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Non-Right-Orderable 3-Manifold Groups

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Abstract. We exhibit infinitely many hyperbolic 3-manifold groups that are not right-orderable.

1 Introduction

Orderability of groups has been studied for some time, and recent attention has been paid to orderability of fundamental groups of 3-manifolds, notably in the paper [BRW02] of Boyer, Rolfsen and Wiest. In that paper, the authors determine exactly which nonhyperbolic, compact, $\mathbb{P}^2$-irreducible 3-manifolds have right orderable fundamental groups. As mentioned in [BRW02], some hyperbolic 3-manifolds have right orderable fundamental groups while others do not. The first examples of hyperbolic 3-manifolds with non-right-orderable fundamental groups appeared in [RSS03, DPT05, Fen07] and an early preprint version of this paper. (A group is right orderable if and only if it is left orderable.)

In both [RSS03] and [Fen07], the groups considered have presentations of the form

$$G = \langle t, a, b | a^t = a^{m-1}b^{-1}a^{-1}, b^t = a^{-1}, t^p[a,b]^q = 1\rangle,$$

where m, p, q are integers and p, q are relatively prime. In [RSS03], the case that $m \leq -3$ and $\frac{p}{q} \in [1, \infty)$ is analyzed. In [Fen07], the case that $m \leq -4$ and $|p - 2q| = 1$ is examined.

In this paper, we investigate the more general case where $G = G(\phi, p, q)$ has presentation

$$G = \langle t, a, b | a^t = a^{\phi_*}, b^t = b^{\phi_*}, t^p[a,b]^q = 1\rangle,$$

where ϕ_* is any automorphism of the rank two free group $F = F(a, b)$ such that

- $[a, b]^{\phi_*} = [a, b]$, and
- the automorphism $\phi_\sharp$ of the abelianization $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$ induced by ϕ_* lies in $SL_2(\mathbb{Z})$, with $|\operatorname{Trace}(\phi_\sharp)| > 2$.

In other words, ϕ_* is induced by an orientation preserving pseudo-Anosov homeomorphism ϕ of a once punctured torus (see [Ni17, FH82, CJR84, Ind06]). We show that if either $\operatorname{Trace}(\phi_\sharp) < -2$ and $\frac{p}{q} \in [1, \infty]$ or $\operatorname{Trace}(\phi_\sharp) > 2$ and $(p, q) = (1, 0)$, then $G(\phi, p, q)$ is not right orderable.

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As noted in [DPT05], there is some overlap between the latter case and the work of Dąbkowski, Przytycki, and Togha. Indeed, if

$$\phi_\sharp = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

then the manifold denoted by $M^{(n)}_{L_{[2,2]}}$ in [DPT05] has fundamental group $G(\phi^n, 1, 0)$.

Let us pause here to explain some notation and conventions we used above and will use throughout the paper. For $g \in F$ and $\psi \in \mathrm{Aut}(F)$, we write g^ψ for the image of g under the action of ψ. The action of $\mathrm{Aut}(F)$ on F, along with all other group actions described in this paper, will be from the right, so if ψ_1 and then ψ_2 from $\mathrm{Aut}(F)$ are applied to $g \in F$, the resulting element is $g^{\psi_1\psi_2}$. For a group G and $g, h \in G$, we write g^h for $h^{-1}gh$. We write $[g, h]$ for $ghg^{-1}h^{-1}$.

A group is called a 3-*manifold group* if it can be realized as the fundamental group of a 3-manifold. The groups $G(\phi, p, q)$ described above are 3-manifold groups and their study in [RSS03, Fen07] and the current paper was motivated by questions arising from the study of Reebless foliations and essential laminations in the associated 3-manifolds. In general, there is an interesting interplay between the existence of Reebless foliations or, more generally, essential laminations in a 3-manifold M and the existence of nontrivial actions of $\pi_1(M)$ on associated (not necessarily Hausdorff) 1-manifolds and trees.

Let us describe in more detail the construction of 3-manifolds with fundamental groups $G(\phi, p, q)$. Let T be a once-punctured torus (a compact surface of genus one with boundary $\partial T \cong S^1$), and let $\phi\colon T \to T$ be a homeomorphism. The *punctured torus bundle* $M(\phi)$ is the quotient space

$$M(\phi) := (T \times [0, 1])/((x, 0) \sim (\phi(x), 1)).$$

The map ϕ induces automorphisms ϕ_* of $F = \pi_1(T)$, well-defined up to an inner automorphism of F, and $\phi_\sharp$ of $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. The following facts are well known (see [Ni17, FH82, CJR84, RSS03, Ind06]).

