Canad. J. Math. Vol. **51** (1), 1999 pp. 96–116

Partial Characters and Signed Quotient Hypergroups

Margit Rösler and Michael Voit

Abstract. If G is a closed subgroup of a commutative hypergroup K, then the coset space K/G carries a quotient hypergroup structure. In this paper, we study related convolution structures on K/G coming from deformations of the quotient hypergroup structure by certain functions on K which we call partial characters with respect to G. They are usually not probability-preserving, but lead to so-called signed hypergroups on K/G. A first example is provided by the Laguerre convolution on $[0,\infty[$, which is interpreted as a signed quotient hypergroup convolution derived from the Heisenberg group. Moreover, signed hypergroups associated with the Gelfand pair $\bigl(U(n,1),U(n)\bigr)$ are discussed.

1 Introduction

There are several constructions which lead from locally compact groups to hypergroups; an important one arises in the context of Gelfand pairs: If H is a compact subgroup of a locally compact group G such that (G,H) is a Gelfand pair, then the double coset space $G//H = \{HgH : g \in G\}$ inherits a commutative hypergroup structure. Roughly speaking, a hypergroup is a locally compact Hausdorff space K with a convolution on the Banach space $M_b(K)$ of regular bounded Borel measures on K with properties similar to those of group convolutions. For an introduction we refer to Jewett [11] and Bloom and Heyer [3]; a good reference to Gelfand pairs is Faraut [5].

Constructions as described above usually lead to probability-preserving convolutions. On the other hand, there exist some not-probability-preserving convolution structures with a group theoretical background. A good example is provided by the Laguerre convolution on $[0,\infty[$ which is closely related to the Heisenberg group. This convolution is discussed, for instance, in [1], [9], [12], [25]. Non-positive convolution structures of this kind are covered by the axiomatic framework of signed hypergroups of Rösler [18]; see also Ross [22].

As already observed by Koornwinder [12], Laguerre convolutions and Heisenberg groups are related as follows: The unitary group $U(n)$ acts in a canonical way on the $(2n+1)$-dimensional Heisenberg group $H_n = \mathbb{C}^n \times \mathbb{R}$, and the orbit space $H_n^{U(n)}$ of $U(n)$-orbits on H_n may be identified with a commutative hypergroup on $\mathbb{R}\times[0,\infty[$. The space $\mathbb{R}\times\{0\}$ is a closed subgroup of this hypergroup, and the quotient space $(\mathbb{R}\times[0,\infty[)/(\mathbb{R}\times\{0\}) \cong [0,\infty[$ carries two different convolution structures: the first one is the usual quotient hypergroup and leads to a Bessel-Kingman hypergroup on $[0,\infty[$. The second one is constructed via some twist by characters of the subgroup $\mathbb{R}\times\{0\}$, and it leads to a signed Laguerre hypergroup on $[0,\infty[$.

Received by the editors July 24, 1997; revised October 19, 1998.
AMS subject classification: Primary: 43A62; secondary: 33C25, 43A20, 43A90.
Keywords: quotient hypergroups, signed hypergroups, Laguerre convolution, Jacobi functions.

https://doi.org/10.4153/CJM-1999-006-6 Published online by Cambridge University Press

The purpose of this paper is to generalize this twisted quotient convolution to closed subgroups G of commutative hypergroups K. We define the twist by means of a continuous function $\sigma\colon K \to \mathbf{T} := \{z \in \mathbb{C} : |z| = 1\}$ satisfying $\sigma(xg) = \sigma(x)\sigma(g)$ and $\sigma(\bar{x}) = \overline{\sigma(x)}$ for $x \in K$, $g \in G$. We prove that such a σ, which we call a partial character of K with respect to G, leads in a natural way to a commutative signed hypergroup structure on the quotient K/G that is closely related to the usual quotient hypergroup structure on K/G (which appears as the special case for $\sigma \equiv 1$). Our approach leads to new signed hypergroup structures for some classes of commutative hypergroups with a group-theoretical background. As a further example, we consider the Gelfand pair $\big(U(n,1), U(n)\big)$. Up to some covering, it leads to a double coset hypergroup on $\mathbb{R} \times [0,\infty[$ with "two-dimensional" Jacobi functions as characters; see Flensted-Jensen [6] and Trimèche [26]. In this case, $\mathbb{R} \times \{0\}$ is again a closed subgroup, the quotient hypergroup is a Jacobi hypergroup of non-compact type, and partial characters lead to new convolution structures on $[0,\infty[$ which are related to product formulas for Jacobi functions.

