Canad. Math. Bull. Vol. **51** (4), 2008 pp. 579–583

Guessing with Mutually Stationary Sets

Pierre Matet

Abstract. We use the mutually stationary sets of Foreman and Magidor as a tool to establish the validity of the two-cardinal version of the diamond principle in some special cases.

Jech [3] introduced the following notions. Let κ be a regular uncountable cardinal and $\lambda > \kappa$ be a cardinal. Then $P_\kappa(\lambda)$ denotes the collection of all subsets of λ of size less than κ. A subset C of $P_\kappa(\lambda)$ is *closed unbounded* if (i) C is cofinal in the partially ordered set $(C, \subset)$, and (ii) for any nonzero ordinal $\delta < \kappa$ and any sequence $\langle a_\alpha : \alpha < \delta \rangle$ of elements of C such that $a_\beta \subseteq a_\alpha$ for all $\beta < \alpha$, $\bigcup_{\alpha<\delta} a_\alpha \in C$. A subset S of $P_\kappa(\lambda)$ is *stationary* if $S \cap C \neq \phi$ for every closed unbounded subset C of $P_\kappa(\lambda)$. The *diamond principle* $\diamondsuit_{\kappa,\lambda}$ asserts the existence of a sequence $\langle s_a : a \in P_\kappa(\lambda) \rangle$ with $s_a \subseteq a$ such that for any $X \subseteq \lambda$, $\{a : s_a = X \cap a\}$ is a stationary subset of $P_\kappa(\lambda)$. Jech established that $\diamondsuit_{\kappa,\lambda}$ holds in the constructible universe L. Moreover, he proved that $\diamondsuit_{\kappa,\lambda}$ could be introduced by forcing.

It was shown in [1] that $2^{<\kappa} < \lambda$ implies $\diamondsuit_{\kappa,\lambda}$. In the present paper we show that if $2^{<\kappa} \leq \mu^{\aleph_0}$ for some cardinal μ such that $\kappa < \mu \leq \lambda$ and $cf(\mu) = \omega$, then $\diamondsuit_{\kappa,\lambda}$ holds. The proof closely follows that of the following result of Foreman and Magidor [2]: for a regular uncountable cardinal ν, let E^ν_ω denote the set of all infinite limit ordinals $\alpha < \nu$ such that $cf(\alpha) = \omega$. Suppose $\langle \mu_n : n < \omega \rangle$ is a sequence of regular cardinals such that $\kappa \leq \mu_0 < \mu_1 < \mu_2 < \cdots < \lambda$. For $n < \omega$, let S_n be a stationary subset of $E^{\mu_n}_\omega$. Then the S_n's are mutually stationary, which means that

$$\{a \in P_\kappa(\lambda) : \forall n \in \omega\ (\sup(a \cap \mu_n) \in S_n)\} \in NS^+_{\kappa,\lambda},$$

where $NS_{\kappa,\lambda}$ denotes the ideal of nonstationary subsets of $P_\kappa(\lambda)$.

Let κ be a regular uncountable cardinal and $\lambda > \kappa$ be a cardinal. For $A \subseteq P_\kappa(\lambda)$, the two-person game $G_{\kappa,\lambda}(A)$ is defined as follows. The game lasts ω moves, with player I making the first move. Players I and II alternately pick elements of $P_\kappa(\lambda)$, thus building a sequence $\langle a_n : n < \omega \rangle$ with the condition that $a_0 \subseteq a_1 \subseteq a_2 \subseteq \cdots$. Player II wins the game just in case $\bigcup_{n<\omega} a_n \in A$. Let $NG_{\kappa,\lambda}$ be the set of all subsets B of $P_\kappa(\lambda)$ such that II has a winning strategy in $G_{\kappa,\lambda}(P_\kappa(\lambda) \setminus B)$.

Lemma 1 **(Matet [4])** *Let κ be a regular uncountable cardinal and $\lambda > \kappa$ be a cardinal. Then $NG_{\kappa,\lambda}$ is a normal ideal on $P_\kappa(\lambda)$.*

Proposition 2 *Let κ, μ and λ be three cardinals such that $\omega_1 \leq \kappa < \mu \leq \lambda, \kappa$ is regular, $cf(\mu) = \omega$ and $2^{<\kappa} \leq \mu^{\aleph_0}$. Then there is a sequence $\langle s_a : a \in P_\kappa(\lambda) \rangle$ with $s_a \subseteq a$ such that for any $X \subseteq \lambda$, $\{a : s_a = X \cap a\} \in NG^+_{\kappa,\lambda}$.*

Received by the editors April 24, 2006; revised July 5, 2007.
AMS subject classification: 03E05.
Keywords: $P_\kappa(\lambda)$, diamond principle.