- The fundamental group $\pi_1(M(\phi))$ has presentation

$$\langle t, a, b \mid a^t = a^{\phi_*}, b^t = b^{\phi_*} \rangle. \tag{1.1}$$

- The automorphism ϕ_* maps $[a, b]$ to one of its conjugates in F if ϕ preserves orientation, and to a conjugate of $[b, a]$ in F otherwise.
- If $\alpha \in Aut(F)$ fixes $[a, b]$, then there is some ϕ such that $\phi_* = \alpha$. Moreover, for each $A \in Aut(H_1(T)) \cong GL_2(\mathbb{Z})$, there is some $\alpha \in Aut(F)$ that fixes $[a, b]$ and induces A on $F/[F, F]$. Therefore, for each $A \in GL_2(\mathbb{Z})$, there is some ϕ with $\phi_\sharp = A$.
- The manifolds $M(\phi)$ and $M(\psi)$ are homeomorphic if and only if $\phi_\sharp$ is conjugate to one of $\psi_\sharp, \psi_\sharp^{-1}$ in $GL_2(\mathbb{Z})$. (This is due to Murasugi.)
- $M(\phi)$ is orientable if and only if $\phi_\sharp \in SL_2(\mathbb{Z})$.

Now $\partial M(\phi)$ is a torus, and we can construct closed 3-manifolds $M(\phi, p, q)$ by performing Dehn filling along $\partial M(\phi)$. We now briefly describe the construction of these

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manifolds, referring the reader to [Rol90] for general facts about Dehn surgery. We will be interested in simple closed curves on $\partial M(\phi)$ that are images under the standard covering map $c\colon \mathbb{R}^2 \to \partial M(\phi)$ of lines with rational slopes. Fixing a coordinate system on $\mathbb{R}^2$, we say that a simple closed curve γ on $\partial M(\phi)$ has slope $p/q \in \mathbb{Q}\cup\{\infty\}$ if $c^{-1}(\gamma)$ is a line of slope p/q in $\mathbb{R}^2$. It is known (see for example [CJR84, Ind06]) that the presentation (1.1) determines a unique choice of coordinate system on $\mathbb{R}^2$ for which the following claims hold true.

- For any $x \in [0,1]$, the fiber $T \times \{x\}$ in $M(\phi)$ intersects $\partial M(\phi)$ in a simple closed curve γ. The line $c^{-1}(\gamma)$ has slope zero in $\mathbb{R}^2$.
- We may assume that the base point x_0 used to determine $\pi_1(M_\phi)$ lies on $\partial M(\phi)$. There is a simple closed curve $\tau \subset \partial M(\phi)$ through x_0 that represents t in the presentation given above. The line $c^{-1}(\tau)$ has infinite slope in $\mathbb{R}^2$.

If l is a line of rational slope p/q in $\mathbb{R}^2$, then $c(l)$ is a simple closed curve on $\partial M(\phi)$. We perform p/q-Dehn surgery on $M(\phi)$, obtaining the closed 3-manifold $M(\phi, p, q)$ as follows. Let $X = D^2 \times S^1$ be a solid torus (here D^2 is a closed disc). Fix $y \in S^1$ and let $f\colon \partial X \to \partial M(\phi)$ be a homeomorphism satisfying $f(\partial D^2 \times \{y\}) = c(l)$. Then the homeomorphism type of the quotient space

$$M(\phi, p, q) := (M(\phi) \cup X)/(x \sim f(x))$$

does not depend on the choice of y or f. When p and q are relatively prime, we have

$$\pi_1(M(\phi, p, q)) \cong G(\phi, p, q).$$

Now we explain how, given a conjugacy class $[C]$ in $SL_2(\mathbb{Z})$, we will choose $A \in [C]$ and $\alpha \in Aut(F)$ that fixes $[a,b]$ and induces A on $F/[F,F]$. By the comments above, there is some ϕ with $\phi_* = \alpha$ (so $\phi_\sharp = A$) and, having fixed $[C]$, the homeomorphism type of $M(\phi)$ does not depend on our choice of A and α. Set

$$N := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, U := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, L := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

So $N, U, L \in SL_2(\mathbb{Z})$. For a sequence $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ of positive integers, set

$$X_\Lambda := \prod_{i=1}^{r} U^{k_i} L^{l_i}.$$

Then (see [FH82, CJR84, Ha92, Ind06]) every $A \in SL_2(\mathbb{Z})$ satisfying $|\operatorname{Trace}(A)| > 2$ is conjugate in $GL_2(\mathbb{Z})$ to X_Λ or NX_Λ for some such sequence. In particular, if $\operatorname{Trace}(A) > 2$, then A is conjugate to some X_Λ, while if $\operatorname{Trace}(A) < -2$, then A is conjugate to some NX_Λ. Define $\mathcal{N}, \mathcal{U}, \mathcal{L} \in \operatorname{Aut}(F)$ by

$$a^{\mathcal{N}} = [a,b]\, a^{-1}, \qquad b^{\mathcal{N}} = ab^{-1}a^{-1},$$
$$a^{\mathcal{U}} = ab, \qquad b^{\mathcal{U}} = b,$$
$$a^{\mathcal{L}} = a, \qquad b^{\mathcal{L}} = ba.$$

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Direct computation shows that

$$[a,b]^{\mathcal{N}} = [a,b]^{\mathcal{U}} = [a,b]^{\mathcal{L}} = [a,b]\,.$$

Moreover, $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{L}$ induce the automorphisms N, U, and L respectively on the abelianization of F.