This paper is organized as follows: Section 2 gives an outline of the relation between Heisenberg groups and Laguerre convolutions. Section 3 contains some facts on commutative signed hypergroups. Here also uniqueness of the pseudo-invariant measure on a commutative signed hypergroup is proved; this measure substitutes the Haar measure and is characterized by some adjoint relation, which is weaker than the usual translation-invariance. In the essential Section 4, partial characters and the associated twisted convolutions are introduced, and the associated twisted signed hypergroups are investigated. In Section 5, we study the dual space of signed quotient hypergroups. Section 6 is devoted to signed quotient hypergroups on $[0,\infty[$ related to the Gelfand pair $\big(U(n,1), U(n)\big)$.

2 Heisenberg Groups and Laguerre Convolutions on $[0,\infty[$

The $(2n+1)$-dimensional Heisenberg group $H_n := \mathbb{C}^n \times \mathbb{R}$ carries the multiplication

$$(z,s)\cdot(w,t) := \big(z+w, s+t-\operatorname{Im}\langle z,w\rangle\big),$$

where $\langle\,.\,,.\rangle$ is the usual Hermitian inner product on $\mathbb{C}^n$. The unitary group $U(n)$ acts on H_n as group of automorphisms via $(z,s) \mapsto (uz,s)$, $u \in U(n)$. The space $K_n := H_n^{U(n)}$ of all $U(n)$-orbits in H_n can be identified with $[0,\infty[\, \times \mathbb{R}$ in the obvious way and carries a orbit hypergroup structure. K_n is also naturally isomorphic to the double coset hypergroup $\big(U(n) \propto H_n\big)//U(n)$ where $U(n)$ is regarded as a subgroup of the semidirect product $U(n) \propto H_n$ (*cf.* [11, Theorem 8.3]). $\big(U(n) \propto H_n, U_n\big)$ is a Gelfand pair (see *e.g.*, Korányi [14]), and the space of all multiplicative symmetric functions in $C_b(K_n)$ can be identified with the space of all symmetric $U(n)$-spherical functions on $U(n) \propto H_n$. The dual $\widehat{K_n}$ decomposes into characters of two different kinds; see [1] and Ch. 1 of [8]:

(1) The characters of the first kind (coming from infinite-dimensional representations of H_n) are parametrized by $(\mathbb{R} \setminus \{0\}) \times \{0,1,2,\dots\}$ and are given by

$$\Lambda_{k,\mu}^{(n-1)}(x,t) = \frac{L_k^{(n-1)}(|\mu|x^2)}{L_k^{(n-1)}(0)} e^{-|\mu|x^2/2+i\mu t} \quad \big(\mu \in \mathbb{R}\setminus\{0\}, k = 0,1,2,\dots\big) \tag{2.1}$$

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where $L_k^{(\alpha)}$ is the k-th Laguerre polynomial of index α.

(2) Characters of the second kind arise from one-dimensional representations of H_n and are parametrized by $[0,\infty[$. They are given in terms of spherical Bessel function by

$$\eta_\tau^{(n-1)}(x,t) := j_{n-1}(\tau x), \quad \text{where} \quad j_\alpha(x) := \Gamma(\alpha+1)\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}. \tag{2.2}$$

In [12], Koornwinder gave a product formula for the $\Lambda_{k,\mu}^{(n-1)}$ and discussed its group theoretical interpretation. As this formula remains valid for the $\eta_\tau^{(n-1)}$, it determines the hypergroup convolution on $K_n = [0,\infty[\times \mathbb{R}$. For $n \geq 2$ we obtain

