \arcsin s, & -1 \le s \le -\dfrac{\sqrt{2}}{2}, \\ -\dfrac{1}{\pi} \arcsin s, & -\dfrac{\sqrt{2}}{2} < s \le \dfrac{\sqrt{2}}{2}, \\ \dfrac{1}{2} - \dfrac{1}{\pi} \arcsin s, & \dfrac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

COROLLARY 3.1. *Let $f : I \to \mathbb{R}$ be absolutely continuous, $I \subset \mathbb{R}$ an open interval, $[-1, 1] \subset I$, (p, q) a pair of conjugate exponents, and $f' \in L_p[-1, 1]$. Then*

$$\left| \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, dt - \frac{\pi}{2} f_0^*\left(\frac{\sqrt{2}}{2}\right) \right| \le \pi \, \|R_1\|_q \, \|f'\|_p. \tag{3.2}$$

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PROOF. This is a special case of Theorem 2.3 for $a=-1$, $b=1$, $x=-\sqrt{2}/2$ and $w(t)=1/(\pi\sqrt{1-t^2})$, $t\in[-1,1]$. □

COROLLARY 3.2. *Suppose that all the assumptions of Corollary 3.1 hold. Then*

$$\left|\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,dt-\frac{\pi}{2}f_0^*\left(\frac{\sqrt{2}}{2}\right)\right|\le\begin{cases}(2\sqrt{2}-2)\,\|f'\|_\infty,\\(\pi\sqrt{2}-4)^{1/2}\,\|f'\|_2,\\\frac{1}{4}\pi\,\|f'\|_1.\end{cases}$$

The constants on the right-hand sides of the first and second inequalities are sharp and the best possible in the third inequality.

PROOF. We apply (3.2) with $p=\infty$:

$$\int_{-1}^{1}|R_1(s)|\,ds=\int_{-1}^{-\sqrt{2}/2}\left|-\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|ds+\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\left|-\frac{1}{\pi}\arcsin s\right|ds$$
$$+\int_{\sqrt{2}/2}^{1}\left|\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|ds=\frac{2\sqrt{2}-2}{\pi}$$

and the first inequality is obtained. To prove the second inequality we take $p=2$:

$$\int_{-1}^{1}|R_1(s)|^2\,ds=\int_{-1}^{-\sqrt{2}/2}\left|-\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|^2ds+\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\left|-\frac{1}{\pi}\arcsin s\right|^2ds$$
$$+\int_{\sqrt{2}/2}^{1}\left|\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|^2ds=\frac{\pi\sqrt{2}-4}{\pi^2}.$$

If $p=1$, then $\sup_{s\in[-1,1]}|R_1(s)|$ equals

$$\max\left\{\sup_{s\in\left[-1,-\frac{\sqrt{2}}{2}\right]}\left|-\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|,\ \sup_{s\in\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]}\left|-\frac{1}{\pi}\arcsin s\right|,\ \sup_{s\in\left[\frac{\sqrt{2}}{2},1\right]}\left|\frac{1}{2}-\frac{1}{\pi}\arcsin s\right|\right\}.$$

By an elementary calculation, each of the three suprema is equal to $1/4$, and the third inequality is proved. □

REMARK 4. The first and third inequality from the Corollary 3.2 have also been obtained in [7].

REMARK 5. If we apply Theorem 2.1 with $n=2$, $a=-1$, $b=1$, $x=-\sqrt{2}/2$ and $w(t)=1/(\pi\sqrt{1-t^2})$, $t\in[-1,1]$, we get

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,dt=\frac{\pi}{2}f_1^*\left(\frac{\sqrt{2}}{2}\right)+\pi\int_{-1}^{1}R_2(s)f''(s)\,ds,$$

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where

$$R_2(s) = \begin{cases} \frac{1}{2}s + \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right), & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right), & -\frac{\sqrt{2}}{2} < s \le \frac{\sqrt{2}}{2}, \\ -\frac{1}{2}s + \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right), & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

COROLLARY 3.3. *Suppose that the assumptions of Theorem 2.3 hold. Then*

$$\left| \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} - \frac{\pi}{2} f_1^*\left(\frac{\sqrt{2}}{2}\right) \right| \le \begin{cases} \frac{1}{2}\pi \, \|f''\|_\infty, \\ \left(\frac{32+3\sqrt{2}\pi}{27}\right)^{1/2} \|f''\|_2, \\ \left(\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2}\right) \|f''\|_1. \end{cases}$$

The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. Similarly to the proof of Corollary 3.2, for the first inequality we have

$$\int_{-1}^{1} |R_2(s)|\,ds = \frac{1}{2}$$

and for the second

$$\int_{-1}^{1} |R_2(s)|^2\,ds = \frac{32+3\sqrt{2}\pi}{27\pi^2}.$$

To prove the third inequality we calculate

$$\sup_{s\in\left[-1,-\sqrt{2}/2\right]} \left| \frac{1}{2}s + \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right) \right| = \frac{(4-\pi)\sqrt{2}}{8\pi},$$

$$\sup_{s\in\left[-\sqrt{2}/2,\sqrt{2}/2\right]} \left| \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right) \right| = \frac{(4+\pi)\sqrt{2}}{8\pi},$$

$$\sup_{s\in\left[\sqrt{2}/2,1\right]} \left| -\frac{1}{2}s + \frac{1}{\pi}\left(s \arcsin s + \sqrt{1-s^2}\right) \right| = \frac{(4-\pi)\sqrt{2}}{8\pi}.$$

Finally

$$\sup_{s\in[-1,1]} |R_2(s)| = \max\left\{ \frac{(4-\pi)\sqrt{2}}{8\pi}, \frac{(4+\pi)\sqrt{2}}{8\pi} \right\} = \frac{(4+\pi)\sqrt{2}}{8\pi}. \qquad \square$$

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REMARK 6. If f'' is a differentiable function on $\langle -1, 1\rangle$, by Theorem 2.4 there exists $\eta \in \langle -1, 1\rangle$ such that

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} - \frac{\pi}{2} f_1^*\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{2} f''(\eta).$$

REMARK 7. If we apply Theorem 2.1 with $n = 3$, $a = -1$, $b = 1$, $x = -\sqrt{2}/2$ and $w(t) = 1/(\pi\sqrt{1-t^2})$, $t \in [-1, 1]$, we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt = \frac{\pi}{2} f_2^*\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{2}\int_{-1}^{1} R_3(s) f'''(s)\,ds,$$

where

$$R_3(s) = \begin{cases} -\frac{3}{2\pi} s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s \\ \quad -\frac{1}{2}\left(\frac{1}{2}+s^2\right), & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ -\frac{3}{2\pi} s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -\frac{\sqrt{2}}{2} < s \le \frac{\sqrt{2}}{2}, \\ -\frac{3}{2\pi} s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s \\ \quad +\frac{1}{2}\left(\frac{1}{2}+s^2\right), & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

COROLLARY 3.4. *Suppose that the assumptions of Theorem 2.3 hold. Then*

$$\left|\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} - \frac{\pi}{2} f_2^*\left(\frac{\sqrt{2}}{2}\right)\right| \le \begin{cases} \frac{1}{36}\left(-8+19\sqrt{2}\right)\|f'''\|_\infty, \\ \frac{1}{2}\left(\frac{-4096+2505\sqrt{2}\pi}{6750}\right)^{1/2}\|f'''\|_2, \\ \frac{1}{8}(3+\pi)\|f'''\|_1. \end{cases}$$

