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Surfaces with $p_g = q = 2$ and an Irrational Pencil

Francesco Zucconi

Abstract. We describe the irrational pencils on surfaces of general type with $p_g = q = 2$.

Introduction

Many authors have contributed to an explicit description of surfaces with small invariants. However if p_g and q are the dimensions of the vector space of global holomorphic 2-forms and respectively 1-forms, even the classification of surfaces with $p_g = q = 2$ is not completed yet.

In [CCM, Theorem B, Theorem 3.23] the case with $p_g = q = 3$ was partially done and two families were found: one is given by the symmetric product of a genus 3 curve, the other is obtained by the $G = \mathbb{Z}/2$-diagonal action on the product $F \times C$ of two genus-3 curves such that F/G is an elliptic curve and C/G has genus 2. In this last case the surfaces have an *irrational pencil*; *i.e.* a fibration over a curve of genus > 0. Recently [Pi], [HP] have ended the classification showing that those families are the only ones.

The state of the art for surfaces with $p_g = q = 2$ can be found in [Ci] and [CM]. Here we only remind the reader that two main cases occur according to the behaviour of the Albanese morphism $\alpha \colon S \to \mathrm{Alb}(S) = A$. In fact α can be surjective, and in this case S is said to be of *Albanese general type*, or it induces a fibration over a genus 2 curve C contained in A. By analogy with the case $p_g = q = 3$, Catanese proposed the following question about surfaces X of Albanese general type with $p_g = q = 2$: if X has no irrational pencil is it true that X is the double cover of a principally polarized Abelian surface branched on a divisor D linearly equivalent to 2Θ? We have been informed by e-mail that A. J. Chen and C. D. Hacon have constructed an example of a minimal surface of general type with $p_g = q = 2$ and $K^2 = 5$, hence the problem has a negative answer.

In any case following that analogy, we were concerned on the other side of the theory. In fact we classify all surfaces of general type with $p_g = q = 2$ carrying on an irrational pencil. The main result is:

***Theorem* [A]** *Every irrational pencil over a surface of Albanese general type with $p_g = q = 2$ is isotrivial if the genus of the curves of the pencil is > 2.*

It relies on the famous result by Simpson [Sim, Theorem 4.2 p. 373] about the locus $V^1(X) = \{L \in \mathrm{Pic}^0(X) \mid h^1(X, -L) \geq 0\}$ of an irregular variety X. In [Z1] there

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is an application to surfaces theory of this important technique, which we have followed almost verbatim to prove the basic Lemma 2.3; incidentally [HP] has a similar approach. Then we have studied irrational isotrivial pencil of genus $g > 2$ obtaining:

***Theorem* [B]** *A surface S of Albanese general type with $p_g = q = 2$ has an irrational isotrivial pencil of genus $g > 2$ if and only if it is the minimal desingularization of $C_1 \times C_2/G$ where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, C_1, C_2 have genus 3, $C_i \to E_i = C_i/G$ is a Galois covering branched over two points of the elliptic curve E_i and G acts diagonally. In this case the general fiber F of the canonical morphism is obtained by the smoothing of two curves of genus 3 which intersect in four points, then it has genus 9.*

This theorem needs the theory showed in [Ca] and [Z2] to deal with isotrivial fibrations.

The theory is very simple when over S there is an irrational pencil of curves of genus 2. In fact S manifests itself as a double covering of the product of two elliptic curves $E_1 \times E_2 = Y$ branched over a reduced divisor $D \in |2\pi_1^\star(P_1) + 2\pi_2^\star(P_2)|$, where $\pi_i \colon Y \to E_i$ are the natural projections, P_i is a point of E_i and $i = 1, 2$. More precisely:

***Proposition* [C]** *A surface S of Albanese general type with $p_g = q = 2$ has an irrational pencil of curves of genus 2 if and only if it is the normalization of the double cover of Y branched along D. In particular the general fiber F of the canonical map is a curve of genus 5.*

To complete the theory we classify the surfaces not of Albanese general type with $p_g = q = 2$. In this case the Albanese morphism induces a fibration $\psi \colon S \to C \subset \mathrm{Alb}(S)$ over a curve of genus 2 and ψ is an étale bundle with genus 2-fiber F. In particular S is the quotient of $C' \times F$ by the diagonal G-actions where $F/G = \mathbb{P}^1$ and $C' \to C = C'/G$ is étale. Then our study relies on the Bolza classification in [Bl] of G-actions on a genus-2 curve and the wanted description of all the irrational pencils is finished in Theorem 4.32. We like to point out that surfaces with $p_g = q = 2$ not of Albanese general type are the first occurrences of the concept of G(eneralized)H(yperelliptic)-Surface studied in [Ca], [Z3]. In particular our final Proposition 4.2 follows by the clear description of the irreducible components of the moduli space of GH-surfaces given in [Ca, Theorem B].

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1 Notation and Vocabulary

Let X be an algebraic variety of dimension n. Let Ω^1_X, $\omega_X^{\otimes m}$ be the sheaves of holomorphic 1-forms and respectively the m-tensor power of the holomorphic n-forms. The graded ring $\bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$ is called the *canonical ring* of X and the transcendency degree of its function field over $\mathbb{C}$ is the Kodaira dimension of X which is denoted by $\mathrm{Kod}(X)$. If $\mathrm{Kod}(X) = \dim_{\mathbb{C}}(X) = n$ then X is said to be a *variety of general type*; essentially this means that there exists an m such that the map $\phi_m \colon X \to \mathbb{P}\big(H^0(X, \omega_X^{\otimes m})^\vee\big)$ is birational over its image. The numbers $q = \dim_{\mathbb{C}} H^0(X, \Omega^1_X)$, and $P_m = \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes m})$ are called the *irregularity* and respectively the *m-plurigenera* of X. If $q > 0$, X is called an *irregular variety*. In the case of an algebraic surface, thanks to its special role, $P_1 = \dim_{\mathbb{C}} H^0(S, \omega_S)$ is called also *geometrical genus* and it is traditionally denoted by p_g. The basic invariants of S are the *Euler-Poincaré characteristic* $\chi(S) = p_g - q + 1$ and the number K_S^2 which is the autointersection of the *canonical class* K_S. In our paper, S will be a surfaces of general type with $p_g = q = 2$. In particular $\chi(S) = 1$.

A *fibration* $f \colon X \to Z$ is a morphism with connected fiber F over a smooth variety Z. In the case of a surface, a fibration $f \colon S \to B$ is also called $g(B)$-*pencil of genus* g due to the fact that B is a smooth curve an it is custom to denote by g the genus of the smooth fibers. If $g(B) > 0$, f is also called *irrational pencil* and if B is an elliptic curve sometimes f has been called *elliptic pencil* even if this can be ambiguous because one could think it refers to the genus g of the fibers, but in our case g is always > 1. A fibration is called *relatively minimal* if there are no -1-curves contained in a fiber. The *dualising sheaf* $\omega_S \otimes f^\star \omega_B^\vee$ is the main tool to study a relatively minimal fibration and the number

$$\chi_f = \chi_S - \big(g(B) - 1\big)(g - 1) = \deg f_\star(\omega_S \otimes f^\star \omega_B^\vee)$$

is one of the most important invariants. A fibration $f \colon S \to B$ is said to be *isotrivial* if there exist a Galois base change $C \to B$ such that the pull back of f can be relatively blown down to a product projection $C \times F \to C$.

The variety $A = \mathrm{Alb}(X) = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z}) /$ Tors is the *the Albanese variety* of X and it turns to be an algebraic tori; we refer to [Be1, p. 81-86] for the main facts about A. Here we only recall that for an irregular variety it remains defined by integrations of the 1-forms along the paths inside X a map $\alpha \colon X \to A$ called *the Albanese morphism*. For a surface S, α can be only generically finite over its image $\alpha(S)$ and in this case S is called of *Albanese general type* or $\alpha(S) = C$ is a smooth curve of genus q and the fibres of α are connected *cf.* [Be1, Proposition V.15]. If $p_g = q = 2$ then A is an algebraic 2-dimensional tori and $\alpha(S) = A$ or $\alpha(S) = C \subset A$ where C is a smooth curve of genus 2.

2 Irrational Pencils Are Isotrivial

From the theory of fibrations we will use the following proposition.

Theorem 2.1 *Let S be a smooth surface of general type and let $f \colon S \to B$ be a fibration with general fiber F of genus $g \geq 2$ over a curve B of genus b. Then $\chi_f \geq 0$. Moreover*

$\chi_f = 0$ if and only if f is isogenous to a product i.e. there exists an unramified base change $B' \to B$ such that the fiber product $S \times_B B'$ is fiberwise isomorphic to the product $F \times B'$.

Proof See [Be2]. ■

We have two different cases for surfaces with $p_g = q = 2$.

Lemma 2.2 *Let S be a surface of general type with $p_g = q = 2$. Then S is of Albanese general type or the Albanese morphism induces an étale bundle $\phi\colon S \to C \subset A$ with fiber F of genus 2.*

Proof Assume that α is not generically finite. Then by [Be1, Proposition V.15] there exists a genus 2 curve $C \subset A$ and a fibration $\phi\colon S \to C$ which factorizes α. Let g be the genus of the smooth fibers of ϕ. Since $\mathrm{Kod}(S) = 2$, by Noether-Enriques theorem *cf.* [Be1, Théorème III.4] and by [Be1, Proposition IX.3] $g \geq 2$. By Theorem 2.1, $1 = \chi(S) \geq g - 1$. Then $g = 2$ and $\chi_f = 0$. By Theorem 2.1 it follows that $\phi\colon S \to C$ is an étale-bundle. ■

From now on in this section S will be a surface of general type with $p_g = q = 2$ whose Albanese morphism $\alpha\colon S \to \mathrm{Alb}(S)$ is surjective.