(2.3)

$$\delta_{(x,s)} * \delta_{(y,t)}(f) := \frac{n-1}{\pi}\int_0^1\int_0^{2\pi} f\bigl(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta\bigr)\cdot r\cdot(1-r^2)^{n-2}\,dr\,d\theta$$

for $x, y \in [0,\infty[$, $s, t \in \mathbb{R}$, $f \in C_b(K_n)$. For $n = 1$, this degenerates to

$$\delta_{(x,s)} * \delta_{(y,t)}(f) := \frac{1}{2\pi}\int_0^{2\pi} f\bigl(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta\bigr)\,d\theta. \tag{2.4}$$

These formulas show that $G := \{0\}\times\mathbb{R}$ is a subgroup of K_n isomorphic to $(\mathbb{R},+)$. The quotient hypergroup $K_n/G = \{\{k\}*G, k \in K_n\}$ can be identified with $[0,\infty[$ and carries the convolution

$$\delta_x * \delta_y(f) := \frac{\Gamma(n)}{\Gamma(n-1/2)\Gamma(1/2)}\int_0^\pi f\bigl(\sqrt{x^2+y^2+2xy\cos\theta}\bigr)\cdot\sin^{2n-2}\theta\,d\theta. \tag{2.5}$$

This is exactly the convolution of the Bessel-Kingman hypergroup of order $n-1$ on $[0,\infty[$; see [3]. The characters of this hypergroup are the modified Bessel functions $x \mapsto j_{n-1}(tx)$ with $t \geq 0$. This is in good agreement with the following fact: If $p\colon [0,\infty[\times\mathbb{R} \to [0,\infty[$ is the canonical projection, then the mapping $\hat{p}\colon (K_n/G)^\wedge \to \widehat{K_n}$, $\rho \mapsto \rho\circ p$ establishes a homeomorphism between the dual space $(K_n/G)^\wedge$ of K_n/G and the annihilator $A(\widehat{K_n}, G) := \{\alpha \in \widehat{K_n} : \alpha|_G = 1\}$ of G in $\widehat{K_n}$ (see Theorem 2.5 of [27]).

On the other hand, (2.3) and (2.4) are closely connected with the Laguerre convolution; in fact, Koornwinder (Eq. (3.4), (3.5) and (4.4) in [12]) observed that

$$\Lambda_{k,\mu}^{(n-1)}(x,s)\cdot\Lambda_{k,\mu}^{(n-1)}(y,t) = (\delta_{(x,s)}*\delta_{(y,t)})(\Lambda_{k,\mu}^{(n-1)}) \quad (x,y\geq 0, s,t\in\mathbb{R}) \tag{2.6}$$

holds by using the product formula

$$\mathcal{L}_k^{(n-1)}(x^2)\cdot\mathcal{L}_k^{(n-1)}(y^2) = \int_{|x-y|}^{x+y}\mathcal{L}_k^{(n-1)}(z^2)\,d(\delta_x *_L \delta_y)(z) \quad (x,y\geq 0) \tag{2.7}$$

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for the Laguerre functions

$$\mathcal{L}_k^{(n-1)}(x) := e^{-x/2} L_k^{(n-1)}(x)/L_k^{(n-1)}(0),$$

where

(2.8)

$$(\delta_x *_L \delta_y)(f) := \frac{\Gamma(n)}{\Gamma(n-1/2)\Gamma(1/2)} \int_0^\pi f\big(\sqrt{x^2+y^2+2xy\cos\theta}\big) \cdot j_{n-3/2}(xy\sin\theta) \cdot \sin^{2n-2}\theta \, d\theta.$$

Involving Poisson's integral representation of Bessel functions, this can also be written as

$$(2.9)\qquad (\delta_x *_L \delta_y)(f) = \frac{n-1}{\pi} \int_0^1 \int_0^{2\pi} f(\sqrt{x^2+y^2+2xyr\cos\theta}) e^{ixyr\sin\theta} \cdot r(1-r^2)^{n-2} \, dr \, d\theta.$$

The convolution generated by (2.8) is called the Laguerre convolution on $[0,\infty[$ (see also [9], [24]) and defines a signed hypergroup structure on $[0,\infty[$ by [18], [19]. Notice that the Laguerre convolution is quite similar to the convolution (2.5), and that the measure $x^{2n-1}dx$ on $[0,\infty[$, which is the Haar measure of the Bessel-Kingman hypergroup, also serves as a substitute for Haar measure on the signed Laguerre hypergroup.