The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. For the first and second inequalities

$$\int_{-1}^{1} |R_3(s)|\,ds = \frac{-8+19\sqrt{2}}{18\pi}, \qquad \int_{-1}^{1} |R_3(s)|^2\,ds = \frac{-4096+2505\sqrt{2}\pi}{6750\pi^2},$$

and for the third

$$\sup_{s\in\left[-1,-\sqrt{2}/2\right]}\left|-\frac{3}{2\pi} s\sqrt{1-s^2} - \frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s - \frac{1}{2}\left(\frac{1}{2}+s^2\right)\right| = \frac{1}{4} - \frac{3}{4\pi},$$

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$$\sup_{s\in\left[-\sqrt{2}/2,\sqrt{2}/2\right]}\left|-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s\right|=\frac{1}{4}+\frac{3}{4\pi},$$

$$\sup_{s\in\left[\sqrt{2}/2,1\right]}\left|-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s+\frac{1}{2}\left(\frac{1}{2}+s^2\right)\right|=\frac{1}{4}-\frac{3}{4\pi}.$$

Finally

$$\sup_{s\in[-1,1]}|R_3(s)|=\max\left\{\frac{1}{4}-\frac{3}{4\pi},\frac{1}{4}+\frac{3}{4\pi}\right\}=\frac{1}{4}+\frac{3}{4\pi}. \qquad \square$$

3.2. The case $\varrho(t)=\sqrt{1-t^2}$, $t\in[-1,1]$ In this case we have a formula of the type

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt=\sum_{i=1}^{k}A_i f(x_i)+E_k(f), \tag{3.3}$$

where the A_i are given by

$$A_i=\frac{\pi}{k+1}\sin^2\frac{i\pi}{k+1}, \qquad i=1,\ldots,k$$

and the x_i are zeros of the *Chebyshev polynomials of the second kind* defined as

$$U_k(x)=\frac{\sin[(k+1)\arccos(x)]}{\sin[\arccos(x)]}.$$

The polynomial $U_k(x)$ has exactly k distinct zeros, all of which lie in the interval $[-1,1]$ (see for instance [13]) and are given by

$$x_i=\cos\left(\frac{i\pi}{k+1}\right).$$

The error of the approximation formula (3.3) is given by

$$E_k(f)=\frac{\pi}{2^{2k+1}(2k)!}f^{(2k)}(\xi), \qquad \xi\in\langle -1,1\rangle.$$

In the case $k=2$ (3.3) reduces to

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt=\frac{\pi}{4}f_0^*\left(\frac{1}{2}\right)+\frac{\pi}{768}f^{(4)}(\xi), \qquad \xi\in\langle -1,1\rangle.$$

REMARK 8. If we apply (2.2) with $a=-1$, $b=1$, $x=-1/2$ and $w(t)=2\sqrt{1-t^2}/\pi$, $t\in[-1,1]$, we get

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt=\frac{\pi}{4}f_0^*\left(\frac{1}{2}\right)+\frac{\pi}{2}\int_{-1}^{1}Q_1(s)f'(s)\,ds,$$

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where

$$Q_1(s) = \begin{cases} -\frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & -1 \le s \le -\frac{1}{2}, \\ -\frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & -\frac{1}{2} < s \le \frac{1}{2}, \\ \frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right), & \frac{1}{2} < s \le 1. \end{cases}$$

COROLLARY 3.5. *Suppose that the assumptions of Corollary 3.1 hold. Then*

$$\left| \int_{-1}^{1} \sqrt{1-t^2} f(t)\, dt - \frac{\pi}{4} f_0^*\left(\frac{1}{2}\right) \right| \le \frac{\pi}{2} \|Q_1\|_q \|f'\|_p . \tag{3.4}$$

PROOF. This is a special case of Theorem 2.3 for $a=-1$, $b=1$, $x=-1/2$ and $w(t) = 2\sqrt{1-t^2}/\pi$, $t \in [-1, 1]$. □

COROLLARY 3.6. *Suppose that the assumptions of Corollary 3.1 hold. Then*

$$\left| \int_{-1}^{1} \sqrt{1-t^2} f(t)\, dt - \frac{\pi}{4} f_0^*\left(\frac{1}{2}\right) \right| \le \begin{cases} \frac{1}{12}\left(-8 + 9\sqrt{3} - \pi\right) \|f'\|_\infty, \\ \frac{1}{6}\left(\frac{-512 + 135\sqrt{3}\pi - 15\pi^2}{20}\right)^{1/2} \|f'\|_2, \\ \frac{1}{24}\left(3\sqrt{3} + 2\pi\right) \|f'\|_1. \end{cases}$$

The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. We apply (3.4) with $p = \infty$:

$$\begin{aligned} &\int_{-1}^{1} |Q_1(s)|\, ds \\ &\quad = \int_{-1}^{-1/2} \left| -\frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right) \right| ds \\ &\quad + \int_{-1/2}^{1/2} \left| -\frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right) \right| ds \\ &\quad + \int_{1/2}^{1} \left| \frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right) \right| ds = \frac{-16 + 18\sqrt{3} - 2\pi}{12\pi} \end{aligned}$$

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and the first inequality is obtained. To prove the second inequality we take $p = 2$:

$$\begin{aligned}\int_{-1}^{1} |Q_1(s)|^2\,ds &= \int_{-1}^{-1/2} \left|-\frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 ds \\ &\quad + \int_{-1/2}^{1/2} \left|-\frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 ds \\ &\quad + \int_{1/2}^{1} \left|\frac{1}{2} - \frac{1}{\pi}\left(s\sqrt{1-s^2} + \arcsin s\right)\right|^2 ds \\ &= \frac{-512 + 135\sqrt{3}\pi - 15\pi^2}{180\pi^2}. \end{aligned} \tag{3.5}$$

If $p = 1$, we have that the arguments of the three integrals in (3.5) have successive suprema $(1/3) - \sqrt{3}/4\pi$, $(1/6) + \sqrt{3}/4\pi$, $(1/3) - \sqrt{3}/4\pi$ so

$$\sup_{s\in[-1,1]} |Q_1(s)| = \max\left\{\frac{1}{3} - \frac{\sqrt{3}}{4\pi}, \frac{1}{6} + \frac{\sqrt{3}}{4\pi}\right\} = \frac{1}{6} + \frac{\sqrt{3}}{4\pi}$$

and the third inequality is proved. □

REMARK 9. The first and third inequalities from Corollary 3.6 have also been obtained in [7].

REMARK 10. If we apply Theorem 2.1 with $n = 2$, $a = -1$, $b = 1$, $x = -1/2$ and $w(t) = 2\sqrt{1-t^2}/\pi$, $t \in [-1, 1]$, we get

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt = \frac{\pi}{4} f_1^*\left(\frac{1}{2}\right) + \frac{\pi}{2}\int_{-1}^{1} Q_2(s) f''(s)\,ds,$$

where

$$Q_2(s) = \begin{cases} \dfrac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \dfrac{1}{\pi} s \arcsin s + \dfrac{s}{2}, & -1 \le s \le -\dfrac{1}{2}, \\ \dfrac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \dfrac{1}{\pi} s \arcsin s, & -\dfrac{1}{2} < s \le \dfrac{1}{2}, \\ \dfrac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \dfrac{1}{\pi} s \arcsin s - \dfrac{s}{2}, & \dfrac{1}{2} < s \le 1. \end{cases}$$

COROLLARY 3.7. *Suppose that the assumptions of Theorem 2.3 hold. Then*

$$\left|\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt - \frac{\pi}{4} f_1^*\left(\frac{1}{2}\right)\right| \le \begin{cases} \dfrac{1}{8}\pi \left\|f''\right\|_\infty, \\ \dfrac{\pi}{2}\left(-\dfrac{1}{144} + \dfrac{3\sqrt{3}}{80\pi} + \dfrac{2048}{4725\pi^2}\right)^{1/2} \left\|f''\right\|_2, \\ \left(\dfrac{\pi}{24} + \dfrac{3\sqrt{3}}{16}\right)\left\|f''\right\|_1. \end{cases}$$