Lemma 2.3 *Let S be a surface of Albanese general type with $p_g = q = 2$. Let $\phi\colon S \to B$ be a fibration of genus g. If the genus b of B is > 0 then $b = 1$ and*

$$\phi_\star \omega_S = \mathcal{L} \oplus \mathcal{O}_B \bigoplus_{i=1}^{g-2} \mathcal{O}_B(\eta_i)$$

where $\eta_i \in \mathrm{Tors}\big(\mathrm{Pic}^0(B)\big)$, $\eta_i \neq 0$, $i = 1, \dots, g-2$ and $\mathcal{L}$ is an invertible sheaf of degree 1.

Proof If $b = 2$ then S is not of Albanese general type by the universal property of α, *cf.* [Be1, Remarques V.14 (2)]. Then B is an elliptic curve. By [Fu, Theorem 3.1 p. 786], $\phi_\star\omega_S = \mathcal{O}_B \oplus \mathcal{F}$ where $\mathcal{F}$ is a nef locally free sheaf of rank $g-1$ such that $h^0(B, \mathcal{F}^\vee) = 0$. Let $\mathcal{F} = \bigoplus_{i=1}^k \mathcal{F}_i$ be the decomposition into indecomposable subvectorbundles. Since $p_g = 2$, $h^0(B, \mathcal{F}) = 1$. Then by Riemann-Roch on B, $\deg \mathcal{F} = 1$.

Let $\mathcal{L}$ be the smallest subvectorbundle of $\mathcal{F}$ containing the subsheaf generically generated by $H^0(B, \mathcal{F})$. In particular $h^0(B, \mathcal{L}) = h^0(B, \mathcal{F}) = 1$, $\deg \mathcal{L} > 0$ and it has rank 1; that is $\mathcal{L}$ is an invertible sheaf of degree 1.

Claim *Up to reordering the $\mathcal{F}_i$'s, it holds that $\mathcal{L} = \mathcal{F}_1$.*

$\subset$: The inclusion $J\colon \mathcal{L} \hookrightarrow \mathcal{F}$ gives for each factor the morphism $\pi_i \circ J = J_i\colon \mathcal{L} \to \mathcal{F}_i$. Set $J_i(\mathcal{L}) = \mathcal{L}_i$ and let $d_i = \deg \mathcal{L}_i$. If $\mathcal{L}_i \neq 0$ then $d_i \geq 1$. Since $1 = h^0(B, \mathcal{F}) \geq \sum_{i=1}^k d_i$ then, up to reorder, we have $d_1 = 1$, $d_i = 0$ and $\mathcal{L}_i = 0$ where $i = 2, \dots, k$.

$\supset$: By contradiction. Assume that $\mathcal{F}_1 \neq \mathcal{L}$. By definition of $\mathcal{L}$ the quotient $\mathcal{F}_1/\mathcal{L}$ is locally free. Let

$$\mathcal{F}_1/\mathcal{L} = \bigoplus_{i=1}^{m} \mathcal{F}_{1,i}$$

be a direct sum of its indecomposable components. Note that: $\deg \mathcal{F}_{1,i} = 0$. Then

$$\text{(1)} \qquad \begin{aligned} H^1\Bigg(B, \mathcal{L} \otimes \Big(\bigoplus_{i=1}^{m} \mathcal{F}_{1,i}\Big)^{\vee}\Bigg) &= \bigoplus_{i=1}^{m} H^1\big(B, \mathcal{L} \otimes (\mathcal{F}_{1,i})^{\vee}\big) \\ &= \bigoplus_{i=1}^{m} H^0(B, \mathcal{L}^{\vee} \otimes \mathcal{F}_{1,i}) = 0. \end{aligned}$$

Now (1) and [Ha, III.6.2] imply that

$$0 \to \mathcal{L} \to \mathcal{F}_1 \to \bigoplus_{i=1}^{m} \mathcal{F}_{1,i} \to 0$$

splits: a contradiction, since $\mathcal{F}_1$ is indecomposable.

Claim *$\mathcal{F}_j \in \operatorname{Tors}\big(\operatorname{Pic}^0(B)\big) \setminus \{0\}$ if $j = 2, \dots, k$.*

By Atiyah classification of vector bundles over an elliptic curve, [At], for the rank 2 case *cf.* [Be1, Proposition III.15(ii)] we have $\mathcal{F}_j = F_{r_j} \otimes T_j$ where the rank-r_j sheaves F_{r_j} are obtained inductively through non trivial extensions by $\mathcal{O}_B$. Besides $T_j \in \operatorname{Pic}^0(B)$. We want to prove that T_j is torsion and $r_j = 1$.

First Step: T_j is Torsion Let $\Lambda \in \operatorname{Pic}^0(B) \setminus \{0\}$. By the Serre duality and by the projection formula it holds:

$$\text{(2)} \qquad h^2\big(S, K_S + \phi^{\star}(\Lambda)\big) = h^0\big(S, -\phi^{\star}(\Lambda)\big) = h^0(B, -\Lambda) = 0.$$

By the Leray spectral sequence for the morphism ϕ,

$$h^1\big(S, K_S + \phi^{\star}(\Lambda)\big) = h^1(B, \Lambda) \oplus h^1(B, \mathcal{L} \otimes \Lambda) \bigoplus_{j=1}^{n} h^1(B, \mathcal{F}_j \otimes \Lambda) + h^0(B, \Lambda).$$

Then by Riemann-Roch on B and by relative duality we have:

$$\text{(3)} \qquad h^1\big(S, K_S + \phi^{\star}(\Lambda)\big) = \bigoplus_{j=1}^{n} h^1(B, \mathcal{F}_j \otimes \Lambda).$$

Choose $\Lambda = -T_i$ where $2 \leq i \leq k$. Then we have a jump in cohomology and by the Simpson solution of the Beauville-Catanese conjecture [Sim, Theorem 4.2 p. 373 and Section V], we have that T_i are torsion sheaves.

Second Step: $r_j = 1$ Only for clarity reasons we assume that the torsion sheaves are of relative prime order. By contradiction assume that there exists $2 \leq i \leq k$ with $r_i > 1$.

Let $\tau_i \colon E_i \to B$ be the unramified covering given by T_i and denote by $f_i \colon S_i = S \times_B E_i \to E_i$ and by $\sigma_i \colon S_i \to S$ the two projections. Then $\tau_i^\star \mathcal{F}_i = F_{r_i}$, $\omega_{S_i} = \sigma_i^\star \omega_S$ and $f_{i\star}\sigma_i^\star \omega_S = \tau_i^\star \phi_\star \omega_S$. In particular, F_{r_i} is a direct summand of $f_{i\star}\omega_{S_i}$. Then $h^1(E_i, f_{i\star}\omega_{S_i}) = 2$. By the just quoted result in [Fu], $\mathcal{O}_B$ must be a direct summand of $f_{i\star}\omega_{S_i}$; this forces F_{r_i} to be decomposable: a contradiction. ■

Corollary 2.4 *Any irrational pencil over S is an elliptic pencil of genus $2 \leq g \leq 5$.*

Proof By Stein factorization, every irrational pencil gives an elliptic fibration; choose $\phi \colon S \to B$ one of these fibrations. Following Lemma 2.3 even in the notation, we have a direct summand $\mathcal{L} \hookrightarrow \phi_\star \omega_S$ where $\mathcal{L}$ is an invertible sheaf of degree 1. Then there exists a section of $\mathcal{L}$ which vanishes only over one point $P \in B$. Then by Xiao's method *cf.* [BZ], it follows that $K_S - D$ is nef where $D = \phi^{-1}(P)$. Then $(K_S - D)K_S \geq 0$. In particular by Miyaoka's inequality $9\chi(S) \geq K_S^2$ it follows $9 \geq K_S D$ since $\chi(S) = 1$. The genus $b > 0$, then $D^2 = 0$ and now the inequality $2 \leq g \leq 5$ easily follows by the adjunction formula. ■

By the universal property of the Albanese morphism once we have an elliptic pencil $f_1 \colon S \to E_1$ it follows that the Albanese surface A is isogenous to a product of elliptic curves $E_1 \times E_2'$. In particular there exists a map $h_2 \colon S \to E_2'$. By the Stein factorization of h_2 we find a fibration $f_2 \colon S \to E_2$ where $E_2 \to E_2'$ is obviously an unramified covering. So we can introduce the following number:

$$N = \min\{m \mid m = \deg(A \to E_1 \times E_2)\}$$

an we will call the set $\{(f_1, E_1), (f_2, E_2)\}$ a *good couple* if $E_1 \times E_2$ realizes the minimum N where f_1 and f_2 are the induced fibrations.

Corollary 2.5 *If S has an elliptic pencil $f_1 \colon S \to E_1$ then there exists a good couple $\{(f_1, E_1), (f_2, E_2)\}$.*

Proof Trivial. ■

Definition 2.6 The two elliptic fibration $f_i \colon S \to E_i$, $i = 1, 2$ associated to a good couple will be called *natural fibrations* of S. Moreover F_i will denote the general fibre of f_i and g_i will denote the genus of F_i where $i = 1, 2$.