The Laguerre convolution has the following background: The function $\sigma \in C_b(K_n)$ with $\sigma(x,t) := e^{it}$ satisfies certain algebraic conditions (see Section 4.1) which ensure that

$$(2.10)\qquad \delta_x *_\sigma \delta_y := \overline{\sigma(x,0)} \cdot \overline{\sigma(y,0)} \cdot p\big(\sigma \cdot (\delta_{(x,0)} * \delta_{(y,0)})\big) \in M_b\big([0,\infty[\big)$$

(with $p\colon [0,\infty[\times \mathbb{R} \to [0,\infty[$ being the canonical projection) generates a convolution structure on $[0,\infty[$ which is just the signed Laguerre hypergroup. Moreover, (2.10) with $\sigma(x,t) := 1$ is the Bessel-Kingman convolution (2.5). In Section 4 we introduce such functions σ on arbitrary commutative hypergroups K with a closed subgroup G, and we will see that (2.10) always leads to a signed hypergroup on K/G.

We finally mention that the Laguerre convolution is related to the twisted convolution

$$(F\#G)(z) = \int_{\mathbb{C}^n} F(w) \cdot G(z-w) \cdot e^{2\pi i \cdot \mathrm{Im}\langle z,w\rangle} \, dw \quad \big(F,G \in L^1(\mathbb{C}^n)\big)$$

on $L^1(\mathbb{C}^n)$ which comes from the convolution on the reduced Heisenberg group $H_n^{red} := H_n/\Gamma$, $\Gamma = \{(0,k), k \in \mathbb{Z}\}$; see pp. 25–27 of [8]. The algebra $\big(L^1([0,\infty[, x^{n-1}dx), *\big)$ associated with the signed Laguerre convolution (2.8) can be regarded as a commutative Banach subalgebra of $\big(L^1(\mathbb{C}^n), \#\big)$ as follows: The Banach space

$$L^1_{U(n)}(\mathbb{C}^n) := \{f \in L^1(\mathbb{C}^n) : f\big(u(z)\big) = f(z) \text{ almost everywhere for all } u \in U(n)\}$$

of all $U(n)$-invariant functions is closed under the convolution #, and the radial mapping

$$\big(L^1_{U(n)}(\mathbb{C}^n), \#\big) \longrightarrow \big(L^1([0,\infty[, x^{n-1}dx), *\big), \quad f \longmapsto \tilde{f}, \quad \tilde{f}(t) := c_n \cdot f(t,0,\ldots,0)$$

is an isometric isomorphism of Banach algebras for some constant $c_n > 0$ (see [25]).

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3 Basic Facts on Commutative Signed Hypergroups

In this section we recapitulate some facts on commutative signed hypergroups from Rösler [18], [19]. We assume that the reader is familiar with usual hypergroups.

For a locally compact Hausdorff space X let $M_b^{\mathbb{R}}(X)$ denote the subspace of real measures from $M_b(X)$, and w_* the $\sigma\big(M_b(X), C_0(X)\big)$-topology on $M_b(X)$.

A *commutative signed hypergroup* is a triple (X, m, ω) consisting of a locally compact, σ-compact Hausdorff space X, a distinguished positive Radon measure $m \in M^+(X)$ with $\operatorname{supp} m = X$ and a commutative w_*-continuous mapping $\omega\colon X \times X \to M_b^{\mathbb{R}}(X)$, $(x, y) \mapsto \delta_x * \delta_y$, satisfying the following axioms:

(A1) For $x \in X$ and $f \in C_b(X)$, the translate $T^x f\colon y \mapsto \delta_x * \delta_y(f)$ again belongs to $C_b(X)$. Furthermore, for $f \in C_c(X)$ and any compactum $K \subset X$, the set $\bigcup_{x\in K} \operatorname{supp}(T^x f)$ is relatively compact in X.