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The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. For the first and second inequalities

$$\int_{-1}^{1} |Q_2(s)|\,ds = \frac{1}{4}, \quad \int_{-1}^{1} |Q_2(s)|^2\,ds = -\frac{1}{144} + \frac{3\sqrt{3}}{80\pi} + \frac{2048}{4725\pi^2},$$

and for the third

$$\sup_{s\in[-1,-1/2]} \left| \frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \frac{1}{\pi} s \arcsin s + \frac{s}{2} \right| = -\frac{1}{6} + \frac{3\sqrt{3}}{8\pi},$$

$$\sup_{s\in[-1/2,1/2]} \left| \frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \frac{1}{\pi} s \arcsin s \right| = \frac{1}{12} + \frac{3\sqrt{3}}{8\pi},$$

$$\sup_{s\in[1/2,1]} \left| \frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2} + \frac{1}{\pi} s \arcsin s - \frac{s}{2} \right| = -\frac{1}{6} + \frac{3\sqrt{3}}{8\pi}.$$

Finally

$$\sup_{s\in[-1,1]} |Q_2(s)| = \max\left\{ -\frac{1}{6} + \frac{3\sqrt{3}}{8\pi}, \frac{1}{12} + \frac{3\sqrt{3}}{8\pi} \right\} = \frac{1}{12} + \frac{3\sqrt{3}}{8\pi}. \qquad \square$$

REMARK 11. If f'' is a differentiable function on $\langle -1, 1\rangle$, by Theorem 2.4, there exists $\eta \in \langle -1, 1\rangle$ such that

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt - \frac{\pi}{4} f_1^*\left(\frac{1}{2}\right) = \frac{\pi}{8} f''(\eta).$$

REMARK 12. If we apply Theorem 2.1 with $n = 3$, $a = -1$, $b = 1$, $x = -1/2$ and $w(t) = 2\sqrt{1-t^2}/\pi$, $t \in [-1, 1]$, we get

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt = \frac{\pi}{4} f_2^*\left(\frac{1}{2}\right) + \frac{\pi}{4} \int_{-1}^{1} Q_3(s) f'''(s)\,ds,$$

where

$$Q_3(s) = \begin{cases} -\frac{1}{12\pi}(13s+2s^3)\sqrt{1-s^2} \\ \quad -\frac{1}{4\pi}(1+4s^2)\arcsin s - \frac{1}{8}(1+4s^2), & -1 \le s \le -\frac{1}{2}, \\ -\frac{1}{12\pi}(13s+2s^3)\sqrt{1-s^2} \\ \quad -\frac{1}{4\pi}(1+4s^2)\arcsin s, & -\frac{1}{2} < s \le \frac{1}{2}, \\ -\frac{1}{12\pi}(13s+2s^3)\sqrt{1-s^2} \\ \quad -\frac{1}{4\pi}(1+4s^2)\arcsin s + \frac{1}{8}(1+4s^2), & \frac{1}{2} < s \le 1. \end{cases}$$

COROLLARY 3.8. *Suppose that the assumptions of Theorem 2.3 hold. Then*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}\,f(t)\,dt-\frac{\pi}{4}f_2^*\left(\frac{1}{2}\right)\right|$$
$$\leq\begin{cases}\frac{1}{2880}\left(-128+297\sqrt{3}-40\pi\right)\|f'''\|_\infty,\\ \frac{\pi}{4}\left(-\frac{7}{720}+\frac{411\sqrt{3}}{5600\pi}-\frac{65536}{496125\pi^2}\right)^{1/2}\|f'''\|_2,\\ \left(\frac{\pi}{48}+\frac{9\sqrt{3}}{128}\right)\|f'''\|_1.\end{cases}$$

The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. For the first and second inequalities

$$\int_{-1}^{1}|Q_3(s)|\,ds=\frac{-128+297\sqrt{3}-40\pi}{720\pi},$$
$$\int_{-1}^{1}|Q_3(s)|^2\,ds=-\frac{7}{720}+\frac{411\sqrt{3}}{5600\pi}-\frac{65536}{496125\pi^2},$$

and for the third

$$\sup_{s\in[-1,-1/2]}\left|-\frac{1}{12\pi}\left[13s+2s^3\right]\sqrt{1-s^2}-\left[1+4s^2\right]\left(\frac{1}{4\pi}\arcsin s+\frac{1}{8}\right)\right|$$
$$=\frac{1}{6}-\frac{9\sqrt{3}}{32\pi},$$
$$\sup_{s\in[-1/2,1/2]}\left|-\frac{1}{12\pi}\left[13s+2s^3\right]\sqrt{1-s^2}-\frac{1}{4\pi}\left[1+4s^2\right]\arcsin s\right|$$
$$=\frac{1}{12}+\frac{9\sqrt{3}}{32\pi},$$
$$\sup_{s\in[1/2,1]}\left|-\frac{1}{12\pi}\left[13s+2s^3\right]\sqrt{1-s^2}+\left[1+4s^2\right]\left(-\frac{1}{4\pi}\arcsin s+\frac{1}{8}\right)\right|$$
$$=\frac{1}{6}-\frac{9\sqrt{3}}{32\pi}.$$

Finally

$$\sup_{s\in[-1,1]}|Q_3(s)|=\max\left\{\frac{1}{6}-\frac{9\sqrt{3}}{32\pi},\frac{1}{12}+\frac{9\sqrt{3}}{32\pi}\right\}=\frac{1}{12}+\frac{9\sqrt{3}}{32\pi}.$$ □

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4. Nonweighted case of a two-point formula and applications

Here we define

$$\begin{aligned}\widehat{t}_n(x) &= \frac{1}{2}\sum_{i=0}^{n-2}\Big[f^{(i+1)}(x)+(-1)^{i+1}f^{(i+1)}(a+b-x)\Big]\\&\quad\times\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)},\\ \widehat{T}_n(x,s) &= -\frac{n}{2}\left[T_n(x,s)+T_n(a+b-x,s)\right]\\ &= \begin{cases}\dfrac{1}{(b-a)}(a-s)^n, & a\le s\le x,\\ \dfrac{1}{2(b-a)}\left[(a-s)^n+(b-s)^n\right], & x<s\le a+b-x,\\ \dfrac{1}{(b-a)}(b-s)^n, & a+b-x<s\le b.\end{cases}\end{aligned}$$

We will use the Beta function and the incomplete Beta function of Euler type defined as

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad B_r(x,y)=\int_0^r t^{x-1}(1-t)^{y-1}\,dt,\ x,y>0.$$

THEOREM 4.1. *Let $f: I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$. Then for each $x\in[a,(a+b)/2]$ we have the identity*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = D(x)+\widehat{t}_n(x)+\frac{1}{n!}\int_a^b \widehat{T}_n(x,s)f^{(n)}(s)\,ds.$$

PROOF. We take $w(t)=1/(b-a)$, $t\in[a,b]$ in (2.1). □

THEOREM 4.2. *Suppose that the assumptions of Theorem 4.1 hold. Additionally assume that (p,q) is a pair of conjugate exponents and that $f^{(n)}\in L_p[a,b]$ for some $n\ge 2$. Then for each $x\in[a,(a+b)/2]$*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - D(x)-\widehat{t}_n(x)\right|\le\frac{1}{n!}\left\|\widehat{T}_n(x,\cdot)\right\|_q\left\|f^{(n)}\right\|_p. \tag{4.1}$$

The constant $(1/n!)\|\widehat{T}_n(x,\cdot)\|_q$ is sharp for $1<p\le\infty$ and the best possible for $p=1$.