For further reference we sum up some results on irrational pencils:

Proposition 2.7 *If $\{(f_1, E_1), (f_2, E_2)\}$ is a good couple then $2g_i - 2 \geq F_1F_2$ where $i = 1, 2$.*

Proof By the proof of Corollary 2.4 we know that $K_S - F_1$ is nef. Then $2g_1 - 2 = K_S F_1 \geq F_2 F_1$. The same holds with 1 and 2 interchanged. ■

The following theorem shows that the Jacobians of the smooth fibers of an irrational pencil $f_1 \colon S \to E_1$ dominate a fixed Abelian variety of dimension $g_1 - 1$; under our hypothesis this forces isotriviality if $g_1 > 2$.

Theorem 2.8 *Every elliptic fibration with fiber of genus $g > 2$ over a surface with $p_g = q = 2$ is an isotrivial pencil.*

Proof Let $f_1 \colon S \to E_1$ be one of the two natural fibrations of Definition 2.6 and assume that $g_1 > 2$. By Lemma 2.3 it holds:

$$f_{1\star}\omega_S = \mathcal{L} \oplus \mathcal{O}_{E_1} \bigoplus_{i=1}^{g-2} \mathcal{O}_{E_1}(\eta_i) \tag{4}$$

where $\eta_i \in \operatorname{Tors}\big(\operatorname{Pic}^0(E_1)\big) \setminus 0$ and $\mathcal{L}$ is an invertible sheaf of degree 1. Let $\sigma \colon E \to E_1$ be the unramified base change given by $\operatorname{lcm}\{\eta_i\}_{i=1}^{g-2}$. $X = S \otimes_{E_1} E$ is connected. Let $\psi \colon X \to E$, $\tau \colon X \to S$ be the projections. It is easy to see that from (4):

$$\psi_\star \omega_X = \tau^\star \mathcal{L} \oplus \bigoplus_{i=1}^{g-1} \mathcal{O}_E. \tag{5}$$

By [Fu, Theorem 3.1] it follows that $q(X) = g$. By the universal property, the Jacobian over the smooth fiber has a surjection $\mu_t \colon J(F_t) \to A$ where $t \in E$, F_t is smooth and A is a *fixed* Abelian subvariety of $\operatorname{Alb}(X)$. In particular, up to isogenies over A, there exists an elliptic curve $E_t \hookrightarrow J(F_t)$ such that

$$0 \to E_t \to J(F_t) \to A \to 0 \tag{6}$$

is exact. Now if E_t is independent of t then the Abelian variety rigidity imply that $J(F_t)$ is fixed. Then the smooth fibers are isomorphic. This means that f_1 is isotrivial.

Assume that the E_t's are a non constant family. By Proposition 2.7 the generic smooth F_t is equipped with two morphisms: $F_t \to E_2$, $F_t \to E_t$ where E_2 is the fixed elliptic curve which is the basis of the other natural fibration $f_2 \colon S \to E_2$. Since E_t moves in a continuous family, the induced morphism into the product $\nu_t \colon F_t \to E_2 \times E_t$ is an immersion. By adjunction, it easily follows that $\rho_a(F_t) \geq 5$. In particular $F_t \to E_2$ is a 2-to-1 morphism. The same argument works interchanging the role between f_1 and f_2. Then $S \to E_1 \times E_2$ is a generically finite 2-to-1 covering whose branch locus $\Delta \in |2\delta|$ satisfies the conditions$\delta(E_1 \times \{y\}) = 4$, $\delta(\{x\} \times E_2) = 4$ where $x \in E_1$ and $y \in E_2$. On the other hand $\delta^2 \leq 2$ since $9 \geq K_S^2 \geq 4\delta^2$. It is easy to see that δ does not exist. ■

We have shown Theorem [A]. For further use we prove:

Proposition 2.9 *Every surface S of Albanese general type with $p_g = q = 2$ equipped with an irrational pencil with fiber of genus $g > 2$ can be realized as the minimal desingularization of the quotient surface X/G, where $X = B_1 \times B_2$, G acts diagonally over X and faithfully over the smooth curves B_1, B_2. Moreover $B_1/G = E_1$, $B_2/G = E_2$ where E_1, E_2 are the elliptic curves of a good couple.*

Proof By Stein factorization the pencil induces on S an elliptic fibration. By Theorem 2.8 and by [Ca, Proposition 3.15] the claim follows since the two pencils are distinct. ■

3 Elliptic Isotrivial Pencils

By Proposition 2.9 the classification task is reduced to classify all the Galoisian G actions over curves B_1 and B_2 of genus at most $2 \le b_i \le 5$, $i = 1, 2$ such that the quotient curves $B_1/G = E_1$, $B_2/G = E_2$ are elliptic and the diagonal G action over $X = B_1 \times B_2$ has quotient with $p_g = q = 2$. We have a linear representation $G \to \mathrm{GL}\big(H^0(B_1, \omega_{B_1})\big)$ *cf.* [Se, Cap.2]. We decompose $H^0(B_1, \omega_{B_1})$ into the direct sum of the irreducible representations and we group the isomorphic representations. Let $G^\star = \{\chi_1, \dots, \chi_h\}$ be the set given by the characters of the irreducible representations of G and let $V^{(1)}_\chi$ be the direct sum of the irreducible representations with character χ. We do the same for B_2.

Lemma 3.1 *Every surface S of Albanese general type with $p_g = q = 2$ equipped with an isotrivial pencil can be realized as the minimal desingularization of a quotient surface X/G if and only if there exists a faithful G action on the two smooth curves B_1, B_2 of genus $2 \le b_i \le 5$, $i = 1, 2$ such that for the two decompositions $H^0(B_1, \omega_{B_1}) = \bigoplus_{\chi \in G^\star} V^{(1)}_\chi$, $H^0(B_2, \omega_{B_2}) = \bigoplus_{\chi \in G^\star} V^{(2)}_\chi$ it holds that there exists a unique nontrivial character χ with $V^{(1)}_\chi \otimes V^{(2)}_{\chi^{-1}} \neq 0$. Moreover the following two numerical conditions hold:*

1) $\dim_{\mathbb{C}} V^{(1)}_{\mathrm{id}} = \dim_{\mathbb{C}} V^{(2)}_{\mathrm{id}} = 1$ *and*
2) $\dim_{\mathbb{C}} V^{(1)}_{\chi} = \dim_{\mathbb{C}} V^{(2)}_{\chi^{-1}} = 1$.

Proof The direct proof is easy; otherwise it follows as in *cf.* [Z2, Lemma 1.3] and [Z2, Theorem 1.4]. ■

We need to compute the G-actions over F where $2 \le g(F) \le 5$ and F/G is elliptic.

Lemma 3.2 *Let $\pi\colon F \to E$ be a Galois morphism with group G such that E is an elliptic curve and F has genus $2 \le g \le 5$. Then the occurring actions are listed in appendix [I], where V^i_χ means that the χ-piece of $H^0(F, \omega_F)$ has dimension i. We have denoted by U^2, U^2_i the irreducible subspaces of dimension 2 for the $\mathcal{S}_3$-representations and for the $\mathcal{D}_4$-representations only (two of) the* linear *characters occur.*

Proof It is an application of the Riemann Hurwitz formula plus a careful analysis on the action over the branch loci. We stress that by Riemann-Hurwitz G is of order ≤ 8.

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Here we show how the quaternion actions over a genus 5 curve with elliptic quotient can be excluded. In fact the orbifold exact sequence *cf.* [Ca, Definition 4.4] for these actions is:

$$0 \to \pi(F) \to \langle a, b, x, y \mid x^2 = y^2 = xy[ab] = 1 \rangle \xrightarrow{\mu} \mathcal{H} \to 0$$

where $\mathcal{H}$ is the quaternion group. Since $\mathcal{H}$ has only one element of order 2, denote it by -1, then $\mu\colon x \mapsto -1$ and $\mu\colon y \mapsto -1$. In particular $[\mu(a), \mu(b)] = 1$ and μ cannot be surjective: a contradiction. Using the fact that the dihedral group of order 8, $\mathcal{D}_4$, has one normal subgroup of order 2 we can compute also this action. ■

Now the final step is to use Proposition 2.9 and Lemma 3.1 to construct the quotient surface.