(A2) $\|\delta_x * \delta_y\| \le C$ for all $x, y \in X$, where $C > 0$ is a constant.

(A3) The canonical continuation $*$ of ω to $M_b(X)$, which is given by

$$\mu * \nu(f) := \int_{X\times X} \delta_x * \delta_y(f)\, d(\mu \otimes \nu)(x, y) \quad \text{for } f \in C_0(X),$$

is associative.

(A4) There exists a neutral element $e \in X$ with $\delta_e * \delta_x = \delta_x$ for all $x \in X$.

(A5) There exists an involutive homeomorphism $^-$ on X such that for all $f, g \in C_c(X)$ and $x \in X$ the following adjoint relation holds:

$$\int_X (T^x f) g\, dm = \int_X f(T^{\bar{x}} g)\, dm.$$

Axiom (A5) implies that $\bar{e} = e$ and $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x, y \in X$, where $\mu^-(A) := \mu(A^-)$ for Borel measures μ on X and Borel sets $A \subseteq X$.

In view of (A5) we call the measure m a "pseudo-invariant" measure w.r.t. $*$ on X. It is $*$-invariant exactly if $\delta_x * \delta_y(X) = 1$ for all $x, y \in X$. Just like the Haar measure of a commutative hypergroup, the pseudo-invariant measure of a commutative signed hypergroup is unique up to a multiplicative constant, see Theorem 3.1 below.

The algebra $\big(M_b(X), *\big)$ becomes a commutative Banach-*-algebra with unit δ_e, the involution $\mu \mapsto \mu^* := \overline{\mu^-}$, and with the norm $\|\mu\|' := \|L_\mu\|$, where $L_\mu(\nu) := \mu * \nu$ for $\mu, \nu \in M_b(X)$. $L^1(X, m)$ with the same multiplication and norm is a closed *-ideal in $\big(M_b(X), *, \|.\|'\big)$. The dual space

$$\widehat{X} := \{\phi \in C_b(X) : \phi \not\equiv 0; \delta_x * \delta_y(\phi) = \phi(x)\phi(y) \text{ and } \phi(\bar{x}) = \overline{\phi(x)} \text{ for all } x, y \in X\}$$

is locally compact w.r.t. the topology of compact-uniform convergence. The Fourier transformation on $M_b(X)$ and $L^1(X, m)$ is given by

$$\widehat{\ }: M_b(X) \to C_b(\widehat{X}), \quad \widehat{\mu}(\varphi) := \int_X \overline{\varphi(x)}\, d\mu(x),$$

and $\widehat{f} := \widehat{fm} \in C_0(\widehat{X})$. The mapping $\mu \mapsto \widehat{\mu}$ on $M_b(X)$ is injective. Finally, the pseudo-invariant measure m admits a Plancherel measure π on $\hat{X}$ which is determined uniquely by the fact that the Fourier transformation becomes an L^2-isometry; see [19].

Theorem 3.1 *Let (X, m, ω) be a commutative signed hypergroup, and suppose that $m' \in M^+(X)$ with $\operatorname{supp} m' = X$ is also pseudo-invariant w.r.t. the same convolution and involution on X. Then $m' = \lambda m$ with a constant $\lambda > 0$.*

Proof Take $\varphi \in \widehat{X}$, $f \in C_c(X)$, and $y \in X$; then the adjoint relation for m' yields

$$\varphi(y) \int_X f|\varphi|^2 \, dm' = \int_X (f\varphi)(x)\varphi(y * \bar{x}) \, dm'(x) = \int_X (f\varphi)(x * y)\overline{\varphi}(x) \, dm'(x). \tag{3.1}$$

Now fix some $g \in C_c(X)$ with $\int_X g|\varphi|^2 \, dm = 1$ and set $\lambda_\varphi := \int_X g|\varphi|^2 \, dm' \geq 0$.