PROOF. We take $w(t)=1/(b-a)$, $t\in[a,b]$ in (2.3). □

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COROLLARY 4.3. *Suppose that the assumptions of Theorem 4.2 hold. Then for each $x \in [a, (a+b)/2]$ we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D(x) - \widehat{t}_n(x) \right| \leq \begin{cases} \Omega_\infty \left\| f^{(n)} \right\|_\infty, \\ \Omega_2 \left\| f^{(n)} \right\|_2, \\ \Omega_1 \left\| f^{(n)} \right\|_1, \end{cases}$$

where

$$\Omega_\infty = \frac{1}{(n+1)!} \left[\frac{(x-a)^{n+1} \left[2 + (-1)^{n+1}\right] + (b-x)^{n+1}}{(b-a)} - \left[\frac{b-a}{2}\right]^n \left[\frac{(-1)^{n+1}+1}{2}\right] \right]$$

$$\Omega_2 = \frac{1}{n!} \left(\frac{(-1)^n (b-a)^{2n-1}}{2} \left[B_{\frac{b-x}{b-a}}(n+1, n+1) - B_{\frac{x-a}{b-a}}(n+1, n+1) \right] + \frac{3(x-a)^{2n+1} + (b-x)^{2n+1}}{2(2n+1)(b-a)^2} \right)^{1/2}$$

$$\Omega_1 = \frac{1}{n!(b-a)} \max\left\{ (x-a)^n \,,\ \frac{(a-x)^n + (b-x)^n}{2} \right\}.$$

The constants on the right-hand sides of the first and second inequalities are sharp and the right-hand side constant in the third inequality is the best possible.

PROOF. We apply (4.1) with $p = \infty$:

$$\begin{aligned} \int_a^b \left| \widehat{T}_n(x, s) \right| ds &= \int_a^x \left| \frac{(a-s)^n}{b-a} \right| ds + \int_x^{a+b-x} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right| ds \\ &\quad + \int_{a+b-x}^b \left| \frac{(b-s)^n}{b-a} \right| ds \\ &= 2\frac{(x-a)^{n+1}}{(n+1)(b-a)} + \frac{(a-x)^{n+1} + (b-x)^{n+1} - \left(\frac{b-a}{2}\right)^{n+1} \left[(-1)^{n+1}+1\right]}{(n+1)(b-a)} \\ &= \frac{(x-a)^{n+1}\left[2+(-1)^{n+1}\right] + (b-x)^{n+1}}{(n+1)(b-a)} - \left(\frac{b-a}{2}\right)^n \left[\frac{(-1)^{n+1}+1}{2(n+1)}\right] \end{aligned}$$

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and the first inequality is obtained. To prove the second inequality we take $p = 2$:

$$\begin{aligned}\int_a^b |\widehat{T}_n(x,s)|^2\,ds &= \int_a^x \left|\frac{(a-s)^n}{b-a}\right|^2 ds + \int_x^{a+b-x} \left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|^2 ds \\ &\quad + \int_{a+b-x}^b \left|\frac{(b-s)^n}{b-a}\right|^2 ds \\ &= \frac{3(x-a)^{2n+1}+(b-x)^{2n+1}}{2(2n+1)(b-a)^2} + \frac{(-1)^n(b-a)^{2n-1}}{2} \\ &\quad \times \left[B_{(b-x)/(b-a)}(n+1,n+1) - B_{(x-a)/(b-a)}(n+1,n+1)\right].\end{aligned}$$

If $p = 1$,

$$\begin{aligned}&\sup_{s\in[a,b]} \left|\widehat{T}_n(x,s)\right| \\ &= \max\left\{\sup_{s\in[a,x]}\left|\frac{(a-s)^n}{b-a}\right|, \sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|, \sup_{s\in[a+b-x,b]}\left|\frac{(b-s)^n}{b-a}\right|\right\}.\end{aligned}$$

By an elementary calculation we obtain

$$\sup_{s\in[a,x]}\left|\frac{(a-s)^n}{b-a}\right| = \frac{(x-a)^n}{(b-a)}, \qquad \sup_{s\in[a+b-x,b]}\left|\frac{(b-s)^n}{b-a}\right| = \frac{(x-a)^n}{(b-a)}.$$

The function $y:[a,b]\to\mathbb{R}$, $y(x) = (a-x)^n + (b-x)^n$, is decreasing on $\langle a, (a+b)/2\rangle$ and increasing on $\langle (a+b)/2, b\rangle$ if n is even, and decreasing on $\langle a, b\rangle$ if n is odd. Thus

$$\sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right| = \frac{(a-x)^n+(b-x)^n}{2(b-a)}. \tag{4.2}$$

Since $x \in [a, (a+b)/2]$

$$\sup_{s\in[a,b]}\left|\widehat{T}_n(x,s)\right| = \max\left\{\frac{(x-a)^n}{(b-a)}, \frac{(a-x)^n+(b-x)^n}{2(b-a)}\right\}$$

and the third inequality is proved. □

COROLLARY 4.4. *Let $f:[a,b]\to\mathbb{R}$ be a L-Lipschitzian function on $[a,b]$. Then for each $x\in[a,(a+b)/2]$*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - D(x)\right| \le \left(\frac{3(x-a)^2+(b-x)^2}{2(b-a)} - \frac{b-a}{4}\right)L. \tag{4.3}$$

PROOF. We apply the first inequality from Corollary 4.3 with $n = 1$. □

REMARK 13. The inequality (4.3) has been proved and generalized for α-L-Lipschitzian functions by Guessab and Schmeisser in [5]. They also proved that this inequality is sharp for each admissible x.

COROLLARY 4.5. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is an L-Lipschitzian function on $[a, b]$. Then for each $x \in [a, (a+b)/2]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - D(x) - \left[f'(x) - f'(a+b-x) \right] \frac{(b-x)^2 - (a-x)^2}{4(b-a)} \right| \leq \frac{(x-a)^3 + (b-x)^3}{6(b-a)} L.$$

PROOF. We apply the first inequality from Corollary 4.3 with $n = 2$. □

COROLLARY 4.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function of bounded variation on $[a, b]$. Then for each $x \in [a, (a+b)/2]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - D(x) \right| \leq \left(\frac{1}{4} + \frac{|3a+b-4x|}{4(b-a)} \right) V_a^b(f). \tag{4.4}$$

More precisely, if $x \in [a, (3a+b)/4]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - D(x) \right| \leq \frac{a+b-2x}{2(b-a)} V_a^b(f)$$

and if $x \in [(3a+b)/4, (a+b)/2]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - D(x) \right| \leq \frac{x-a}{b-a} V_a^b(f).$$

PROOF. We apply the third inequality from Corollary 4.3 with $n = 1$ to get

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - D(x) \right| \leq \frac{1}{(b-a)} \max\left\{ x-a, \frac{a+b}{2} - x \right\} V_a^b(f).$$

Using the formula $\max\{A, B\} = (1/2)\,(A + B + |A - B|)$ the proof for the first inequality follows. Since

$$\max\left\{ x-a, \frac{a+b}{2} - x \right\} = \begin{cases} \dfrac{a+b}{2} - x, & \text{if } x \in \left[a, \dfrac{3a+b}{4} \right], \\ x-a, & \text{if } x \in \left[\dfrac{3a+b}{4}, \dfrac{a+b}{2} \right], \end{cases}$$

the proofs of the second and third inequalities follow. □

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REMARK 14. The inequalities (4.3) and (4.4) and their generalizations based on extended Euler formulae via Bernoulli polynomials have been proved by Pečarić, Perić and Vukelić on the interval $[0, 1]$ in [11].