Proof Pick an increasing sequence $\langle \mu_n : n < \omega \rangle$ of regular cardinals so that $\kappa \le \mu_0$ and $\sup\{\mu_n : n < \omega\} = \mu$. For $n < \omega$, select a one-to-one function $\varphi_n \colon \bigcup_{\zeta<\kappa} {}^{\zeta}2 \to \prod_{n \le p < \omega} \mu_p$ and a sequence $\langle S_n(z) : z \in \prod_{j \le n} {}^{(j+1)}\mu_n \rangle$ of pairwise disjoint stationary subsets of $E^{\mu_n}_{\omega}$. For $b \subseteq \lambda$, let o.t.(b) denote the order type of b, and $e(b) \colon \text{o.t.}(b) \to b$ be the function that enumerates the elements of b in increasing order. For $a, b \in P_\kappa(\lambda)$ with $a \subseteq b$, let $\chi(a, b) \colon \text{o.t.}(b) \to 2$ be defined by $(\chi(a, b))(\alpha) = 1$ if and only if $(e(b))(\alpha) \in a$.

The proof will proceed as follows. Given $A \in NG^*_{\kappa,\lambda}$ and $X \subseteq \lambda$, we will construct a_n and z_n for $n < \omega$, and f^i_n and g^i_n for $i \le n < \omega$ so that

- $a_0, a_1, \dots \in P_\kappa(\lambda)$ and $a_0 \subseteq a_1 \subseteq \cdots$,
- $f^i_n = \chi(a_i, a_n)$ for $i < n$, and $f^n_n = \chi(X \cap a_n, a_n)$,
- $g^i_n = \varphi_n(f^i_n)$,
- $z_n \in \prod_{j \le n} {}^{(j+1)}\mu_n$ and $(z_n(j))(i) = g^i_j(n)$ for $i \le j \le n$,
- setting $a = \bigcup_{n<\omega} a_n$, $a \in A$ and for every $n < \omega$, $\sup(a \cap \mu_n) \in S_n(z_n)$.

The point is that the a_n's are coded by the z_n's. In fact, let $\theta < \text{o.t.}(a)$. For $n < \omega$, let $a^\theta_n = a_n \cap \{(e(a))(\zeta) : \zeta < \theta\}$. Suppose j is the least r such that $(e(a))(\theta) \in a_r$. Then (i) for $j < n < \omega, \text{o.t.}(a^\theta_n) \in \text{dom}(f^j_n)$ and $f^j_n(\text{o.t.}(a^\theta_n)) = 1$, and (ii) for $\ell < j \le n < \omega, \text{o.t.}(a^\theta_n) \in \text{dom}(f^\ell_n)$ and $f^\ell_n(\text{o.t.}(a^\theta_n)) = 0$.

The guessing sequence $\langle s_a : a \in P_\kappa(\lambda) \rangle$ is defined as follows. Suppose $a \in P_\kappa(\lambda)$ and z_n for $n < \omega$ are such that $\sup(a \cap \mu_n) \in S_n(z_n)$ for any n. For $i \le n < \omega$, define $g^i_n \in \prod_{n \le p < \omega} \mu_p$ by $g^i_n(p) = (z_p(n))(i)$, and let $g^i_n = \varphi_n(f^i_n)$. Put $\xi = \text{o.t.}(a)$. By induction on θ, define a^θ_n for $\theta \le \xi$ and $n < \omega$ as follows. Set $a^0_n = \phi$ for all $n < \omega$. If θ is an infinite limit ordinal, set $a^\theta_n = \bigcup_{\eta<\theta} a^\eta_n$ for all $n < \omega$. Assuming a^θ_n has been defined for every n, look for a $j < \omega$ such that (α) for $j < n < \omega, \text{o.t.}(a^\theta_n) \in \text{dom}(f^j_n)$ and $f^j_n(\text{o.t.}(a^\theta_n)) = 1$, and (β) for $\ell < j \le n < \omega, \text{o.t.}(a^\theta_n) \in \text{dom}(f^\ell_n)$ and $f^\ell_n(\text{o.t.}(a^\theta_n)) = 0$. If there is no such j, put $a^{\theta+1}_n = a^\theta_n$ for all $n < \omega$. If there is one, it is unique. Set $a^{\theta+1}_n = a^\theta_n$ for $n < j$, and $a^{\theta+1}_n = a^\theta_n \cup \{(e(a))(\theta)\}$ for $j \le n < \omega$. Finally, letting $a_n = a^\xi_n$ for each $n < \omega$, set $s_a = \bigcup_{n<\omega} s_n$, where

$$s_n = \{(e(a_n))(\eta) : \eta \in \text{dom}(f^n_n) \cap \text{o.t.}(a_n) \text{ and } f^n_n(\eta) = 1\}.$$