Construction To exclude the non Abelian cases it requires only to couple the possible actions to see that it never happens that $p_g = q = 2$. The computation for the Abelian case can be easily done. The two solutions correspond to a $\mathbb{Z}_2$ diagonal action on the product of two genus-2 curves and to a diagonal $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ action over the product of two genus 3 curves; in this last case G acts on the two factors via the action $V^1_{\chi_1} \oplus V^1_{\chi_2}$ and respectively $V^1_{\chi_1} \oplus V^1_{\chi_{12}}$. A more geometrical construction can be achieved following [Z2]. We have:

Theorem 3.3 *There are only two classes $\mathcal{M}_{\mathbb{Z}_2}$, $\mathcal{M}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ of Albanese general type surfaces with $p_g = q = 2$ and with an isotrivial pencil. An element $S \in \mathcal{M}_{\mathbb{Z}_2}$ is the minimal desingularization of the quotient surface $B_1 \times B_2/G$ where $G = \mathbb{Z}_2$, B_1, B_2 have genus $b_1 = b_2 = 2$ and $B_i \to E_i = B_i/G$ is a Galois covering branched over two points of the elliptic curve E_i. In this case the general fiber F of the canonical morphism is obtained by the smoothing of two curves of genus 2 which intersect in two points, then it has genus 5. An element $S \in \mathcal{M}_{\mathbb{Z}_2\times\mathbb{Z}_2}$ is the minimal desingularization of $C_1 \times C_2/G$ where $G = (\mathbb{Z}_2)^2$, C_1, C_2 have genus $c_1 = c_2 = 3$ and $C_i \to E_i = C_i/G$ is a Galois covering branched over two points of E_i. In this case the general fiber F of the canonical map is obtained by the smoothing of two curves of genus 3 which intersect in four points, then it has genus 9.*

In particular we have proved Theorem [B] and Proposition [C].

4 Surfaces With $p_g = q = 2$ and Non Surjective Albanese Morphism

In this section S will be a surface of general type with $p_g = q = 2$ and with non surjective Albanese morphism. We will show that the fibration given by Lemma 2.2 $\phi\colon S \to C \subset \mathrm{Alb}(S)$ is isogenous to a product *cf.* Theorem 2.1 and that S is a Generalized Hyperelliptic surface. In fact these surfaces are baby examples of GH-surfaces.

Generalized Hyperelliptic Surfaces The following definition is in [Ca], see also [Z3]. Let C_1, C_2 be two smooth curves with the corresponding automorphisms groups: $\mathrm{Aut}(C_1)$, $\mathrm{Aut}(C_2)$. Let G be a non trivial finite group G with two injections: $G \hookrightarrow \mathrm{Aut}(C_1)$, $G \hookrightarrow \mathrm{Aut}(C_2)$.

Definition 4.1 The quotient surface $S = C_1 \times C_2/G$ by the diagonal action of G over $C_1 \times C_2$ is said to be of *Generalized Hyperelliptic type* (GH) if
(i) the Galois morphism $\pi_1 \colon C_1 \to C = C_1/G$ is unramified;
(ii) the quotient curve C_2/G is isomorphic to $\mathbb{P}^1$.

Proposition 4.2 *If S is not of Albanese general type with $p_g = q = 2$ then S is* GH.

Proof By Lemma 2.2 we know that $\alpha \colon S \mapsto \mathrm{Alb}(S)$ induces a fibre bundle of genus 2 over a curve C of genus 2: $\phi \colon S \to C \subset \mathrm{Alb}\, S$. Now let us call C_2 the fiber of ϕ. Then there exists a group G acting on two curves C_1, C_2 such that $\pi_1 \colon C_1 \to C_1/G = C$ is étale, $g(C_2) = 2$ and the pull-back $Y = C_1 \times_C S$ is isomorphic to $C_1 \times C_2$. In particular $g(C_1) > g(C) = g(C_2) = 2$ and G acts diagonally on Y. Then $g(C_2/G) = 0$ since [S, Proposition 2.2] and S is GH. ■

The *Viceversa* of Proposition 4.2 holds in a strong form:

Proposition 4.3 *Let C_1, C_2 and G as above. If $g(C_2) = 2$, $C_2/G = \mathbb{P}^1$ and $\pi_1 \colon C_1 \to C_1/G = C$ is an étale morphism where $g(C) = 2$ then the quotient $S = C_1 \times C_2/G$ by the diagonal action is a minimal smooth surface of general type with $p_g(S) = q(S) = 2$ and non surjective Albanese morphism.*

Proof Since G acts freely over $C_1 \times C_2 = Y$ then S is minimal, smooth, of general type and $\chi(Y) = n\chi(S)$, $K_S^2 = 1/nK_Y^2$ being n the order of G. By Riemann-Hurwitz $g(C_1) = n + 1$ then $p_g(Y) = 2(n + 1)$ and $q(Y) = n + 3$; that is $\chi(Y) = n$. Then $\chi(S) = 1$ and by [S, Proposition 2.2] it follows $p_g(S) = q(S) = 2$ since $g(C) = 2$. In particular $g(F/G) = 0$. Let us consider the natural fibration $S \to C$. Since

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d] \arrow[dr] & \mathrm{Alb}(S) \arrow[d] \\ C \arrow[r] & J(C) \end{tikzcd}$$

is commutative, then α is not surjective. ■

From Proposition 4.3 we can extract the following property which is a standard feature of GH-surfaces.

Remark 4.4 Let S be a minimal surface with $p_g = q = 2$ such that $\alpha(S) = C$ is a curve. Then $\mathrm{Alb}(S) = J(C)$.

Notice that since S is GH, $S \to C_2/G = \mathbb{P}^1$ *is not the canonical map.*
By Proposition 4.3 we have the following classification criteria

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Remark 4.5 To classify S is equivalent (i) to classify all the G actions over a genus 2 curve C_2 such that $C_2/G = \mathbb{P}^1$ and (ii) for each occurrence of G in (i) to classify all the étale morphisms $\pi_1 \colon C_1 \to C_1/G = C$ where $g(C) = 2$.

We give a modern way to solve (i) and so we will have a classification of S following [Ca, Theorem B].

The classification of all the couples (C, G) where C is a curve of genus 2 and G is a subgroup of $\operatorname{Aut}(C)$ was obtained by Bolza [Bl]. However, probably because he assumed it to be trivial (at least for our understanding of his proof) he did not specify the different dihedral actions. In every case we rewrite it again because we adopt a completely different approach which should be generalizable to the other hyperelliptic curves and also because we give a recipe to compute easily each action.

Weighted Projective Spaces Let C be a curve of genus 2. By the embedding $C \hookrightarrow \operatorname{Proj}(\mathcal{R})$ where $\mathcal{R} = \bigoplus_{m\geq 0} H^0(C, \omega_C^{\otimes m})$, C can be seen as an hypersurface $C = \operatorname{Proj}(\mathcal{R})$ of the weighted projective space $\mathbb{P}(1,1,3) = \operatorname{Proj} \mathbb{C}[x_0, x_1, z]$ where x_0, x_1 have degree 1 and z has degree 3. We want to find explicitly the action of $\operatorname{Aut}(C)$ over $\operatorname{Proj}(\mathcal{R})$. This description requires some well known facts that we recall since they can be useful for further generalizations.

The genus 2 curve C is the normalization of the projective closure $\bar{C}_0 \subset \mathbb{P}^2$ of $C_0 = \{(x, y) \in \mathbb{A}^2 \mid y^2 = \beta(x)\}$ where $\beta \in \mathbb{C}[x]$, $\deg \beta = 6$ and it has 6 distinct roots. Let $\nu \colon C \to \bar{C}_0$ be the normalization morphism then the hyperelliptic involution $i \colon C \to C$ is induced by the affine automorphism $(x, y) \to (x, -y)$ and

$$\omega_0 = \nu^\star \frac{dx}{y}, \quad \omega_1 = \nu^\star x \frac{dx}{y} \tag{7}$$

give a basis of $H^0(C, \omega_C)$. Set $\eta = \nu^\star y (\frac{dx}{y})^3$. Then

$$\langle \omega_0^3, \omega_0^2\omega_1, \omega_0\omega_1^2, \omega_1^3, \eta \rangle \tag{8}$$

is a basis of $H^0(C, \omega_C^{\otimes 3})$. By (7) it holds

$$i^\star \omega_0 = i^\star \circ \nu^\star \frac{dx}{y} = \nu^\star \circ i^\star \frac{dx}{y} = -\nu^\star \frac{dx}{y} = -\omega_0$$

and in the same way $i^\star\omega_1 = -\omega_1$. In particular if $\rho\colon \operatorname{Aut}(C) \to \operatorname{GL}\big(H^0(C, \omega_C)\big)$ is the natural faithful representation we have that $\rho(i) = -\operatorname{Id}$ and i commutes with every $g \in G$. By the tricanonical morphism there is also a faithful representation $\rho_3 \colon \operatorname{Aut}(C) \to \operatorname{GL}\big(H^0(C, \omega_C^{\otimes 3})\big)$.

Lemma 4.6 *The hyperelliptic involution i is in the center of* $\operatorname{Aut}(C)$. *The action of i splits:*

$$H^0(C, \omega_C^{\otimes 3}) = \mathcal{S}^3 H^0(C, \omega_C) \oplus \eta\mathbb{C}.$$

Moreover the decomposition is preserved by every $g \in \operatorname{Aut}(C)$.