COROLLARY 4.7. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then for each $x \in [a, (a+b)/2]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D(x) - \left[f'(x) - f'(a+b-x) \right] \frac{(b-x)^2 - (a-x)^2}{4(b-a)} \right| \\ \leq \frac{(x-a)^2 + (b-x)^2}{4(b-a)} V_a^b(f').$$

PROOF. We apply the third inequality from Corollary 4.3 with $n = 2$ to get

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D(x) - \left[f'(x) - f'(a+b-x) \right] \frac{(b-x)^2 - (a-x)^2}{4(b-a)} \right| \\ \leq \frac{1}{2(b-a)} \max \left\{ (x-a)^2 , \frac{(a-x)^2 + (b-x)^2}{2} \right\} V_a^b(f') \\ = \frac{(x-a)^2 + (b-x)^2}{4(b-a)} V_a^b(f')$$

and the proof follows. □

COROLLARY 4.8. *Suppose that the assumptions of Theorem 4.2 hold. Then for each $x \in [a, (a+b)/2]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - D(x) - \widehat{t_n}(x) \right| \\ \leq \frac{1}{n!} \left(\frac{2(x-a)^{nq+1}}{(nq+1)(b-a)^q} + \frac{((a-x)^n + (b-x)^n)^q}{2^q (b-a)^q} (a+b-2x) \right)^{1/q} \left\| f^{(n)} \right\|_p .$$

PROOF. We have

$$\int_a^b \left| \widehat{T}_n(x, s) \right|^q ds = \int_a^x \left| \frac{(a-s)^n}{b-a} \right|^q ds + \int_x^{a+b-x} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right|^q ds \\ + \int_{a+b-x}^b \left| \frac{(b-s)^n}{b-a} \right|^q ds.$$

Since

$$\int_a^x \left| \frac{(a-s)^n}{b-a} \right|^q ds = \int_{a+b-x}^b \left| \frac{(b-s)^n}{b-a} \right|^q ds = \frac{(x-a)^{nq+1}}{(nq+1)(b-a)^q},$$

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and, by applying (4.2),

$$\int_x^{a+b-x} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right|^q ds \le \int_x^{a+b-x} \left(\frac{(a-x)^n + (b-x)^n}{2(b-a)} \right)^q ds$$
$$= \frac{((a-x)^n + (b-x)^n)^q}{2^q (b-a)^q} (a+b-2x)$$

we obtain

$$\int_a^b \left| \widehat{T}_n(x,s) \right|^q ds \le \frac{2(x-a)^{nq+1}}{(nq+1)(b-a)^q} + \frac{\left((a-x)^n + (b-x)^n\right)^q}{2^q (b-a)^q}(a+b-2x). \quad \square$$

REMARK 15. If we set $x = a$, $(2a+b)/3$, $(3a+b)/4$, $(a+b)/2$ in Theorem 4.2 and Corollaries 4.3–4.8, we get the generalized trapezoid, two-point Newton–Cotes, two-point Maclaurin and midpoint inequalities.

REMARK 16. For some related results see [3, 8, 12].

5. Bullen-type inequalities

In this section we use identity (5.1) to prove a generalization of Bullen-type inequalities (1.6) for $(2n)$-convex functions $(n \in \mathbb{N})$. Also we study for $x \in [a, (a+b)/2]$ the general weighted quadrature formula

$$\int_a^b w(t) f(t)\, dt = \frac{1}{4}(f(a) + f(x) + f(a+b-x) + f(b)) + G(f, w; x),$$

where $G(f, w; x)$ is the remainder.

Again, let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a, b]$ for some $n \ge 2$. We introduce the following notation for each $x \in [a, (a+b)/2]$:

$$\widetilde{D}(x) = \frac{D(x) + D(a)}{2} = \frac{f(a) + f(x) + f(a+b-x) + f(b)}{4},$$
$$\widetilde{T}_{w,n}(x,s) = \frac{\widehat{T}_{w,n}(a,s) + \widehat{T}_{w,n}(x,s)}{2} \quad \text{and} \quad \widetilde{t}_{w,n}(x) = \frac{t_{w,n}(x) + t_{w,n}(a)}{2},$$

where $D(x)$, $\widehat{T}_{w,n}(x,s)$ and $t_{w,n}(x)$ are as in Section 2.

THEOREM 5.1. *Suppose that the assumptions of Theorem 2.1 hold. Then for each $x \in [a, (a+b)/2]$ the following identity holds:*

$$\int_a^b w(t) f(t)\, dt = \widetilde{D}(x) + \widetilde{t}_{w,n}(x) + \frac{1}{(n-1)!} \int_a^b \widetilde{T}_{w,n}(x,s) f^{(n)}(s)\, ds. \tag{5.1}$$

PROOF. We put $x \equiv x$, $x \equiv a+b-x$, $x \equiv a$ and $x \equiv b$ in (1.3) to obtain four new formulae. After adding these four formulae and multiplying by $1/4$, we obtain (5.1). $\square$

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REMARK 17. If in Theorem 5.1 we choose $x = (2a + b)/3$, $(a + b)/2$ we obtain closed Newton–Cotes formulae with the same nodes as Simpson's 3/8 rule and Simpson's rule respectively.

THEOREM 5.2. *Suppose that the assumptions of Theorem 2.3 hold. Then for each $x \in [a, (a + b)/2]$ the following inequality holds:*

$$\left| \int_a^b w(t) f(t)\, dt - \widetilde{D}(x) - \widetilde{t}_{w,n}(x) \right| \leq \frac{1}{(n-1)!} \left\| \widetilde{T}_{w,n}(x, \cdot) \right\|_q \left\| f^{(n)} \right\|_p. \tag{5.2}$$

The constant $(1/(n-1)!) \|\widehat{T}_{w,n}(x, \cdot)\|_q$ is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

PROOF. The proof is similar to the proof of Theorem 2.3. □

THEOREM 5.3. *Suppose that the assumptions of Theorem 2.3 hold. Additionally assume that $f^{(2n)}$ is a differentiable function on $\langle a, b \rangle$. Then for every $x \in [a, (a + b)/2]$ there exists $\eta \in \langle a, b \rangle$ such that*

$$\int_a^b w(t) f(t)\, dt - \widetilde{D}(x) - \widetilde{t}_{w,2n}(x) = \frac{f^{(2n)}(\eta)}{(2n-1)!} \int_a^b \widetilde{T}_{w,2n}(x, s)\, ds.$$

PROOF. Similarly to Theorem 2.4, we have $\widetilde{T}_{w,2n}(x, s) \geq 0$ for $s \in [a, b]$. Thus we can apply the integral mean value theorem to $\int_a^b \widetilde{T}_{w,2n}(x, s) f^{(2n)}(s)\, ds$. □

THEOREM 5.4 (Weighted generalization of Bullen-type inequality). *Suppose that the assumptions of Theorem 5.1 hold for $2n$, $n \geq 1$. If f is $(2n)$-convex, then for each $x \in [a, (a + b)/2]$ the following inequality holds:*

$$\begin{aligned} &\int_a^b w(t) f(t)\, dt - \frac{f(x) + f(a + b - x)}{2} - t_{w,2n}(x) \\ &\quad \geq \frac{f(a) + f(b)}{2} - \int_a^b w(t) f(t)\, dt + t_{w,2n}(a). \end{aligned} \tag{5.3}$$

If f is $(2n)$-concave, then the inequality (5.3) is reversed.