Given a sequence Λ as above, set

$$\phi_\Lambda^+ := \prod_{i=1}^{r} \mathcal{U}^{k_i}\mathcal{L}^{l_i} \in \operatorname{Aut}(F), \quad \text{and} \quad \phi_\Lambda^- := \mathcal{N}\phi_\Lambda^+ \in \operatorname{Aut}(F).$$

According to the discussion above, we may make the following assumption without loss of generality, and the coordinate system on $\mathbb{R}^2$ described above is always chosen having fixed ϕ_* as described therein.

Assumption 1.1 *Let ϕ be an orientation preserving homeomorphism of the once-punctured torus T.*

(i) *If* $\operatorname{Trace}(\phi_\sharp) > 2$, *then there is some $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ such that $\phi_* = \phi_\Lambda^+$.*

(ii) *If* $\operatorname{Trace}(\phi_\sharp) < -2$, *then there is some $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ such that $\phi_* = \phi_\Lambda^-$.*

Let $\operatorname{Homeo}^+(\mathbb{R})$ be the group of orientation preserving homeomorphisms of $\mathbb{R}$. Our main results are as follows.

Theorem 1.2 *If $\phi_\sharp \in SL_2(\mathbb{Z})$ and* $\operatorname{Trace}(\phi_\sharp) > 2$, *then there is no nontrivial homeomorphism from $\pi_1(M(\phi, 1, 0))$ to* $\operatorname{Homeo}^+(\mathbb{R})$.

Theorem 1.3 *If $\phi_\sharp \in SL_2(\mathbb{Z})$ and* $\operatorname{Trace}(\phi_\sharp) < -2$ *and $\frac{p}{q} \in [1, \infty]$, then there is no nontrivial homomorphism from $\pi_1(M(\phi, p, q))$ to* $\operatorname{Homeo}^+(\mathbb{R})$.

By Thurston's hyperbolic Dehn surgery Theorem (see [Th79]), infinitely many of the manifolds $M(\phi, p, q)$ appearing in Theorem 1.3 are hyperbolic. In fact, the work of Bleiler and Hodgson (see [BH96]) shows that for infinitely many ϕ, the manifold $M(\phi, p, q)$ is hyperbolic whenever $(p, q) \neq (0, 1)$. On the other hand, there exist infinitely many hyperbolic 3-manifolds $M = M(\phi, p, q)$ admitting a nontrivial homomorphism from $\pi_1(M)$ to $\operatorname{Homeo}^+(\mathbb{R})$. Indeed, such a homomorphism was shown to exist when $\operatorname{Trace}(\phi_\sharp) > 2$ and $q = 1$ by Fenley in [Fen94].

Now we describe in more detail the applications of Theorems 1.2 and 1.3. Recall that a foliation (see [CaCo99] for definitions and basic results on foliations) $\mathcal{F}$ of a manifold M is called $\mathbb{R}$-*covered* (see [Pl83]) if the leaf space $\widetilde{L}$ of the foliation $\widetilde{\mathcal{F}}$ of the universal cover $\widetilde{M}$ of M obtained by lifting $\mathcal{F}$ is homeomorphic to $\mathbb{R}$. The foliation $\mathcal{F}$ is *transversely orientable* (sometimes called *coorientable*) if each leaf of $\mathcal{F}$ admits an oriented transversal in such a manner that the given orientations are locally consistent. In any case, the action of $\pi_1(M)$ on $\widetilde{M}$ determines an action of $\pi_1(M)$ on $\widetilde{L}$ by homeomorphisms, and if $\mathcal{F}$ is $\mathbb{R}$-covered and transversely orientable this action gives a nontrivial homomorphism from $\pi_1(M)$ to $\operatorname{Homeo}^+(\mathbb{R})$. Thus we obtain Corollary 1.4. It is not hard to show that if $\phi_\sharp \in SL_2(\mathbb{Z})$ then the abelianization of $G(\phi, p, q)$ has order $|p(\operatorname{Trace}(\phi_\sharp) - 2)|$. It follows that if p and $\operatorname{Trace}(\phi_\sharp)$ are both odd, then $G(\phi, p, q)$ has no subgroup of index two, and Corollary 1.4 still holds when we remove the phrase "transversely orientable".

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Corollary 1.4 *If $\phi_\sharp \in SL_2(\mathbb{Z})$ and either*

- $\mathrm{Trace}(\phi_\sharp) < -2$ *and* $\frac{p}{q} \in [1, \infty]$, *or*
- $\mathrm{Trace}(\phi_\sharp) > 2$ *and* $(p, q) = (1, 0)$,

then $M(\phi, p, q)$ admits no transversely orientable $\mathbb{R}$-covered foliation.

As mentioned above, it was shown in [RSS03] that certain of the $M(\phi, p, q)$ described in Theorem 1.3 admit no Reebless foliation and therefore no transversely orientable $\mathbb{R}$-covered foliation. On the other hand, it is known (see [Ha92]) that if $\phi_\sharp \in SL_2(\mathbb{Z})$ and either

(i) $\mathrm{Trace}(\phi_\sharp) > 2$ and $q \neq 0$ or
(ii) $\frac{p}{q} < 1$,

then $M(\phi, p, q)$ does admit a Reebless foliation.