Proof We have just seen that i is central. Obviously $i^\star(\omega_0^s\omega_1^j) = -\omega_0^s\omega_1^j$ with $s + j = 3$ and $i^\star\eta = \eta$; and then we have the claimed decomposition for i. Let $g \in \mathrm{Aut}(C)$ we want to show that $g^\star\eta = \chi(g)\eta$, where $\chi\colon \mathrm{Aut}(C) \to \mathbb{C}$ is a character of $\mathrm{Aut}(C)$. By (8), $g^\star\eta = \sum_{i+j=3} a_{ij}\omega_0^i\omega_1^j + \chi(g)\eta$. Since $g^\star\eta = g^\star i^\star\eta = i^\star g^\star\eta$ then $\sum_{i+j=3} a_{ij}\omega_0^i\omega_1^j + \chi(g)\eta = -\sum_{i+j=3} a_{ij}\omega_0^i\omega_1^j + \chi(g)\eta$; that is $a_{ij} = 0$ for every i, j. ■

Let us consider $\mathbb{P}(1,1,3) = \mathrm{Proj}(\mathbb{C}[x_0,x_1,z])$. The map

$$j\colon \mathcal{R} \to \frac{\mathbb{C}[x_0,x_1,z]}{\big(z^2 - \beta(x_0,x_1)\big)}$$

defined by $\omega_0 \mapsto x_0, \omega_1 \mapsto x_1\ \eta \mapsto z$ is an isomorphism; that is $C = \mathrm{Proj}(\mathcal{R}) = C_6 \subset \mathbb{P}(1,1,3)$. In particular, by j, we can fix once for all an identification $\mathrm{GL}(2,\mathbb{C}) \sim \mathrm{GL}(\mathcal{R}_1)$. Thanks to the interpretation of C as an hypersurface in $\mathbb{P}(1,1,3)$ we have another description of $H^0(C,\omega_C)$ and of $H^0(C,\omega_C^{\otimes 3})$.

Lemma 4.7 *If $C \subset \mathbb{P}(1,1,3)$ and ω is the regular differential induced by $x_0dx_1 - x_1dx_0$ then:*

(i) $H^0(C,\omega_C) = \{(\frac{\omega}{z})P_1(x_0,x_1) \mid P_1(x_0,x_1) \in j(\mathcal{R}), \deg(P_1) = 1\}$;
(ii) $H^0(C,\omega_C^{\otimes 3}) = \{(\frac{\omega}{z})^3P_3(x_0,x_1,z) \mid P_3(x_0,x_1,z) \in \mathbb{C}[x_0,x_1,z], \deg(P_3) = 3\}$.

Proof A local computation. ■

Corollary 4.8 *Via the identification* $\mathrm{GL}(2,\mathbb{C}) \sim \mathrm{GL}(\mathcal{R}_1)$ *it holds:*

(i) *If $G \subset Aut(C)$ then $G \subset \mathrm{GL}(2,\mathbb{C})$;*
(ii) *If $\langle i\rangle$ is the group generated by the hyperelliptic involution then $\langle i\rangle = \{\mathrm{Id}, -\mathrm{Id}\}$;*
(iii) *G acts over z by the character $\chi = \det$.*

Proof Trivial exercise in representation theory. ■

Corollary 4.9

(i) $\chi(i) = 1$.
(ii) *G acts over β via the character $\lambda = \chi^2 = \det^2$.*

Subgroups of $\mathrm{GL}(2,\mathbb{C})$ We have seen that $G \subset \mathrm{GL}(2,\mathbb{C})$. Let $\mathrm{GL}(2,\mathbb{C}) \xrightarrow{\pi} \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ be the canonical projection, $\pi\colon \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto \left[\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right]$ and we set $K = \pi(G)$. We consider the following exact sequences:

$$0 \to \Delta \to \mathrm{GL}(2,\mathbb{C}) \xrightarrow{\pi} \mathbb{P}\,\mathrm{GL}(2,\mathbb{C}) \to 0 \tag{9}$$

$$0 \to \{\pm\,\mathrm{Id}\} \to \mathrm{SL}(2,\mathbb{C}) \xrightarrow{\pi_1} \mathbb{P}\,\mathrm{GL}(2,\mathbb{C}) \to 0. \tag{10}$$

By Corollary 4.8 we can write $\mathrm{Aut}(C) \subset \mathrm{GL}(2,\mathbb{C})$. In particular $\mathrm{Aut}(C) \cap \Delta = \{\pm\,\mathrm{Id}\}$ and this is a motivation for the following definition.

Definition 4.10 Let K be a subgroup of $\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$. Let G be a subgroup of $\mathrm{GL}(2,\mathbb{C})$. G is said to be *extendable* if

$$0 \to \{\pm\,\mathrm{Id}\} \to G \xrightarrow{\pi} K \to 0$$

is exact, and G is said to be *non-extendable* if $\pi|_G\colon G \to K$ is an isomorphism.

Let $B = \{(x_0, x_1) \in \mathbb{P}^1 \mid \beta(x_0, x_1) = 0\}$.

Remark 4.11 Let $G \subset \mathrm{Aut}(C)$ and $K = \pi(G)$. It holds (1) G is extendable iff $i \in G$; (2) B is K-invariant.

Obvious. ∎

The finite subgroups $K \subset \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ are well known [Kl], and it is easy to find the K-invariant polynomials β.

Then we will show that the data K and β *uniquely determine an extendable group* G acting on C.

Proposition 4.12 *Let $K \subset \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ be a finite subgroup and $B = \{\beta(x_0, x_1) = 0\}$ a K-invariant reduced divisor of degree* 6. *There exists a* unique $G \subset \mathrm{GL}(2,\mathbb{C})$ *such that*

(i) *G is a group of automorphisms of $\frac{\mathbb{C}[x_0,x_1,z]}{\left(z^2-\beta(x_0,x_1)\right)}$ and*

(ii) *G is extendable.*

Proof. Unicity If $G_1, G_2 \subset \mathrm{GL}(2)$ satisfy the claim and $h_1 \in G_1 - G_2$, there exists $h_2 \in G_2$ such that $\pi h_1 = \pi h_2$, since $K = \pi G_1 = \pi G_2$. But $i \in G_1 \cap G_2$, then $h_1 = i h_2 \in G_2$: a contradiction.

Existence By (10) we have:

$$0 \to \{\pm\,\mathrm{Id}\} \to \hat{K} \xrightarrow{\pi_1} K \to 0. \tag{11}$$

Using the K-invariance of B, *define* $\lambda\colon \hat{K} \to \mathbb{C}^\star$ by $\left(\pi(\hat{k})\right)^\star(\beta) = \lambda(\hat{k})\beta$. Since $\deg(\beta)$ is even, λ descends to the quotient $K = \frac{\hat{K}}{\{\pm\,\mathrm{Id}\}}$ and we will not distinguish this latter character from λ.

Define

$$G = \left\{ \pm\sqrt{\left(\lambda(k)\right)^{-1}}\hat{k} \mid \hat{k} \in \hat{K}, k = \pi(\hat{k}) \right\}. \tag{12}$$

G is the claimed group. Notice that $G \subset \mathrm{GL}(2,\mathbb{C})$ and G is a group. We can write the G-action over $\mathcal{R}$:

$$\begin{cases} g(x_0) = \pm\sqrt{\left(\lambda(k)\right)^{-1}}k(x_0) \\ g(x_1) = \pm\sqrt{\left(\lambda(k)\right)^{-1}}k(x_1) \\ g(z) = \left(\lambda(k)\right)^{-1} z = \det(g) z \end{cases} \tag{13}$$

where $g = \pm\sqrt{\big(\lambda(k)\big)^{-1}}\hat{k}$. It remains to prove that

$$0 \to \{\pm\,\mathrm{Id}\} \to G \xrightarrow{\pi} K \to 0$$

is exact. By definition π is surjective. Let $g \in \ker\pi$. By (12) there exists $\hat{k} \in \hat{K}$ such that $g = \pm\sqrt{\big(\lambda(k)\big)^{-1}}\hat{k}$. Then $\hat{k} = \pm\,\mathrm{Id}$; in particular $\lambda(k) = 1$ and it follows $g = \pm\,\mathrm{Id}$. ∎

Actually we have shown:

Theorem 4.13 *The couples (C, G) where C is a fixed curve of genus 2 and $G \subset \mathrm{Aut}(C)$ is extendable are in bijection with the classes (K, B) up to $\mathrm{Aut}(\mathbb{P}^1)$ where $K \subset \mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$ is a finite subgroup and B is a degree 6, K-invariant, reduced divisor.*

We want to classify non-extendable groups. If G is non-extendable we need to understand how G fits into $G' = \mathrm{Aut}(C)$. To this end, set $K' = \pi(G')$, and restrict (9) to G':

$$0 \to \{\pm\,\mathrm{Id}\} \to G' \xrightarrow{\pi} K' \to 0. \tag{14}$$

Notice that in general it is *not* true that $G \subset \mathrm{SL}(2, \mathbb{C})$. We have to consider subgroups $K \subset K'$ and their liftings to G'.

Definition 4.14 A subgroup $K \subset K'$ is said to be *of splitting-type* if $(\pi)^{-1}(K)$ is splitted (*i.e.* $(\pi)^{-1}(K) = K \times \{\pm\,\mathrm{Id}\}$). Otherwise it is of non splitting type.

Obviously we have the following remarks that we write for further reference:

Remark 4.15 K is splitting if and only if there exist a lifting $\epsilon\colon K \to G'$ such that $\pi \circ \epsilon = \mathrm{Id}_K$. In particular K is splitting if and only if there exists a nontrivial homomorphism $\epsilon\colon K \to \{\pm\,\mathrm{Id}\}$.

Remark 4.16 If G in (C, G) is non-extendable then $K = \pi(G)$ is of splitting type.

On the other hand if G is extendable both cases for $K = \pi(G)$ may occur. The following case is easy to describe:

Remark 4.17 If G in (C, G) is extendable and $\pi(G) = K$ is of splitting type then $G = K \times \{\pm\,\mathrm{Id}\}$.