PROOF. From (5.1) we have that

$$\begin{aligned} &2 \int_a^b w(t) f(t)\, d - \frac{f(a) + f(x) + f(a + b - x) + f(b)}{2} - t_{w,2n}(x) - t_{w,2n}(a) \\ &\quad = \frac{1}{(2n-1)!} \int_a^b \widetilde{T}_{w,2n}(x, s) f^{(2n)}(s)\, ds. \end{aligned}$$

Similarly to Theorem 2.5, we have $\widetilde{T}_{2n}(x, s) \geq 0$ and $\int_a^b \widetilde{T}_{2n}(x, s) f^{(2n)}(s)\, ds \geq 0$, from which (5.3) follows immediately. □

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In the special case $w(t) = 1/(b-a)$, $t \in [a, b]$, we define

$$\widetilde{T}_n(x,s) = -\frac{n}{4}\left[T_n(a,s) + T_n(x,s) + T_n(a+b-x,s) + T_n(b,s)\right]$$
$$= \begin{cases} \dfrac{1}{4(b-a)}\left[3(a-s)^n + (b-s)^n\right], & a \le s \le x, \\ \dfrac{1}{2(b-a)}\left[(a-s)^n + (b-s)^n\right], & x < s \le a+b-x, \\ \dfrac{1}{4(b-a)}\left[(a-s)^n + 3(b-s)^n\right], & a+b-x < s \le b, \end{cases}$$

and

$$\widetilde{t}_n(x) = \frac{\widehat{t}_n(x) + \widehat{t}_n(a)}{2}. \qquad \square$$

THEOREM 5.5. *Suppose that the assumptions of Theorem 4.1 hold. Then for each* $x \in [a, (a+b)/2]$ *we have the identity*

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \widetilde{D}(x) + \widetilde{t}_n(x) + \frac{1}{n!}\int_a^b \widetilde{T}_n(x,s) f^{(n)}(s)\,ds.$$

PROOF. We take $w(t) = 1/(b-a)$, $t \in [a, b]$ in (5.1). $\square$

THEOREM 5.6. *Suppose that the assumptions of Theorem 4.2 hold. Then for each* $x \in [a, (a+b)/2]$ *we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x) - \widetilde{t}_n(x)\right| \le \frac{1}{n!}\left\|\widetilde{T}_n(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{5.4}$$

The constant $(1/n!)\|\widetilde{T}_n(x,\cdot)\|_q$ *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

PROOF. We take $w(t) = 1/(b-a)$, $t \in [a, b]$ in (5.2). $\square$

COROLLARY 5.7. *Suppose that the assumptions of Theorem 5.6 hold. Then for each* $x \in [a, (a\sqrt[n]{3} + b)/(1 + \sqrt[n]{3})]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x) - \widetilde{t}_n(x)\right|$$
$$\le \frac{1}{(n+1)!}\left(\frac{(-1)^n\,(x-a)^{n+1} + (b-x)^{n+1} + (b-a)^{n+1}}{2(b-a)}\right.$$
$$\left. - (b-a)^n\left[\frac{(-1)^{n+1}+1}{2^{n+1}}\right]\right)\left\|f^{(n)}\right\|_\infty$$

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and for $x \in [(a\sqrt[n]{3}+b)/(1+\sqrt[n]{3}),\ (a+b)/2]$

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\widetilde{D}(x)-\widetilde{t}_n(x)\right|\\ &\quad\le\Bigg(\frac{(x-a)^{n+1}\left[3+2(-1)^{n+1}\right]+(b-x)^{n+1}\left[2+(-1)^{n+1}\right]+(b-a)^{n+1}}{2(b-a)}\\ &\qquad-(b-a)^n\left[3\left(\frac{1}{1+\sqrt[n]{3}}\right)^n+\frac{1}{2^n}\right]\left[\frac{(-1)^{n+1}+1}{2}\right]\Bigg)\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}.\end{aligned}$$

Also, for each $x \in [a,\ (a+b)/2]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\widetilde{D}(x)-\widetilde{t}_n(x)\right|\le\begin{cases}\Phi_2\left\|f^{(n)}\right\|_2,\\ \Phi_1\left\|f^{(n)}\right\|_1,\end{cases}$$

where

$$\begin{aligned}\Phi_2=\frac{1}{n!}\Bigg(&\frac{5(x-a)^{2n+1}+3(b-x)^{2n+1}+(b-a)^{2n+1}}{8(2n+1)(b-a)^2}+\frac{(-1)^n(b-a)^{2n-1}}{4}\\ &\times\left[2B_{(b-x)/(b-a)}(n+1,n+1)+B_{(x-a)/(b-a)}(n+1,n+1)\right]\Bigg)^{1/2},\end{aligned}$$

$$\Phi_1=\frac{1}{n!(b-a)}\max\left\{\frac{(b-a)^n}{4},\ \frac{(a-x)^n+(b-x)^n}{2},\ \frac{\left|3(x-a)^n+(x-b)^n\right|}{4}\right\}.$$

The constants on the right-hand sides of the first, second and third inequalities are sharp and the right-hand side constant in the last inequality is the best possible.

PROOF. We apply (5.4) with $p=\infty$:

$$\begin{aligned}\int_a^b\left|\widetilde{T}_n(x,s)\right|ds=&\int_a^x\left|\frac{3(a-s)^n+(b-s)^n}{4(b-a)}\right|ds\\ &+\int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|ds\\ &+\int_{a+b-x}^b\left|\frac{(a-s)^n+3(b-s)^n}{4(b-a)}\right|ds.\end{aligned}$$

The second integral is

$$\begin{aligned}&\int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|ds\\ &\quad=\frac{(a-x)^{n+1}+(b-x)^{n+1}-((b-a)/2)^{n+1}\left[(-1)^{n+1}+1\right]}{(n+1)(b-a)}.\end{aligned}$$

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Now, we suppose n is even. The first and the third integrals are

$$\frac{3(x-a)^{n+1}-(b-x)^{n+1}+(b-a)^{n+1}}{4(n+1)(b-a)}.$$

Now, we suppose n is odd. There are two possible cases.

(1) If $x \in [a, (a\sqrt[n]{3}+b)/(1+\sqrt[n]{3})]$, the first and third integrals are

$$\frac{-3(x-a)^{n+1}-(b-x)^{n+1}+(b-a)^{n+1}}{4(n+1)(b-a)}.$$

(2) If $x \in [(a\sqrt[n]{3}+b)/(1+\sqrt[n]{3}), (a+b)/2]$

$$\begin{aligned}
\int_a^x \left|\frac{3(a-s)^n+(b-s)^n}{4(b-a)}\right| ds
&= \int_a^{(a\sqrt[n]{3}+b)/(1+\sqrt[n]{3})} \frac{-3(s-a)^n+(b-s)^n}{4(b-a)}\, ds \\
&\quad + \int_{(a\sqrt[n]{3}+b)/(1+\sqrt[n]{3})}^{x} \frac{3(s-a)^n-(b-s)^n}{4(b-a)}\, ds \\
&= \frac{3(x-a)^{n+1}+(b-x)^{n+1}+(b-a)^{n+1}-6(1+\sqrt[n]{3})\left(\frac{b-a}{1+\sqrt[n]{3}}\right)^{n+1}}{4(n+1)(b-a)},
\end{aligned}$$

and the transformation $s \to t = a+b-s$ shows that

$$\int_{a+b-x}^{b} \left|\frac{(a-s)^n+3(b-s)^n}{4(b-a)}\right| ds$$

has the same value.