A group G is *right orderable* if there exists a total ordering $\prec$ on G such that for all $x, y, g \in G$, we have $x \prec y$ if and only if $xg \prec yg$. It is known ([Li99]) that a countable group G is right orderable if and only if there is an injective homomorphism from G to $\mathrm{Homeo}^+(\mathbb{R})$. Thus we have the following result.

Corollary 1.5 *If $\phi_\sharp \in SL_2(\mathbb{Z})$ and either*

- $\mathrm{Trace}(\phi_\sharp) < -2$ *and* $p \geq q \geq 1$ *or* $(p, q) = (1, 0)$ *or*
- $\mathrm{Trace}(\phi_\sharp) > 2$ *and* $(p, q) = (1, 0)$,

then $\pi_1(M(\phi, p, q))$ is not right orderable.

We prove Theorems 1.2 and 1.3 as follows. Given coprime integers p, q, and $e \in \{+, -\}$, and a sequence $\Lambda = (k_1, l_1, \dots, k_r, l_r)$, let $G^e(\Lambda, p, q)$ be the group with generators t, a, b subject to the relations

(R1) $t^{-1}at = a^{\phi^e_\Lambda}$,
(R2) $t^{-1}bt = b^{\phi^e_\Lambda}$,
(R3) $t^p [a, b]^q = 1$.

Thus, if ϕ is a homeomorphism of the once punctured torus S and $\phi_* = \phi^e_\Lambda$, then $\pi_1(M(\phi, p, q)) = G^e(\Lambda, p, q)$. Note that $G^e(\Lambda, 1, 0)$ is the group with generators a, b subject to the relations

(S1) $a = a^{\phi^e_\Lambda}$,
(S2) $b = b^{\phi^e_\Lambda}$.

We say that a subgroup G of $\mathrm{Homeo}(\mathbb{R})$ has a global fixed point if there is some $x \in \mathbb{R}$ such that $x\sigma = x$ for each $\sigma \in G$. Let G be a nontrivial subgroup of $\mathrm{Homeo}^+(\mathbb{R})$. Then the set of global fixed points of G is not dense in $\mathbb{R}$. Therefore, there is some G-invariant interval $(x, y) \subseteq \mathbb{R}$ such that G has no global fixed point in (x, y). Thus, Theorems 1.2 and 1.3 follow immediately from the following results, whose proofs appear in the next two sections.

Theorem 1.6 *Let $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ be a sequence of positive integers and let $\phi = \phi^+_\Lambda$. Let $f \colon F \to \mathrm{Homeo}^+(\mathbb{R})$ be a homomorphism such that $f(a) = f(a^\phi)$ and $f(b) = f(b^\phi)$. The* $\mathrm{Image}(f)$ *has a global fixed point.*

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Theorem 1.7 *Let $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ be a sequence of positive integers. Let p, q be relatively prime integers with $p \geq q \geq 1$ or $(p, q) = (1, 0)$. Let $f\colon G^-(\Lambda, p, q) \to$ $\mathrm{Homeo}^+(\mathbb{R})$ be a homomorphism. Then* $\mathrm{Image}(f)$ *has a global fixed point.*

Our proofs of Theorems 1.6 and 1.7 use induction on the parameter r appearing in the sequence $\Lambda = (k_1, l_1, \dots, k_r, l_r)$. We find it interesting that this technique works, as it is unclear that there is any close algebraic similarity between $G^e(\Lambda, p, q)$ and $G^e(\Gamma, p, q)$ when Γ is obtained from $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ by appending k_{r+1}, l_{r+1}.

2 The Proof of Theorem 1.6

To prove Theorem 1.6, let us assume for contradiction that, with $\phi = \phi_\Lambda^+$ as in the theorem, there exists a homomorphism $f\colon F \to \mathrm{Homeo}^+(\mathbb{R})$ satisfying $f(a) = f(a^\phi)$ and $f(b) = f(b^\phi)$, whose image has no global fixed point. For $x \in \mathbb{R}$ and $g \in F$, we write xg for $xf(g)$.

Lemma 2.1 *For each $x \in \mathbb{R}$, we have $xa \neq x$ and $xb \neq x$.*

Proof Fix $x \in \mathbb{R}$ and assume for contradiction that $xa = x$. If $xb = x$, then x is a global fixed point for $\mathrm{Image}(f)$, a contradiction. Say $xb > x$. Since a^ϕ is a product of positive powers of a and b, we have $xa^\phi > x$, contradicting $f(a) = f(a^\phi)$. A similar argument shows that we cannot have $xb < x$, and further arguments of the same type supply contradictions under the initial assumption that $xb = x$. ■

Using Lemma 2.1 and the Intermediate Value Theorem, we see that either $xa > x$ for all $x \in \mathbb{R}$ or $xa < x$ for all $x \in \mathbb{R}$, and the same holds for b. We cannot have $xa > x$ and $xb > x$ for all $x \in \mathbb{R}$, since from this we can derive $xa^\phi > xa$ for all $x \in \mathbb{R}$, contradicting $f(a) = f(a^\phi)$. Similarly, we cannot have $xa < x$ and $xb < x$ for all $x \in \mathbb{R}$. If $xa < x$ and $xb > x$ for all $x \in \mathbb{R}$, we may conjugate $\mathrm{Image}(f)$ by any orientation reversing homeomorphism of $\mathbb{R}$ to get a homomorphism $f^-\colon F \to \mathrm{Homeo}^+(\mathbb{R})$ satisfying $f^-(a) = f^-(a^\phi)$, $f^-(b) = f^-(b^\phi)$, $xf^-(a) > x$, and $xf^-(b) < x$, whose image has no global fixed point. Therefore, we may continue under the following assumption without loss of generality.