Now fix $A \in NG^*_{\kappa,\lambda}$ and $X \subseteq \lambda$. We will find $a \in A$ such that $s_a = X \cap a$. Let τ be a winning strategy for player II in the game $G_{\kappa,\lambda}(A)$. Let $\langle \nu_i : i < \omega \rangle$ be an enumeration of the set $\{\mu_n : n < \omega\}$ such that (0) for each n, there are infinitely many i's with $\nu_i = \mu_n$, and (1) $\ell(r) < \ell(s)$ whenever $r < s < \omega$, where $\ell \colon \omega \to \omega$ is defined by $\ell(j) =$ the least i such that $\mu_j = \nu_i$. (For instance, enumerate $\{\mu_n : n < \omega\}$ as $\mu_0, \mu_1, \mu_0, \mu_1, \mu_2, \mu_0, \mu_1, \mu_2, \mu_3, \mu_0, \mu_1, \mu_2, \mu_3, \mu_4 \cdots$). Let T_0 be the tree $\bigcup_{m<\omega} \prod_{i<m} \nu_i$, ordered by inclusion. For a subtree T of T_0, let $[T]$ denote the set of all branches of T, *i.e.*,

$$[T] = \Big\{ f \in \prod_{i<\omega} \nu_i : \forall m < \omega \, (f \restriction m \in T) \Big\},$$

and set $\text{suc}_T(t) = \{\alpha : t \cup \{(\text{dom}(t), \alpha)\} \in T\}$ for every $t \in T$. Define $k \colon T_0 \cup [T_0] \to P_\kappa(\lambda)$ as follows. Put $k(\phi) = \tau(\phi)$. Given $m < \omega$ and $t \in T_0$ with $\text{dom}(t) =$

$m+1$, define c_i and d_i for $i \leq m+1$ by $c_0 = \phi$ and $d_0 = \tau(c_0)$, and for $i > 0$, $c_i = d_{i-1} \cup \{t(i-1)\}$ and $d_i = \tau(c_0, c_1, \dots, c_i)$, and set $k(t) = d_{m+1}$. Finally, let $k(f) = \bigcup_{m<\omega} k(f \restriction m)$ for any $f \in [T_0]$. Note that $\{k(f) : f \in [T_0]\} \subseteq A$.

We will define $T_{n+1}, z_n, \langle g^i_n : i \leq n\rangle, \langle f^i_n : i \leq n\rangle$ and a_n for $n < \omega$ so that

- T_{n+1} is a subtree of T_n;
- for any $t \in T_{n+1}$, $|\operatorname{suc}_{T_{n+1}}(t)|$ equals 1 if $\nu_{\operatorname{dom}(t)} \leq \mu_n$, and $\nu_{\operatorname{dom}(t)}$ otherwise;
- for any $f \in [T_{n+1}]$, $\sup(\mu_n \cap k(f)) \in S_n(z_n)$;
- $z_n \in \prod_{j \leq n} {}^{(j+1)}\mu_n$ and $(z_n(j))(i) = g^i_j(n)$ for $i \leq j \leq n$;
- $g^i_n = \varphi_n(f^i_n)$;
- $f^i_n = \chi(a_i, a_n)$ for $i < n$, and $f^n_n = \chi(X \cap a_n, a_n)$;
- $a_n = k(s(T_n))$, where $s(T_n)$ is the unique $t \in T_n$ such that $\operatorname{dom}(t) = \ell(n)$.

From here on we follow the proof of the Foreman–Magidor result, mentioned above, as it was written up by Shioya [6]. The only significant difference is that our mutually stationary sets are not given in advance, but defined one after the other as we go down the tree.

Suppose T_n has been constructed. For $\gamma < \mu_n$, let $\mathcal{W}_\gamma$ be the collection of all subtrees W of T_0 such that for any $w \in W$, $\operatorname{suc}_W(w)$ equals $\nu_{\operatorname{dom}(w)}$ if $\nu_{\operatorname{dom}(w)} < \mu_n, \gamma \setminus \alpha$ for some $\alpha < \gamma$ if $\nu_{\operatorname{dom}(w)} = \mu_n$, and $\nu_{\operatorname{dom}(w)} \setminus \beta$ for some $\beta < \nu_{\operatorname{dom}(w)}$ if $\nu_{\operatorname{dom}(w)} > \mu_n$. Let C be the set of all $\gamma < \mu_n$ such that for every $W \in \mathcal{W}_\gamma$, there is $f \in [T_n] \cap [W]$ with $\mu_n \cap k(f) \subseteq \gamma$.