The following corollary gives the analogue of Proposition 4.12 for the non-extendable groups:

Corollary 4.18 *Let K and B as in Proposition 4.12. The set of the couples (C, G) where G is non-extendable is in bijection with the set of the liftings $\epsilon\colon K \to G_s$ where G_s is the unique extendable group constructed in the proof of Proposition 4.12 through the data K and B.*

Proof Trivial. ∎

If G is of splitting type there exists a lifting $K \xrightarrow{\epsilon} \mathrm{GL}(2,\mathbb{C})$ and let us denote by $\mu\colon K \to \mathbb{C}^\star$ the character uniquely defined by the relation $\mu(k)\beta = \big(\epsilon(k)\big)^\star(\beta)$.

Proposition 4.19 *The isomorphism class (K, B) up to $\mathrm{Aut}(\mathbb{P}^1)$ induces (C, G_s) where G_s is splitted if and only if there exists*

(i) *a lifting $K \xrightarrow{\epsilon} \mathrm{GL}(2,\mathbb{C})$ and*
(ii) *a square root ν of the character μ associated to ϵ and β.*

Proof Assume that the procedure described in Proposition 4.12 gives a splitted couple (C, G_s). By Remark 4.17, (i) follows. From the proof of Proposition 4.12 and by $\epsilon\colon K \to \mathrm{GL}(2,\mathbb{C})$ we have that in (11)

$$\hat{K} = \left\{ \hat{k} = \pm\sqrt{\Big(\det\big(\epsilon(k)\big)\Big)^{-1}}\epsilon(k) \mid k \in K \right\}. \tag{15}$$

Moreover the character λ to construct the claimed G_s is by definition

$$\lambda(k) = \Big(\det\big(\epsilon(k)\big)\Big)^{-3}\mu(k).$$

By Proposition 4.12

$$G_s = \left\{ \pm\sqrt{\Big(\det\big(\lambda(k)\big)\Big)^{-1}}\hat{k} \mid \hat{k} \in \hat{K} \right\},$$

then by the form $\hat{k}$ in (15) it holds

$$\pm\sqrt{\big(\lambda(k)\big)^{-1}}\sqrt{\Big(\det\big(\epsilon(k)\big)\Big)^{-1}}\epsilon(k) = \pm\sqrt{\bigg(\Big(\det\big(\epsilon(k)\big)\Big)^{-3}\mu(k)\bigg)^{-1}}\sqrt{\Big(\det\big(\epsilon(k)\big)\Big)^{-1}}\epsilon(k)$$

that is, if we set $\rho(k) = \sqrt{\lambda(k)\det(\epsilon(k))^{-1}}$ we can write the claimed square root $\nu(k) = \frac{\det(\epsilon(k))}{\rho(k)}$.

Viceversa Assume that (i) and (ii) hold. We *define* $\rho(k) = \frac{\det(\epsilon(k))}{\nu(k)}$ and the same computation in reverse order shows that $(\epsilon)' = \rho\epsilon$ is a lifting $\epsilon'\colon K \to G_s$. The by Remark 4.15 G_s is splitted. ∎

Bolza Classification In [Kl] Klein shows the finite groups acting on $\mathbb{P}^1$:

Group K	order special orbits	Order of K
$\mathbb{Z}_n$	1, 1	n Cyclic
$\mathcal{D}_n$	n, n, 2	2n Dihedral
$\mathcal{A}_4$	6, 4, 4	12 tetrahedral
$\mathcal{S}_4$	12, 8, 6	24 esahedral or octahedral
$\mathcal{A}_5$	30, 30, 12	60 Icosahedral or dodecahedral

Table 1

In our case B is a reduced K-invariant divisor then $\mathcal{A}_5$ does not occur. In the same book we find the groups $\hat{K}$ such that $0 \to \{\pm \mathrm{Id}\} \to \hat{K} \to K \to 0$ is exact:

Group $\hat{K}$	Generator	Relations
$\mathbb{Z}_{2n}$	$\zeta = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix}$	$\langle \zeta \mid \zeta^{2n} = 1 \rangle$
$\mathcal{D}_n \rtimes \mathbb{Z}_2$	$\zeta = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix}, \eta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\left\langle \zeta, \eta \,\middle\vert\, \begin{matrix} \zeta^{2n}=\eta^4=1 \\ \zeta^n=\eta^2 \\ \eta^2\zeta=\zeta\eta^2 \end{matrix} \right\rangle$
$\hat{\mathcal{A}}_4$	$\zeta=1/2(i-1)\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ $\eta=(i+1)\begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}$	$\left\langle \zeta, \eta \,\middle\vert\, \begin{matrix} \zeta^3=\eta^3=\eta\zeta^4=1 \\ (\zeta\eta)^2=(\eta\zeta)^2 \\ (\eta\zeta)^2\zeta=\zeta(\eta\zeta)^2 \\ (\eta\zeta)^2\eta=\eta(\eta\zeta)^2 \end{matrix} \right\rangle$
$\hat{\mathcal{S}}_4$	$\zeta=1/2(i-1)\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ $\eta=\frac{1}{\sqrt{2}}\begin{pmatrix} i+1 & 0 \\ 0 & 1-i \end{pmatrix}$	$\left\langle \zeta, \eta \,\middle\vert\, \begin{matrix} \zeta^3=\eta^8=\eta\zeta^4=1 \\ \eta^4\zeta=\zeta\eta)^4 \end{matrix} \right\rangle$

Table 2

If one wants to look directly to the quoted book, notice that if $\hat{K} = \hat{\mathcal{A}}_4$ then, in the book's notation, we have:

$$\zeta\eta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \eta\zeta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

while if $\hat{K} = \hat{\mathcal{S}}_4$ then

$$\eta^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (\eta\zeta)^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\zeta\eta)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have just noticed that the case $\mathcal{A}_5$ does not occur but it is not the unique one:

Lemma 4.20 *The groups of Table 1 which give a couple (C, G) where G is extendable are completely classified in Table 3.*

K	G	C
$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$z^2 = x_1^6 - x_0^6$
$\mathbb{Z}_5$	$\mathbb{Z}_1 0$	$z^2 = x_0(x_1^5 - x_0^5)$
$\mathbb{Z}_4$	$\mathbb{Z}_8$	$z^2 = x_1 x_0 (x_1^4 - x_0^4)$
$\mathbb{Z}_3$	$\mathbb{Z}_6$	$z^2 = (x_1^3 - x_0^3)(x_1^3 + x_0^3)$
$\mathbb{Z}_2$	$\mathbb{Z}_4$	$z^2 = (x_1^2 - x_0^2)(x_1^2 - 4x_0^2)(x_1^2 - 9x_0^2)$
$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$z^2 = x_1 x_0 (x_1^2 - x_0^2)(x_1^2 - 4x_0^2)$
$\langle \mathrm{id} \rangle$	$\mathbb{Z}_2$	$z^2 =$ generic polynomial of degree 6
$\mathcal{D}_6$	$\left\langle TU \;\middle\vert\; \begin{matrix} T^2 = U^6 = (TU)^4 = 1 \\ T(TU)^2 = TU^2T \\ U(TU)^2 = (TU)^2U \end{matrix} \right\rangle$	$z^2 = (x_1^6 - x_0^6)$
$\mathcal{D}_4$	$\left\langle TU \;\middle\vert\; \begin{matrix} T^2 = U^8 = (UT)^4 = 1 \\ TU^4 = U^4T \end{matrix} \right\rangle$	$z^2 = x_0 x_1 (x_1^4 - x_0^4)$
$\mathcal{D}_3$	$\mathcal{D}_6$	$z^2 = (x_0^3 - 2x_1^3)(x_1^3 - 2x_0^3)$
$\mathcal{D}_3$	$\left\langle TU \;\middle\vert\; \begin{matrix} T^3 = U^4 = (TU)^4 = 1 \\ (TU)^2 = U^2 \\ TU^2 = U^2T \end{matrix} \right\rangle$	$z^2 = (x_1^6 - x_0^6)$
$\mathcal{D}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$z^2 = x_0 x_1 (x_1^2 - 4x_0^2)(x_0^2 - 4x_1^2)$
$\mathcal{D}_2$	$\mathcal{H}$	$z^2 = x_0 x_1 (x_0^4 - x_1^4)$
$\mathcal{A}_4$	$\hat{\mathcal{A}}_4$	$z^2 = x_0 x_1 (x_0^4 - x_1^4)$
$\mathcal{S}_4$	$\left\langle TU \;\middle\vert\; \begin{matrix} T^3 = U^8 = (UT)^2 = 1 \\ TU^4 = U^4T \end{matrix} \right\rangle$	$z^2 = x_0 x_1 (x_0^4 - x_1^4)$

Table 3

Proof

First Step: To Find the Occurrences of β This is easy since for each K in Table 1 we have to find which unions of orbits have order 6.