Assumption 2.2 *For all $x \in \mathbb{R}$, we have $xa > x$ and $xb < x$.*

Let us now examine the case $r = 1$, so $\phi = \mathcal{U}^{k_1}\mathcal{L}^{l_1}$. In this case, we calculate that

$$a^\phi = a(ba^{l_1})^{k_1} \tag{2.1}$$

and

$$b^\phi = ba^{l_1}. \tag{2.2}$$

Since $f(b^\phi) = f(b)$, it follows from (2.2) that $f(a^{l_1}) = 1$, and we cannot have $xa > x$ for all $x \in \mathbb{R}$, contradicting Assumption 2.2. Thus we proceed under the following assumption.

Assumption 2.3 *We have $r \geq 2$.*

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Now we introduce some useful notation. Having fixed $\phi = \phi^+_\Lambda$, we define, for $1 \le i \le j \le r$,

$$\phi^{(i,j)} := \prod_{h=i}^{j} \mathcal{U}^{k_h}\mathcal{L}^{l_h}.$$

Lemma 2.4 *We have*

$$a^\phi = a^{\phi^{(2,r)}}(b^\phi)^{k_1} \tag{2.3}$$

and

$$b^\phi = b^{\phi^{(2,r)}}(a^{\phi^{(2,r)}})^{l_1}. \tag{2.4}$$

Proof We proceed by induction on r. The base case is $r = 2$. In this case, we use (2.1) and (2.2) to get

$$\begin{aligned} a^\phi &= (a(ba^{l_1})^{k_1})^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}} = a^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}}((ba^{l_1})^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}})^{k_1} \\ &= a^{\phi^{(2,2)}}((b^{\mathcal{U}^{k_1}\mathcal{L}^{l_1}})^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}})^{k_1} = a^{\phi^{(2,2)}}(b^\phi)^{k_1} \end{aligned}$$

and

$$b^\phi = (ba^{l_1})^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}} = b^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}}(a^{\mathcal{U}^{k_2}\mathcal{L}^{l_2}})^{l_1} = b^{\phi^{(2,2)}}(a^{\phi^{(2,2)}})^{l_1}$$

as claimed. Now assume $r > 2$. Using our inductive hypothesis, we get

$$\begin{aligned} a^\phi &= (a^{\phi^{(1,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = (a^{\phi^{(2,r-1)}}(b^{\phi^{(1,r-1)}})^{k_1})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} \\ &= (a^{\phi^{(2,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}((b^{\phi^{(1,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})^{k_1} = a^{\phi^{(2,r)}}(b^\phi)^{k_1} \end{aligned}$$

and

$$\begin{aligned} b^\phi &= (b^{\phi^{(1,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = (b^{\phi^{(2,r-1)}}(a^{\phi^{(2,r-1)}})^{l_1})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} \\ &= (b^{\phi^{(2,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}((a^{\phi^{(2,r-1)}})^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})^{l_1} = b^{\phi^{(2,r)}}(a^{\phi^{(2,r)}})^{l_1}. \end{aligned}$$

■

Corollary 2.5 *For all $x \in \mathbb{R}$, we have*

$$xa^{l_r}\prod_{m=r}^{2}(a^{\phi^{(m,r)}})^{l_{m-1}} = x. \tag{2.5}$$

(By $\prod_{m=r}^{2} c_m$ we mean the product $c_r c_{r-1} \cdots c_2$, for any $c_2, \dots, c_r$.)

Proof Applying (2.4) repeatedly, we get

$$\begin{aligned} b^\phi &= b^{\phi^{(2,r)}}(a^{\phi^{(2,r)}})^{l_1} \\ &= b^{\phi^{(3,r)}}(a^{\phi^{(3,r)}})^{l_2}(a^{\phi^{(2,r)}})^{l_1} \\ &= \dots \\ &= b^{\phi^{(r,r)}}\prod_{m=r}^{2}(a^{\phi^{(m,r)}})^{l_{m-1}}. \end{aligned}$$

Now $b^{\phi^{(r,r)}} = ba^{l_r}$ by (2.2), so (2.5) follows from $f(b^\phi) = f(b)$. ■

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Corollary 2.6 *For $1 \le m \le r$ and all $x \in \mathbb{R}$, we have*

$$xa^{\phi^{(m,r)}} > x \tag{2.6}$$

and $xb^{\phi^{(m,r)}} < x$.