Now, in case (1), $\int_a^b \left|\widetilde{T}_n(x,s)\right| ds$ has value

$$\begin{aligned}
&\frac{3(-1)^n(x-a)^{n+1}-(b-x)^{n+1}+(b-a)^{n+1}}{2(n+1)(b-a)} \\
&+ \frac{(a-x)^{n+1}+(b-x)^{n+1}-((b-a)/2)^{n+1}\left[(-1)^{n+1}+1\right]}{(n+1)(b-a)} \\
&= \frac{(-1)^n\,(x-a)^{n+1}+(b-x)^{n+1}+(b-a)^{n+1}}{2(n+1)(b-a)} - (b-a)^n\left[\frac{(-1)^{n+1}+1}{2^{n+1}(n+1)}\right]
\end{aligned}$$

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while, in case (2), its value is

$$\frac{3(x-a)^{n+1}+(-1)^{n+1}(b-x)^{n+1}+(b-a)^{n+1}\left[1-3\left(\frac{1}{1+\sqrt[n]{3}}\right)^{n}\left((-1)^{n+1}+1\right)\right]}{2(n+1)(b-a)}$$
$$+\ \frac{(a-x)^{n+1}+(b-x)^{n+1}-((b-a)/2)^{n+1}\left[(-1)^{n+1}+1\right]}{(n+1)(b-a)}$$
$$=\frac{(x-a)^{n+1}\left[3+2(-1)^{n+1}\right]+(b-x)^{n+1}\left[2+(-1)^{n+1}\right]+(b-a)^{n+1}}{2(n+1)(b-a)}$$
$$-\ (b-a)^{n}\left[3\left(\frac{1}{1+\sqrt[n]{3}}\right)^{n}+\frac{1}{2^{n}}\right]\left[\frac{(-1)^{n+1}+1}{2\,(n+1)}\right].$$

Therefore, the first and the second inequalities are obtained. To prove the third inequality we take $p=2$:

$$\int_a^b \left|\widetilde{T}_n(x,s)\right|^2 ds$$
$$=\int_a^x \left|\frac{3(a-s)^n+(b-s)^n}{4(b-a)}\right|^2 ds+\int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|^2 ds$$
$$+\int_{a+b-x}^{b}\left|\frac{(a-s)^n+3(b-s)^n}{4(b-a)}\right|^2 ds$$
$$=\frac{5(x-a)^{2n+1}+3(b-x)^{2n+1}+(b-a)^{2n+1}}{8(2n+1)(b-a)^2}$$
$$+\ \frac{(-1)^n(b-a)^{2n-1}}{4}$$
$$\times\ \left[2B_{(b-x)/(b-a)}(n+1,n+1)+B_{(x-a)/(b-a)}(n+1,n+1)\right].$$

If $p=1$,

$$\sup_{s\in[a,b]}\left|\widetilde{T}_n(x,s)\right|$$
$$=\max\left\{\sup_{s\in[a,x]}\left|\frac{3(a-s)^n+(b-s)^n}{4(b-a)}\right|,\ \sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|,\right.$$
$$\left.\sup_{s\in[a+b-x,b]}\left|\frac{(a-s)^n+3(b-s)^n}{4(b-a)}\right|\right\}.$$

Carrying out the same analysis as in Corollary 4.3 we obtain that the first and last suprema take the common values

$$\max\left\{\frac{(b-a)^n}{4(b-a)},\ \frac{3(x-a)^n+(x-b)^n}{4(b-a)}\right\},$$

$$\max\left\{\frac{(b-a)^n}{4(b-a)}, \frac{\left|3(x-a)^n+(x-b)^n\right|}{4(b-a)}\right\}$$

according as n is even or odd. Also

$$\sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right| = \frac{(a-x)^n+(b-x)^n}{2(b-a)}$$

for each n. Since $x \in [a, (a+b)/2]$ we have

$$\sup_{s\in[a,b]}\left|\widetilde{T}_n(x,s)\right| = \max\left\{\frac{(b-a)^{n-1}}{4}, \frac{(a-x)^n+(b-x)^n}{2(b-a)}, \frac{\left|3(x-a)^n+(x-b)^n\right|}{4(b-a)}\right\}$$

and the last inequality is proved. □

COROLLARY 5.8. *Let $f : [a, b] \to \mathbb{R}$ be an L-Lipschitzian function on $[a, b]$. Then for each $x \in [a, (3a+b)/4]$*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \frac{-(x-a)^2+(b-x)^2}{4(b-a)}L$$

and for each $x \in [(3a+b)/4, (a+b)/2]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \left(\frac{5(x-a)^2+3(b-x)^2}{4(b-a)} - \frac{3(b-a)}{8}\right)L.$$

PROOF. We apply the first and second inequality from Corollary 5.7 with $n = 1$. □

COROLLARY 5.9. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is an L-Lipschitzian function on $[a, b]$. Then for each $x \in [a, (a+b)/2]$*

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x) - \left[f'(x)-f'(a+b-x)\right]\frac{(b-x)^2-(a-x)^2}{8(b-a)}\\ - \left[f'(a)-f'(b)\right]\frac{(b-a)}{8}\Bigg|\\ \le \frac{(x-a)^3+(b-x)^3+(b-a)^3}{12(b-a)}L.\end{aligned}$$

PROOF. We apply the first and second inequality from Corollary 5.7 with $n = 2$. □

COROLLARY 5.10. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function of bounded variation on $[a, b]$. Then for each $x \in [a, (a+b)/2]$*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \max\left\{\frac{1}{4}, \frac{a+b-2x}{2(b-a)}, \frac{|4x-3a-b|}{4(b-a)}\right\}V_a^b(f).$$

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More precisely, if $x \in [a, (3a+b)/4]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \frac{a+b-2x}{2(b-a)} V_a^b(f)$$

and if $x \in [(3a+b)/4, (a+b)/2]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \frac{1}{4} V_a^b(f).$$

PROOF. We apply the last inequality from Corollary 5.7 with $n = 1$ to get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x)\right| \le \max\left\{\frac{1}{4}, \frac{a+b-2x}{2(b-a)}, \frac{|4x-3a-b|}{4(b-a)}\right\} V_a^b(f).$$

Now, carrying out the same analysis as in Corollary 4.6 we obtain the second and the third inequality.

COROLLARY 5.11. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* f' *is a continuous function of bounded variation on* $[a, b]$. *Then for each* $x \in [a, (a+b)/2]$

$$\begin{aligned}&\Bigg|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x) - \left[f'(x) - f'(a+b-x)\right]\frac{(b-x)^2-(a-x)^2}{8(b-a)}\\ &\quad - \left[f'(a) - f'(b)\right]\frac{(b-a)}{8}\Bigg|\\ &\quad \le \frac{(a-x)^2+(b-x)^2}{4(b-a)} V_a^b(f').\end{aligned}$$

PROOF. We apply the last inequality from Corollary 5.7 with $n = 2$ to get

$$\begin{aligned}&\Bigg|\frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{D}(x) - \left[f'(x) - f'(a+b-x)\right]\frac{(b-x)^2-(a-x)^2}{8(b-a)}\\ &\quad - \left[f'(a) - f'(b)\right]\frac{(b-a)}{8}\Bigg|\\ &\quad \le \frac{1}{2(b-a)}\max\Bigg\{\frac{(b-a)^2}{4}, \frac{(a-x)^2+(b-x)^2}{2},\\ &\qquad \frac{3(x-a)^2+(x-b)^2}{4}\Bigg\} V_a^b(f')\\ &\quad = \frac{(a-x)^2+(b-x)^2}{4(b-a)} V_a^b(f')\end{aligned}$$

and the proof follows. □

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COROLLARY 5.12. *Suppose that the assumptions of Theorem 4.2 hold. Then for each* $x \in [a, (a+b)/2]$ *we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \widetilde{D}(x) - \widetilde{t}_n(x) \right| \le \frac{(b-a)^{n-1}}{2 \cdot n!} \left(2\left(\frac{3}{2}\right)^q (x-a) + (a+b-2x) \right)^{1/q} \|f^{(n)}\|_p.$$

PROOF. We have

$$\int_a^b |\widetilde{T}_n(x,s)|^q\,ds = \int_a^x \left| \frac{3(a-s)^n + (b-s)^n}{4(b-a)} \right|^q ds + \int_x^{a+b-x} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right|^q ds + \int_{a+b-x}^b \left| \frac{(a-s)^n + 3(b-s)^n}{4(b-a)} \right|^q ds.$$