Second Step: To Find G By the first step we know B and K. The procedure described in Proposition 4.12 gives the result. For example, we show how to obtain the extendable group associated to $\mathcal{S}_4$. In particular we will see that it is *different* from $\hat{\mathcal{S}}_4$. Let us consider Table 2. We find two generators of $\hat{\mathcal{S}}_4$:

$$\zeta = 1/2(i-1)\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\eta = \frac{1}{\sqrt{2}}\begin{pmatrix} i+1 & 0 \\ 0 & 1-i \end{pmatrix}.$$

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The character λ, to start with Proposition 4.12, can be obtained by the following game:

$$\lambda(\zeta)\beta = (\zeta)^\star(\beta) = \beta$$
$$\lambda(\eta)\beta = (\eta)^\star(\beta) = -\beta.$$

By construction, the solution is $G = \left\{ \pm\sqrt{\left(\lambda(k)\right)^{-1}}\hat{k} \mid \hat{k} \in \hat{\mathcal{S}}_4 \right\}$ and in G there are:

$$T = (\zeta)^2 = i/2(i-1)\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
$$U = i\eta = \frac{1}{\sqrt{2}}\begin{pmatrix} i-1 & 0 \\ 0 & i+1 \end{pmatrix}.$$

Let $H = \langle T, U\rangle$ be the subgroup generated by T and U. We want to show that $H = G$. Obviously: $H \subset G$ and $U^4 = -\operatorname{Id}$. Then we can restrict (9) to H and it gives $K_1 \subset \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ such that

$$0 \to \langle U^4\rangle \to H \xrightarrow{\pi} K_1 \to 0$$

is exact. The task is to show that $K_1 = \mathcal{S}_4$. That is, we have to show that $\mathcal{S}_4 \subset K_1$. Set $u = \pi(U)$ and $t = \pi(T)$. From Table 2 we have $u = \pi(\eta)$ and $t = \pi(\zeta^2)$. We conclude by (10) restricted to $\mathcal{S}_4$. ■

Remark 4.21 The way we prove Lemma 4.20 gives an explicit description of the extendable groups as subgroups of $\mathrm{GL}(2,\mathbb{C})$. They are listed in Table 4: see appendix [II].

To end the classification we have to find which groups in Table 3 are splitted and in the affirmative case to classify all the liftings $\epsilon\colon K \to G$ *such that* $C/\epsilon(K) = \mathbb{P}^1$. It requires only a few basic facts on curves theory: essentially that if C is a genus 2 curve, $G \subset \operatorname{Aut}(C)$ and C/G is elliptic then $G = \mathbb{Z}_2$; this follows from the Hurwitz formula and the easy monodromy argument that an Abelian covering over an elliptic curve has at least two branch points. In some cases to find the splitted group G_s, instead of Proposition 4.19, we can use a more direct argument.

Lemma 4.22 *If $K = \mathcal{A}_4$ or $K = \mathcal{S}_4$ then K is non splitting.*

Proof The proofs are similar. We only do the case $K = \mathcal{S}_4$. If $\mathcal{S}_4$ were splitting, then the corresponding extendable group in Table 3 would be $G = \mathcal{S}_4 \times \mathbb{Z}_2$; a contradiction, because in G there is an element of order 8. ■

The Dihedral Case If $K = \mathcal{D}_n$ we like to consider two cases depending on the parity of n.

Lemma 4.23 *Let $K = \mathcal{D}_n$, then K is splitting if and only if n is odd.*

Proof $\mathcal{D}_n \subset \mathbb{P}\,\mathrm{SL}(2)$ is given by:

$$\left\langle \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}\right] \right\rangle.$$

If $\mathcal{D}_n \xrightarrow{\epsilon} \mathrm{GL}(2)$ is a lifting, the preimage of $\left[\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right]$ is $\left\{\pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\}$ and every liftings of $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & \xi \end{smallmatrix}\right)\right]$ has the following form: $\xi^i\left(\begin{smallmatrix} 1 & 0 \\ 0 & \xi \end{smallmatrix}\right)$. By definition, ϵ is an isomorphism over its image then the relation defining $\epsilon(\mathcal{D}_n)$ forces

$$\xi^i \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} = \left(\xi^i \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}\right)^{-1}$$

to hold. Then $\xi^{2i+1} = 1$, which has a solution if and only if n is odd. ∎

By Lemma 4.23 and by Table 3 the case $K = \mathcal{D}_n$ splitting is achieved applying Proposition 4.19 to $K = \mathcal{D}_3$. We recall:

Remark 4.24 If n is odd then $\mathbb{Z}_2$ is the group of $\mathcal{D}_n$-linear characters.

Let $\mathcal{D}_3 \xrightarrow{\epsilon} \mathrm{GL}(2)$ be a lifting, for example:

$$\epsilon\left[\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \quad \epsilon\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\xi = e^{2i\pi/3}$. There are two cases. If $\beta = x_1^6 - x_0^6$ the character μ induced by ϵ, β is:

$$\mu \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} = 1, \quad \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

By Remark 4.24, it does not exist a character ν such that $\nu^2 = \mu$. In the other case $\beta = (x_0^3 - 2x_1^3)(x_1^3 - 2x_0^3)$, and we have

$$\mu \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} = 1, \quad \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1.$$

We easily see that $\nu = 1$ satisfy the condition of Proposition 4.19. If $\epsilon_1 = \det(\epsilon)\epsilon$, $\epsilon_1 \colon \mathcal{D}_3 \to \mathcal{D}_6 \subset \mathrm{GL}(2)$ is a lifting and

$$\epsilon_1(\mathcal{D}_3) = G_1 = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

To conclude the case $\mathcal{D}_3$ we need:

Remark 4.25 Let $K \subset \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ with a fixed lifting $K \xrightarrow{\epsilon_1} \mathrm{GL}(2,\mathbb{C})$. Then for every lifting $\epsilon_2 \colon K \to \mathrm{GL}(2,\mathbb{C})$ it holds $\epsilon_2 = \rho\epsilon_1$ where ρ is a character of K.

Then by Remarks 4.25 and 4.24, we have another lifting $\epsilon_2 = \rho\epsilon_1$, with

$$\rho\left[\begin{pmatrix}1 & 0\\ 0 & \xi\end{pmatrix}\right] = 1, \quad \rho\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = -1,$$

and

$$\epsilon_2(\mathcal{D}_3) = G_2 = \left\langle \begin{pmatrix}\xi & 0\\ 0 & \xi^2\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle.$$

We sum up the dihedral case in the following lemma:

Lemma 4.26 *If $K = \mathcal{D}_n$ is splitting then $K = \mathcal{D}_3$, $\beta = (x_0^3 - 2x_1^3)(x_1^3 - 2x_0^3)$ and there are two liftings $\epsilon_i \colon \mathcal{D}_3 \to \mathcal{D}_6 \subset \mathrm{GL}(2,\mathbb{C})$ where $i = 1, 2$ such that if $G_i = \epsilon_i(\mathcal{D}_3)$ then $C/G_i = \mathbb{P}^1$.*

The Cyclic Case Even in the case $K = \mathbb{Z}_n$ the behaviour depends on the parity.

Lemma 4.27 *Let $K = \mathbb{Z}_n$ with n odd. If G is the corresponding extendable group then $G = \mathbb{Z}_{2n}$, and it is splitted. Moreover every lifting $\epsilon \colon K \to G$ gives* the same *subgroup of G and $C/\epsilon(K) = \mathbb{P}^1$.*

Proof An easy computation with the cases $n = 5$, $n = 3$ in Table 3. ■

If n is even we have more cases.

Lemma 4.28 *If n is even, $K = \mathbb{Z}_n$ and B contains some special orbits then the corresponding extendable group is non-splitted.*

Proof The special orbit in $B = \{\beta = 0\}$ is given by $x_0x_1 = 0$. By Table 1 and Table 3 we have to consider only the case with $n = 4$ or $n = 2$. The cyclic subgroup of $\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ is $\left\langle \left[\left(\begin{smallmatrix}1 & 0\\ 0 & \xi\end{smallmatrix}\right)\right]\right\rangle$, where $\xi = e^{2i\pi/n}$. Let $\epsilon\left[\left(\begin{smallmatrix}1 & 0\\ 0 & \xi\end{smallmatrix}\right)\right] = \left(\begin{smallmatrix}1 & 0\\ 0 & \xi\end{smallmatrix}\right)$ be a lifting. Using the notation of Proposition 4.19, $\mu\left[\left(\begin{smallmatrix}1 & 0\\ 0 & \xi\end{smallmatrix}\right)\right] = \xi$. Then it does not exist a character ν of $\mathbb{Z}_n$, such that $\nu^2 = \mu$. By Proposition 4.19 we conclude. ■

By Table 3 we have only two cases to consider: $(K, B) = (\mathbb{Z}_6, x_1^6 - x_0^6)$ and $(K, B) = (\mathbb{Z}_2, x_1^2 - x_0^2)(x_1^2 - x_0^2)(x_1^2 - x_0^2)$.