Proof We proceed by induction on m, the base case $m = 1$ being a restatement of Assumption 2.2. Now assume $m > 1$. By inductive hypothesis, we have $xa^{\phi^{(m-1,r)}} > x$ and $xb^{\phi^{(m-1,r)}} < x$ for all $x \in \mathbb{R}$. Now by (2.3), we have (for each $x \in \mathbb{R}$)

$$xa^{\phi^{(m-1,r)}} = xa^{\phi^{(m,r)}}(b^{\phi^{(m-1,r)}})^{k_{m-1}} < xa^{\phi^{(m,r)}},$$

so we must have $xa^{\phi^{(m,r)}} > x$ for all x. Hence by (2.4), we have

$$xb^{\phi^{(m-1,r)}} = xb^{\phi^{(m,r)}}(a^{\phi^{(m,r)}})^{l_1} > xb^{\phi^{(m,r)}},$$

so we must have $xb^{\phi^{(m,r)}} < x$. ■

Combining Corollaries 2.5 and 2.6, we obtain the contradiction that proves Theorem 1.6. Indeed, by Assumption 2.2 and (2.6), we have

$$xa^{l_r}\prod_{m=r}^{2}(a^{\phi^{(m,r)}})^{l_{m-1}} > x$$

for all $x \in \mathbb{R}$, contradicting (2.5).

3 The Proof of Theorem 1.7

We will begin by proving Theorem 1.7 under the following assumption and then explain how to adjust the given proof to handle the case $(p,q) = (1,0)$.

Assumption 3.1 *We have $p \ge q \ge 1$.*

Now we introduce some additional notation. Fix $\Lambda = (k_1, l_1, \dots, k_r, l_r)$ and let $\phi = \phi_\Lambda^-$. For $1 \le i \le j \le r$, set

$$\Lambda^{(i,j)} := (k_i, l_i, \dots, k_j, l_j)$$

and

$$\phi_{(i,j)} := \phi^-_{\Lambda^{(i,j)}} = \mathcal{N}\phi^{(i,j)}.$$

For $i > j$, set $\phi_{(i,j)} = 1$. Now define

$$u_\Lambda := (b^{-1})^{\phi_{(2,r)}}, \quad v_\Lambda := (a^{-1})^{\phi_{(2,r)}}, \quad \text{and} \quad w_\Lambda := v_\Lambda^{l_1-1}u_\Lambda v_\Lambda.$$

We call a nonidentity element g of the free group $F(a,b)$ *totally negative* if we can write g in reduced form as

$$g = \prod_{i=1}^{s} a^{\rho_i}b^{\theta_i}$$

with $\rho_i, \theta_i \le 0$ for all $i \in [s]$.

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Lemma 3.2 *Each of* $u_\Lambda, v_\Lambda, w_\Lambda \in F(a,b)$ *is totally negative. Also,*

$$a^\phi = [a,b]\, v_\Lambda w_\Lambda^{k_1} \quad \textit{and} \tag{3.1}$$

$$b^\phi = w_\Lambda. \tag{3.2}$$

Proof We proceed by induction on r. If $r = 1$, direct calculation gives

$$a^\phi = [a,b]\, a^{-1}(a^{1-l_1}b^{-1}a^{-1})^{k_1}$$

and (since $b^{\mathcal{N}} = b^{-1}[a,b]^{-1}$)

$$b^\phi = a^{1-l_1}b^{-1}a^{-1}.$$

Hence the claim holds in this case. Now assume that $r > 1$. We have

$$\phi = \phi_{(1,r-1)}\mathcal{U}^{k_r}\mathcal{L}^{l_r}$$

and $\phi_{(2,r)} = \phi_{(2,r-1)}\mathcal{U}^{k_r}\mathcal{L}^{l_r}$. It follows immediately that

$$u_\Lambda = u_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} \quad \text{and} \tag{3.3}$$

$$v_\Lambda = v_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}. \tag{3.4}$$

It follows from (3.3) and (3.4) that $w_\Lambda = w_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}$. If $g \in F(a,b)$ is totally negative, then $g^{\mathcal{U}}$ and $g^{\mathcal{L}}$ are totally negative. It now follows from the inductive hypothesis that u_Λ, v_Λ and w_Λ are totally negative. Our inductive hypothesis also gives

$$a^\phi = a^{\phi_{(1,r-1)}\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = ([a,b]^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})v_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}(w_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})^{k_1} = [a,b]\, v_\Lambda w_\Lambda^{k_1}$$

and

$$b^\phi = b^{\phi_{(1,r-1)}\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = w_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = w_\Lambda$$

as claimed. ■

Lemma 3.3 *If* $r \geq 2$, *then*

$$u_\Lambda = v_{\Lambda_{(2,r)}}^{l_2} u_{\Lambda_{(2,r)}} \tag{3.5}$$

and

$$v_\Lambda = u_\Lambda^{k_2} v_{\Lambda_{(2,r)}} \tag{3.6}$$

Proof Again we use induction on r. If $r = 2$, then $u_{\Lambda_{(2,r)}} = (b^{-1})^{\phi_{(3,2)}} = b^{-1}$, and similarly, $v_{\Lambda_{(2,r)}} = a^{-1}$. Now direct calculation gives $u_\Lambda = a^{-l_2}b^{-1}$ and $v_\Lambda = (a^{-l_2}b^{-1})^{k_2}a^{-1}$, and the claim of the lemma holds in this case. Now assume $r > 2$. Using the inductive hypothesis, we get