Claim 1 C contains a closed unbounded subset of μ_n.

Proof Suppose otherwise. For $\gamma \in \mu_n \setminus C$, pick $W_\gamma \in \mathcal{W}_\gamma$ so that $(\mu_n \cap k(f)) \setminus \gamma \neq \phi$ for all $f \in [T_n] \cap [W_\gamma]$. Construct a subtree T of T_n so that for any $t \in T$, $\operatorname{suc}_T(t)$ equals $\operatorname{suc}_{T_n}(t)$ if $\nu_{\operatorname{dom}(t)} \leq \mu_n$, and $\{\alpha\}$ for some $\alpha \in \bigcap\{\operatorname{suc}_{W_\gamma}(t) : t \in W_\gamma$ and $\gamma \in \mu_n \setminus C\}$ otherwise. For $\gamma < \mu_n$, let Y_γ be the set of all $t \in T$ such that $\nu_{\operatorname{dom}(t)} = \mu_n$ and $t(i) < \gamma$ for every $i \in \operatorname{dom}(t)$ with $\nu_i = \mu_n$. Note that $\{t \in T \cap W_\gamma : \nu_{\operatorname{dom}(t)} = \mu_n\} \subseteq Y_\gamma$. Let D be the set of all $\gamma < \mu_n$ such that for any $t \in Y_\gamma$, $\mu_n \cap k(t) \subseteq \gamma$ and $\gamma \cap \operatorname{suc}_T(t)$ is cofinal in γ. Since D contains a closed unbounded subset of μ_n, we can find $\gamma \in D \setminus C$. It is simple to see that $\operatorname{suc}_T(t) \cap \operatorname{suc}_{W_\gamma}(t) \neq \phi$ for all $t \in T \cap W_\gamma$. Pick $f \in [T] \cap [W_\gamma]$. Then setting $H = \{j < \omega : \nu_j = \mu_n\}$,

$$\mu_n \cap k(f) = \bigcup_{j \in H} (\mu_n \cap k(f \restriction j)) \subseteq \gamma.$$

This contradiction completes the proof of Claim 1. ■

Now use Claim 1 to select $\gamma \in C \cap S_n(z_n)$. Let T be the set of all $t \in T_n$ such that $(\alpha)\{W \in \mathcal{W}_\gamma : t \in W\} \neq \phi$, and (β) for any $m \leq \operatorname{dom}(t)$ and any $W \in \mathcal{W}_\gamma$ with $t \restriction m \in W$, there is $f \in [T_n] \cap [W]$ such that $t \restriction m \subseteq f$ and $\mu_n \cap k(f) \subseteq \gamma$. Clearly, T is a subtree of T_n. Moreover, $\phi \in T$ and $\mu_n \cap k(t) \subseteq \gamma$ for every $t \in T$. It is simple to see that $\operatorname{suc}_T(t) \neq \phi$ for any $t \in T$ such that $\nu_{\operatorname{dom}(t)} < \mu_n$.

Claim 2 Let $t \in T$ be such that $\nu_{\operatorname{dom}(t)} = \mu_n$. Then $\gamma \cap \operatorname{suc}_T(t)$ is cofinal in γ.

https://doi.org/10.4153/CMB-2008-057-5 Published online by Cambridge University Press

Proof Suppose otherwise. Then $(\gamma \setminus \delta) \cap \mathrm{suc}_T(t) = \phi$ for some $\delta < \gamma$. Set $Q = (\gamma \setminus \delta) \cap \mathrm{suc}_{T_n}(t)$. For $\alpha \in Q$ put $t_\alpha = t \cup \{(\mathrm{dom}(t), \alpha)\}$ and pick $W_\alpha \in \mathcal{W}_\gamma$ so that $t_\alpha \in W_\alpha$ and $(\mu_n \cap k(f)) \setminus \gamma \neq \phi$ for every $f \in [T_n] \cap [W_\alpha]$ with $t_\alpha \subseteq f$. Now select $W \in \mathcal{W}_\gamma$ so that

- given $\alpha \in Q, t_\alpha \in W$ and for any $w \in T_0$ with $t_\alpha \subset w, w \in W$ if and only if $w \in W_\alpha$,
- $\mathrm{suc}_W(t) = \gamma \setminus \delta$.