It is easy to check that the function $y : [a, b] \to \mathbb{R}$, $y(x) = (x-a)^n + (b-x)^n$ attains its maximal values on the boundary, so $(x-a)^n + (b-x)^n \le (b-a)^n$. Using this fact we obtain

$$|3(a-s)^n + (b-s)^n| \le 3|(s-a)^n + (b-s)^n| \le 3(b-a)^n$$

and thus

$$\int_a^x \left| \frac{3(a-s)^n + (b-s)^n}{4(b-a)} \right|^q ds \le \left(\frac{3}{4}(b-a)^{n-1} \right)^q (x-a).$$

Similarly, we have

$$\int_{a+b-x}^b \left| \frac{(a-s)^n + 3(b-s)^n}{4(b-a)} \right|^q ds \le \left(\frac{3}{4}(b-a)^{n-1} \right)^q (x-a)$$

and

$$\int_x^{a+b-x} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right|^q ds \le \left(\frac{1}{2}(b-a)^{n-1} \right)^q (a+b-2x).$$

Now,

$$\int_a^b |\widetilde{T}_n(x,s)|^q\,ds \le \left(\frac{1}{2}(b-a)^{n-1} \right)^q \left(2\left(\frac{3}{2}\right)^q (x-a) + (a+b-2x) \right)$$

and the proof follows. □

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THEOREM 5.13 (Nonweighted generalization of Bullen-type inequalities). *Suppose that the assumptions of Theorem 5.5 hold for* $2n$, $n \geq 1$. *If* f *is* $(2n)$*-convex, then for each* $x \in [a, (a+b)/2]$ *we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - \widehat{t_{2n}}(x) \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt + \widehat{t_{2n}}(a). \tag{5.5}$$

If f *is* $(2n)$*-concave, then inequality* (5.5) *is reversed.*

PROOF. We take $w(t) = 1/(b-a)$, $t \in [a, b]$ in (5.3). □

REMARK 18. Generalizations of Bullen-type inequalities (1.6) for $(2n)$-convex functions ($n \in \mathbb{N}$) and $x \in [a, (a+b)/2 - (b-a)/4\sqrt{6}] \cup \{(a+b)/2\}$ (of the same type as in Theorem 5.13) were first proved by Klaričić and Pečarić in [6].

COROLLARY 5.14. *Suppose that the assumptions of Theorem 5.13 hold. If* f *is* 2*-convex, then for each* $x \in [a, (a+b)/2]$ *the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - r(x) \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{5.6}$$

where

$$r(x) = \big(f'(x) - f'(a+b-x)\big)\,\frac{a+b-2x}{4} + \big(f'(a) - f'(b)\big)\,\frac{b-a}{4}.$$

If f *is* 2*-concave, then inequality* (5.6) *is reversed.*

PROOF. This is a special case of Theorem 5.13 for $n = 1$. □

COROLLARY 5.15. *Suppose that the assumptions of Theorem 5.13 hold. If* f *is* 4*-convex, then for each* $x \in [a, (a+b)/2]$ *we have the inequality*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - r(x) \geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{5.7}$$

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where

$$
\begin{aligned}
r(x) &= \big(f'(x) - f'(a+b-x)\big)\,\frac{a+b-2x}{4} + \big(f'(a) - f'(b)\big)\,\frac{b-a}{4} \\
&\quad + \big(f''(x) + f''(a+b-x)\big)\,\frac{(a-x)^2 + (a-x)(b-x) + (b-x)^2}{12} \\
&\quad + \big(f''(a) + f''(b)\big)\,\frac{(b-a)^2}{12} + \big(f'''(a) - f'''(b)\big)\,\frac{(b-a)^3}{48} \\
&\quad + \big(f'''(x) - f'''(a+b-x)\big)(a+b-2x)\,\frac{(a-x)^2 + (b-x)^2}{48}.
\end{aligned}
$$

If f is 4*-concave, then inequality* (5.7) *is reversed.*

PROOF. This is a special case of Theorem 5.13 for $n = 2$. □

REMARK 19. If we apply Theorem 5.4 with $n = 1$, $a = -1$, $b = 1$, $x = -\sqrt{2}/2$ and $w(t) = 1/(\pi\sqrt{1-t^2})$, $t \in [-1, 1]$, inequality (5.3) reduces to

$$
\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt - \frac{\pi}{2} f_0^*\left(\frac{\sqrt{2}}{2}\right) - r(x) \ge \frac{\pi}{2} f_0^*(1) - \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt,
$$

where

$$
r(x) = \frac{\pi\sqrt{2}}{4}\left[f'\left(-\frac{\sqrt{2}}{2}\right) - f'\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{2}\left[f'(-1) - f'(1)\right].
$$

REMARK 20. If we apply Theorem 5.4 with $n = 1$, $a = -1$, $b = 1$, $x = -1/2$ and $w(t) = 2\sqrt{1-t^2}/\pi$, $t \in [-1, 1]$, inequality (5.3) reduces to

$$
\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt - \frac{\pi}{4} f_0^*\left(\frac{1}{2}\right) - r(x) \ge \frac{\pi}{4} f_0^*(1) - \int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt,
$$

where

$$
r(x) = \frac{\pi}{8}\left[f'\left(-\frac{1}{2}\right) - f'\left(\frac{1}{2}\right)\right] + \frac{\pi}{4}\left[f'(-1) - f'(1)\right].
$$

REMARK 21. If we apply Theorem 5.4 with $n = 2$, $a = -1$, $b = 1$, $x = -\sqrt{2}/2$ and $w(t) = 1/\pi\sqrt{1-t^2}$, $t \in [-1, 1]$, inequality (5.3) reduces to

$$
\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt - \frac{\pi}{2} f_0^*\left(\frac{\sqrt{2}}{2}\right) - r(x) \ge \frac{\pi}{2} f_0^*(1) - \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt,
$$

where

$$r(x) = \frac{\pi\sqrt{2}}{4}\left[f'\left(-\frac{\sqrt{2}}{2}\right) - f'\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{2}\left[f'(-1) - f'(1)\right]$$
$$+ \frac{\pi}{4}\left[f''\left(-\frac{\sqrt{2}}{2}\right) + f''\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{3\pi}{8}\left[f''(-1) + f''(1)\right]$$
$$+ \frac{\pi\sqrt{2}}{12}\left[f'''\left(-\frac{\sqrt{2}}{2}\right) - f'''\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{5\pi}{24}\left[f'''(-1) - f'''(1)\right].$$

REMARK 22. If we apply Theorem 5.4 with $n = 2$, $a = -1$, $b = 1$, $x = -1/2$ and $w(t) = 2\sqrt{1-t^2}/\pi$, $t \in [-1, 1]$, inequality (5.3) reduces to

$$\int_{-1}^{1} \sqrt{1-t^2}\, f(t)\, dt - \frac{\pi}{4} f_0^*\left(\frac{1}{2}\right) - r(x) \geq \frac{\pi}{4} f_0^*(1) - \int_{-1}^{1} \sqrt{1-t^2}\, f(t)\, dt,$$

where

$$r(x) = \frac{\pi}{8}\left[f'\left(-\frac{1}{2}\right) - f'\left(\frac{1}{2}\right)\right] + \frac{\pi}{4}\left[f'(-1) - f'(1)\right]$$
$$+ \frac{\pi}{16}\left[f''\left(-\frac{1}{2}\right) + f''\left(\frac{1}{2}\right)\right] + \frac{5\pi}{32}\left[f''(-1) + f''(1)\right]$$
$$+ \frac{\pi}{48}\left[f'''\left(-\frac{1}{2}\right) - f'''\left(\frac{1}{2}\right)\right] + \frac{7\pi}{96}\left[f'''(-1) - f'''(1)\right].$$

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