$$u_\Lambda = u_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = (v_{\Lambda_{(2,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})^{l_2}(u_{\Lambda_{(2,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}}) = v_{\Lambda_{(2,r)}}^{l_2} u_{\Lambda_{(2,r)}}$$

and

$$v_\Lambda = v_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = (u_{\Lambda_{(1,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}})^{k_2} v_{\Lambda_{(2,r-1)}}^{\mathcal{U}^{k_r}\mathcal{L}^{l_r}} = u_\Lambda^{k_2} v_{\Lambda_{(2,r)}}$$

as claimed. ■

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Corollary 3.4 *There is no homomorphism* $\psi\colon F(a,b) \to \mathrm{Homeo}^+(\mathbb{R})$ *satisfying all of the conditions*

(i) $x\psi(a) > x$ *for all* $x \in \mathbb{R}$,
(ii) $x\psi(b) < x$ *for all* $x \in \mathbb{R}$,
(iii) $x\psi(u_\Lambda) < x$ *for all* $x \in \mathbb{R}$, *and*
(iv) $x\psi(v_\Lambda) > x$ *for all* $x \in \mathbb{R}$.

Proof Again we use induction on r. As noted above, if $r = 1$, we have $u_\Lambda = b^{-1}$ and conditions (ii) and (iii) cannot be satisfied simultaneously. Now assume $r > 1$. By equation (3.6) of Lemma 3.3, if (iii) and (iv) are both satisfied, then we have $x\psi(v_{\Lambda_{(2,r)}}) > x$ for all $x \in \mathbb{R}$. Now (iii) and equation (3.5) of Lemma 3.3 force $x\psi(u_{\Lambda_{(2,r)}}) < x$ for all $x \in \mathbb{R}$. This means that conditions (i)–(iv) are satisfied if we replace Λ with $\Lambda_{(2,r)}$, which contradicts our inductive hypothesis. ■

Now we prove Theorem 1.7. Assume (for contradiction) that $f\colon G^-(\Lambda,p,q) \to \mathrm{Homeo}^+(\mathbb{R})$ is a homomorphism whose image has no global fixed point. We write $t, a, b, u_\Lambda, v_\Lambda, w_\Lambda$ for the respective images of $t, a, b, u_\Lambda, v_\Lambda, w_\Lambda \in F(a,b,t)$ in $G^-(\Lambda,p,q)$. Set

$$\alpha := f(a), \quad \beta := f(b), \quad \gamma := f([a,b]), \quad \tau := f(t),$$
$$\mu := f(u_\Lambda), \quad \nu := f(v_\Lambda), \quad \text{and} \quad \omega := f(w_\Lambda).$$

Since ϕ fixes $[a,b]$, we know that τ and γ commute. The following simple result will be of great use.

Lemma 3.5 *Let* g, h *be elements of a group* G *such that* $gh = hg$ *and there exist relatively prime integers* p, q *with* $g^p = h^{-q}$. *Then there is some* $k \in G$ *such that* $g = k^q$ *and* $h = k^{-p}$.

To prove Lemma 3.5, we simply take integers r, s with $rp + sq = 1$ and verify that $k = g^s h^{-r}$ has the desired properties. The next corollary, which we will use repeatedly, follows immediately.

Corollary 3.6 *There is some* $\kappa \in \mathrm{Homeo}^+(\mathbb{R})$ *such that* $\tau = \kappa^q$ *and* $\gamma = \kappa^{-p}$. *In particular, if* $p \geq q \geq 1$, *then, for any* $x \in \mathbb{R}$, *one of the following conditions holds.*

(K1) $x\kappa = x\tau = x\gamma = x$,
(K2) $x\gamma^{-1} \leq x\tau \leq x\kappa < x < x\kappa^{-1} \leq x\tau^{-1} \leq x\gamma$, *or*
(K3) $x\gamma \leq x\tau^{-1} \leq x\kappa^{-1} < x < x\kappa \leq x\tau \leq x\gamma^{-1}$

Lemma 3.7 *There is no* $x \in \mathbb{R}$ *such that* $x\alpha = x$.

Proof Assume for contradiction that there is some $x \in \mathbb{R}$ satisfying $x\alpha = x$. Note first that we cannot have $x\beta = x$, since this would force $x\gamma = x$, which in turn would force $x\tau = x$ (by Corollary 3.6), making x a global fixed point for Image(f).

Moreover, since v_Λ and w_Λ are totally negative words in a, b, we see that

- if $x\beta < x$ then $x\omega > x$ and $x\nu > x$, and
- if $x\beta > x$ then $x\omega < x$ and $x\nu < x$.

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We can therefore conclude that $x\tau \neq x$. Indeed, if $x\tau = x$, then $x\beta\tau = x\tau^{-1}\beta\tau = x\omega$. However, this is impossible, because if $x\beta > x$, then $x\omega = x\beta\tau > x\tau = x$, and similarly, if $x\beta < x$, then $x\omega < x$.