There is $f \in [T_n] \cap [W]$ such that $t \subseteq f$ and $\mu_n \cap k(f) \subseteq \gamma$. Then $t_\alpha \subseteq f$ for some $\alpha \in Q$. Clearly, $f \in [T_n] \cap [W_\alpha]$. This contradiction completes the proof of Claim 2. ■

A similar argument proves that $|\mathrm{suc}_T(t)| = \nu_{\mathrm{dom}(t)}$ for every $t \in T$ such that $\nu_{\mathrm{dom}(t)} > \mu_n$.

Now pick an increasing sequence $\langle \gamma_r : r < \omega \rangle$ of ordinals with $\sup\{\gamma_r : r < \omega\} = \gamma$. Construct a subtree K of T so that for any $t \in K, \mathrm{suc}_K(t)$ equals $\mathrm{suc}_T(t)$ if $\nu_{\mathrm{dom}(t)} \neq \mu_n$, and $\{\alpha\}$ for some α such that $\gamma_{u_t} \leq \alpha < \gamma$ otherwise, where $u_t = |\{i < \mathrm{dom}(t) : \nu_i = \mu_n\}|$. Note that $\sup(\mu_n \cap k(f)) = \gamma$ for any $f \in [K]$. Put $T_{n+1} = K$.

Finally, let $a = k(f)$ where $\{f\} = \left[\bigcap_{n<\omega} T_n\right]$. Clearly $a \in A, a = \bigcup_{n<\omega} a_n$ and $\sup(\mu_n \cap a) \in S_n(z_n)$ for all $n < \omega$. Moreover,

$$s_a = \bigcup_{n<\omega} \{(e(a_n))(\eta) : \eta \in \mathrm{o.t.}(a_n) \text{ and } f_n^n(\eta) = 1\} = \bigcup_{n<\omega} (X \cap a_n) = X \cap a.$$

This completes the proof of Proposition 2. ■

The reader has probably noticed that we proved more than what is asserted by Proposition 2. Here is the full statement of our result.

Proposition 3 *Suppose that κ and μ_n for $n < \omega$ are regular cardinals such that $\omega_1 \leq \kappa \leq \mu_0 < \mu_1 < \cdots$ and $2^{<\kappa} \leq (\sup\{\mu_n : n < \omega\})^{\aleph_0}$, and λ is a cardinal such that $\lambda > \mu_n$ for any $n < \omega$. Suppose further that for each $n < \omega$, T_n is a stationary subset of $E^{\mu_n}_\omega$. Then letting $S = \{a \in P_\kappa(\lambda) : \forall n < \omega\ (\sup(a \cap \mu_n) \in T_n)\}$, there is a sequence $\langle s_a : a \in P_\kappa(\lambda)\rangle$ with $s_a \subseteq a$ such that for every $X \subseteq \lambda$, $\{a \in S : s_a = X \cap a\} \in NG^+_{\kappa,\lambda}$.*

Proposition 3 can be seen as a variation on the Foreman–Magidor result mentioned above. At the cost of assuming some inequality (that is trivially verified if $\kappa = \omega_1$) we obtain a stronger conclusion where the ideal $ND_{\kappa,\lambda}$ of those sets such that $\diamondsuit_{\kappa,\lambda}(S)$ does not hold is substituted for the ideal $NS_{\kappa,\lambda}$ (see [1]).

For more on $NG_{\kappa,\lambda}$ and $\diamondsuit_{\kappa,\lambda}$, see [5].

References

[1] H. D. Donder and P. Matet, *Two cardinal versions of diamond.* Israel J. Math. **83**(1993), no. 1-2, 1–43.

[2] M. Foreman and M. Magidor, *Mutually stationary sequence of sets and the non-saturation of the non-stationary ideal on $P_\kappa(\lambda)$.* Acta Math. **186**(2001), no. 2, 271–300.

[3] T. J. Jech, *Some combinatorial problems concerning uncountable cardinals.* Ann. Math. Logic **5**(1972/73), 165–198.

https://doi.org/10.4153/CMB-2008-057-5 Published online by Cambridge University Press

[4] P. Matet, *Concerning stationary subsets of* $[\lambda]^{<\kappa}$. In: Set Theory and Its Applications. Lecture Notes in Mathematics 1401, Springer, Berlin, 1989, pp. 119–127.
[5] P. Matet, *Game ideals.* Ann. Pure Appl. Logic, to appear.
[6] M. Shioya, *Splitting* $\mathcal{P}_\kappa\lambda$ *into maximally many stationary sets.* Israel J. Math. **114**(1999), 347–357.

Université de Caen - CNRS, Laboratoire de Mathématiques, 14032 Caen Cedex, France
e-mail: matet@math.unicaen.fr