Lemma 4.29 *Let $(K, B) = (\mathbb{Z}_6, x_1^6 - x_0^6)$ then K is splitting and it has two liftings $\epsilon_i : \mathbb{Z}_6 \to \mathbb{Z}_{12} \subset \mathrm{GL}(2,\mathbb{C})$, $i = 1, 2$. Letting $G_i = \epsilon_i(\mathbb{Z}_6)$ it holds that $C/G_i = \mathbb{P}^1$ and*

$$G_1 = \left\langle \begin{pmatrix}e^{2i\pi/3} & 0\\ 0 & e^{i\pi/3}\end{pmatrix}\right\rangle, \quad G_2 = \left\langle \begin{pmatrix}-e^{2\pi/3} & 0\\ 0 & -e^{i\pi/3}\end{pmatrix}\right\rangle.$$

Proof An easy computation. ■

Lemma 4.30 *Let $(K, B) = (\mathbb{Z}_2, x_1^2 - x_0^2)(x_1^2 - x_0^2)(x_1^2 - x_0^2)$. Then K is splitting and it has two liftings $\epsilon_i \colon \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{GL}(2)$, $i = 1, 2$. Moreover if $G_i = \epsilon_i(\mathbb{Z}_2)$ then C/G_i is an elliptic curve.*

Proof The group $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathrm{GL}(2, \mathbb{C})$ is $G = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$. K is generated by $\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right]$, and its two liftings are $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Finally if $H_0 = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle$ and $H_1 = \left\langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$ then x_0, x_1 in the canonical ring correspond to the invariant 1-forms by H_0 and respectively, by H_1. In particular C/H_i has genus 1 where $i = 0, 1$. ∎

The classification result is:

Theorem 4.31 *Let C be a curve of genus 2 and let $G \subset \mathrm{Aut}(C)$ be a non trivial subgroup such that $C/G = \mathbb{P}^1$. There are only 21 types of couples (C, G). Moreover 15 have extendable type and they are listed in Table 3. The remaining 6 types are listed below:*

$K \sim \epsilon(K) = G$	G_s	*generators of* $\epsilon(K)$
$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{4i\pi/3} \end{pmatrix}$
$\mathbb{Z}_5$	$\mathbb{Z}_5 \times \mathbb{Z}_2$	$\begin{pmatrix} e^{2i\pi/5} & 0 \\ 0 & e^{4i\pi/5} \end{pmatrix}$
$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}$
$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}/2$	$\begin{pmatrix} -e^{2i\pi/3} & 0 \\ 0 & -e^{i\pi/3} \end{pmatrix}$
$\mathcal{D}_3$	$\mathcal{D}_6$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}$
$\mathcal{D}_3$	$\mathcal{D}_6$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -e^{2i\pi/3} & 0 \\ 0 & -e^{i\pi/3} \end{pmatrix}$

Proof It follows from Lemmas 4.20, 4.22, 4.26, 4.27, 4.29, 4.30. ∎

The full classification of the case with $p_g = q = 2$ and S not of Albanese general type would require to compute all the unramified G actions $C_1 \to C_1/G$ where $g(C_1/G) = 2$, for each occurrence of G mentioned in Theorem 4.31. We think that the outcome is not worthy of the effort. However since S is GH, by [Ca, Theorem B], [Ca, Theorem C] then we can say:

Theorem 4.32 *Each irreducible component of the moduli space of surfaces with $p_g = q = 2$ and not of Albanese general type is given by $\mathcal{M}(\Pi, 4)$, the moduli space of surfaces isogenus to a product with fundamental group Π, Euler number 4 and each component is specified by a fixed isomorphism $\Pi(S) \to \Pi$ which fits into the exact sequence*

$$0 \to \Pi_1(C_1) \times \Pi_1(C_2) \to \Pi \to G \to 0$$

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such that the factors $\Pi_1(C_i)$ are normal in Π, the orbifold exact sequences of the coverings $C_1 \to C = C_1/G$, $C_2 \to \mathbb{P}^1 = C_2/G$,

$$0 \to \Pi_1(C_i) \to \Pi(i) \to G \to 0$$

are such that there is no element of Π mapping in each $\Pi(i)$ to an element of finite order and G embeds in $\mathrm{Out}\big(\Pi(C_i)\big) = \mathrm{Aut}\big(\Pi(C_i)\big) / \mathrm{Int}\big(\Pi(C_i)\big)$ by the above sequence where $i = 1, 2$ and G is one of the groups classified in Theorem 4.31.

Appendix [I]

g	G	$\bigoplus_{\chi\in G^*\setminus \mathrm{id}} V^i_\chi$
2	$\mathbb{Z}_2$	V^1_-
3	$\mathbb{Z}_2$	V^2_-
3	$\mathbb{Z}_3$	$V^1_\chi \oplus V^1_{\chi^2}$
3	$\mathbb{Z}_4$	$V^1_\chi \oplus V^1_{\chi^3}$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$V^1_{\chi_1} \oplus V^1_{\chi_{12}}$
4	$\mathbb{Z}_2$	V^3_-
4	$\mathbb{Z}_3$	$V^1_\chi \oplus V^2_{\chi^2}$
4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$V^1_{\chi_1} \oplus V^1_{\chi_{12}} \oplus V^1_{\chi_2}$
4	$\mathbb{Z}_4$	$V^1_\chi \oplus V^1_{\chi^2} \oplus V^1_{\chi^3}$
4	$\mathbb{Z}_6$	$V^1_\chi \oplus V^1_{\chi^3} \oplus V^1_{\chi^5}$
4	$\mathcal{S}_3$	$U^2 \oplus W^1$
4	$\mathcal{S}_3$	W^3
5	$\mathbb{Z}_2$	V^4_-
5	$\mathbb{Z}_3$	$V^2_\chi \oplus V^2_{\chi^2}$
5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$V^2_{\chi_1} \oplus V^2_{\chi_{12}}$
5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$V^2_{\chi_1} \oplus V^1_{\chi_{12}} \oplus V^1_{\chi_2}$
5	$\mathbb{Z}_4$	$V^2_\chi \oplus V^1_{\chi^2} \oplus V^1_{\chi^3}$
5	$\mathbb{Z}_4$	$V^2_\chi \oplus V^2_{\chi^3}$
5	$\mathbb{Z}_5$	$\oplus_{i=1}^4 V^1_{\chi^i}$
5	$\mathcal{S}_3$	$U^2_1 \oplus U^2_2$
5	$\mathcal{S}_3$	$U^2 \oplus W^2$
5	$\mathcal{S}_3$	W^4
5	$(\mathbb{Z}_2)^3$	$V^1_{\chi_1} \oplus V^1_{\chi_2} \oplus V^1_{\chi_3} \oplus V^1_{\chi_{123}}$
5	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$V^1_{\chi_1} \oplus V^1_{\chi_1\chi_2} \oplus V^1_{\chi_1\chi_2^2} \oplus V^1_{\chi_1\chi_2^3}$
5	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$V^1_{\chi_1} \oplus V^1_{\chi_2} \oplus V^1_{\chi_2^3} \oplus V^1_{\chi_1\chi_2^2}$
5	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$V^1_{\chi_2} \oplus V^1_{\chi_2^3} \oplus V^1_{\chi_2} \oplus V^1_{\chi_1\chi_2^2}$
5	$\mathbb{Z}_8$	$V^1_\chi \oplus V^1_{\chi^3} \oplus V^1_{\chi^5} \oplus V^1_{\chi^7}$
5	$\mathcal{D}_4$	$V^2_{\chi_1} \oplus V^2_{\chi^{12}}$

Actions over a curve of genus $2 \leq g \leq 5$ with elliptic quotient.

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Appendix [II]

K	G	generators of G
$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbb{Z}_5$	$\mathbb{Z}_{10}$	$\begin{pmatrix} e^{3i\pi/5} & 0 \\ 0 & e^{i\pi/5} \end{pmatrix}$
$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{3i\pi/4} \end{pmatrix}$
$\mathbb{Z}_3$	$\mathbb{Z}_6$	$\begin{pmatrix} e^{-i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}$
$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$
$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$\langle \mathrm{id} \rangle$	$\mathbb{Z}_2$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathcal{D}_6$	$\left\langle TU \,\middle\vert\, \begin{pmatrix} T^2 = U^6 = (TU)^4 = 1 \\ T(TU)^2 = TU^2T \\ U(TU)^2 = (TU)^2U \end{pmatrix} \right\rangle$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{pmatrix}$
$\mathcal{D}_4$	$\left\langle TU \,\middle\vert\, \begin{pmatrix} T^2 = U^8 = (UT)^4 = 1 \\ TU^4 = U^4T \end{pmatrix} \right\rangle$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} e^{3i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
$\mathcal{D}_3$	$\mathcal{D}_6$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -e^{-2i\pi/3} & 0 \\ 0 & -e^{2i\pi/3} \end{pmatrix}$
$\mathcal{D}_3$	$\left\langle TU \,\middle\vert\, \begin{pmatrix} T^3 = U^4 = (TU)^4 = 1 \\ (TU)^2 = U^2 \\ TU^2 = U^2T \end{pmatrix} \right\rangle$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} e^{2i\pi/3} & 0 \\ 0 & e^{-2i\pi/3} \end{pmatrix}$
$\mathcal{D}_2$	$\mathbb{Z}_4 \rtimes \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$
$\mathcal{D}_2$	$\mathcal{H}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$\mathcal{A}_4$	$\hat{\mathcal{A}}_4$	$\zeta = 1/2(i-1) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ $\eta = (i+1) \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix}$
$\mathcal{S}_4$	$\left\langle TU \,\middle\vert\, \begin{matrix} T^3 = U^8 = (UT)^2 = 1 \\ TU^4 = U^4T \end{matrix} \right\rangle$	$T = (\zeta)^2 = i/2(i-1) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ $U = i\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} i-1 & 0 \\ 0 & i+1 \end{pmatrix}$

Table 4: Extendable groups as subgroups of $\mathrm{GL}(2, \mathbb{C})$.

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Università di Udine
Dipartimento di Matematica e Informatica
Via delle Scienze 206
33100 Udine
Italia
e-mail: zucconi@dimi.uniud.it