We may now assume without loss of generality that $x\tau > x$. (As we argued earlier, if $x\tau < x$, we may conjugate Image(f) by an orientation reversing homeomorphism.) Now case (K3) of Corollary 3.6 holds.

If $x\beta > x$, then $x\tau^{-1}\beta\tau = x\omega < x$, so $(x\tau^{-1})\beta < x\tau^{-1}$. Similarly, if $x\beta < x$, then $(x\tau^{-1})\beta > x\tau^{-1}$. In either case, the Intermediate Value Theorem guarantees that there is some $y \in (x\tau^{-1}, x)$ such that $y\beta = y$.

Note that $y\alpha \neq y$. In fact, since $y\gamma^{-1} = y\beta\alpha\beta^{-1}\alpha^{-1} = (y\alpha)\beta^{-1}\alpha^{-1}$, we must have $y\alpha > y$. (Otherwise, $y\gamma^{-1} < x$, which gives $y < x\gamma \leq x\tau^{-1}$, a contradiction.) Similarly, $x\gamma = (x\beta)\alpha^{-1}\beta^{-1}$ forces $x\beta < x$. In summary, we have

- $x\tau^{-1} < y < y\alpha < x$.

Now relation (R1), along with (3.1) and (3.2), gives

$$\begin{aligned} x\nu^{-1}\gamma^{-1}\tau^{-1}\alpha\tau = x\omega^{k_1} &= (x\tau^{-1})\beta^{k_1}\tau \\ &< y\beta^{k_1}\tau = y\tau. \end{aligned}$$

Since ν_Λ is totally negative in a and b, we have $y < y\nu^{-1}$. It now follows that

$$\begin{aligned} y\gamma^{-1}\tau^{-1} < (y\nu^{-1})\gamma^{-1}\tau^{-1} &< x\nu^{-1}\gamma^{-1}\tau^{-1} \\ &< y\alpha^{-1} < y. \end{aligned}$$

It follows that $y\gamma^{-1} < y\tau$. Thus case (K2) of Corollary 3.6 holds, and we have $y\tau < y$. Now $x\tau^{-1} < y$ forces $x < y\tau < y$, giving the desired contradiction. ∎

By Lemma 3.7 and the Intermediate Value Theorem, either $x\alpha > x$ for all $x \in \mathbb{R}$ or $x\alpha < x$ for all $x \in \mathbb{R}$. Now we may assume without loss of generality (once again using conjugation by an orientation reversing homeomorphism if necessary) that we have

(1) $x\alpha > x$ for all $x \in \mathbb{R}$.

Relation (R1) and (3.1) give

$$\begin{aligned} \nu\omega^{k_1} &= \gamma^{-1}\tau^{-1}\alpha\tau = \tau^{-1}\gamma^{-1}\alpha\tau \\ &= \tau^{-1}\beta\alpha\beta^{-1}\tau = (\beta^{-1}\tau)^{-1}\alpha(\beta^{-1}\tau). \end{aligned}$$

So, since α and $\nu\omega^{k_1}$ are conjugate in $\mathrm{Homeo}^+(\mathbb{R})$, it follows from our assumption $x\alpha > x$ for all $x \in \mathbb{R}$ that

$$x\nu\omega^{k_1} > x \tag{3.7}$$

for all $x \in \mathbb{R}$. Since $\nu\omega^{k_1}$ is totally negative and $x\alpha > x$ for all $x \in \mathbb{R}$ we must have

(2) $x\beta < x$ for all $x \in \mathbb{R}$.

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By (3.2), ω and β are conjugate in $\text{Homeo}^+(\mathbb{R})$. Therefore,

$$x\omega < x \tag{3.8}$$

for all $x \in \mathbb{R}$. Combining this with (3.7), we get

(4) $x\nu > x$ for all $x \in \mathbb{R}$.

We know that $\omega = \nu^{l_1 - 1}\mu\nu$, and combining this with (4) and (3.8) gives

(3) $x\mu < x$ for all $x \in \mathbb{R}$.

Now (i)–(iv) give a contradiction to Corollary 3.4, and this completes the proof of Theorem 1.7 under Assumption 3.1.

Finally, assume that $(p, q) = (1, 0)$. We retain the notation introduced above. Note that Corollary 3.4 and its proof do not involve (p, q). Therefore, the corollary holds under our assumption. Lemma 3.7 also holds. Indeed, if $x\alpha = x$, then we cannot have $x\beta = x$, as $G^-(\Lambda, p, q)$ is generated by a and b. We may assume that $x\beta > x$. Since b^ϕ is totally negative, we have $xf(b^\phi) < x$. This contradicts $b = b^\phi$. So, we may assume that $x\alpha > x$ for all $x \in \mathbb{R}$. Now we may proceed as we did when $p \geq q \geq 1$. We have (since $\tau = 1$) $\nu\omega^{k_1} = \beta\alpha\beta^{-1}$, so $x\nu\omega^{k_1} > x$ for all x. This forces $x\beta < x$ for all x, which in turn forces $x\nu > x$ and $x\mu < x$ for all x as it did before, and our proof is complete. ■

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