As f and g have compact support, axiom (A1) implies that the function $(x, y) \mapsto g(y)(f\varphi)(x * y)$ belongs to $C_c(X \times X)$. (3.1) and the adjoint relation for m now lead to

$$\begin{aligned}\int_X f|\varphi|^2 \, dm' &= \int_X g(y)|\varphi|^2(y) \, dm(y) \cdot \int_X f|\varphi|^2 \, dm' \\ &= \int_X g(y)\overline{\varphi}(y)\Big(\int_X (f\varphi)(x * y)\bar{\varphi}(x) \, dm'(x)\Big) \, dm(y) \\ &= \int_X \Big(\int_X (g\bar{\varphi})(\bar{x} * y)\bar{\varphi}(x) \, dm'(x)\Big) f(y)\varphi(y) \, dm(y).\end{aligned}$$

The inner integral in this last expression equals $\bar{\varphi}(y) \int_X g|\varphi|^2 \, dm' = \lambda_\varphi \bar{\varphi}(y)$; this results from (3.1) with g and $\bar{\varphi}$ instead of f and φ (note that $(m')^- = m'$ by commutativity of $*$.) Hence $\int_X f|\varphi|^2 \, dm' = \lambda_\varphi \cdot \int_X f|\varphi|^2 \, dm$ for all $f \in C_c(X)$, and this implies

$$|\varphi|^2 \, dm' = \lambda_\varphi |\varphi|^2 \, dm. \tag{3.2}$$

But λ_φ is independent of $\varphi \in \widehat{X}$: in fact, let $\varphi, \psi \in \widehat{X}$. As $\varphi(e) = \psi(e) = 1$, there is an open neighbourhood U of e such that φ and ψ have no zeros on U. Hence, by (3.2), $m'|_U = \lambda_\varphi m|_U$ and $m'|_U = \lambda_\psi m|_U$, and thus $\lambda_\varphi = \lambda_\psi$. It follows that there is a $\lambda \geq 0$ with $|\varphi|^2 m' = \lambda|\varphi|^2 m$ for all $\varphi \in \widehat{X}$. By injectivity of the Fourier-Stieltjes transform on $M_b(X)$, $\widehat{X}$ separates points on X. Hence for any $x \in X$ there is an open neighbourhood V of x and a $\varphi \in \widehat{X}$ without zeros on V. Hence, $m'|_V = \lambda \cdot m|_V$. As X is σ-compact and m' has full support, we get $m' = \lambda m$ with $\lambda > 0$.

4 Partial Characters and Signed Quotient Hypergroups

Let G be a closed subgroup of a commutative hypergroup K. Then, by Theorem 4.1 of [21], the space $K/G = \{\{x\} * G, x \in K\}$ of left cosets of G in K, equipped with the quotient topology, is a commutative hypergroup with convolution

$$\delta_{xG} * \delta_{yG} = \int_K \delta_{zG} \, d(\delta_x * \delta_y)(z);$$

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the trivial coset $G \in K/G$ is the neutral element and the involution is $(xG)^- = \bar{x}G$. We now introduce a principle to construct further convolution structures on K/G via deformations by so-called partial characters.

Definition 4.1 A function $\sigma \in C_b(K)$ is called a partial character of K with respect to G, for short: of (K, G), if for all $x \in K$ and $g \in G$,

$$|\sigma(x)| = 1, \quad \sigma(\bar{x}) = \overline{\sigma(x)} \quad \text{and} \quad \sigma(xg) = \sigma(x) \cdot \sigma(g).$$

(As usual, xg is defined as the unique element of K with $\delta_x * \delta_g = \delta_{xg}$.)

Remarks 4.2

(1) The restriction $\sigma|_G$ of a partial character σ of (K, G) is a character of G.
(2) The partial characters of (K, G) form a group w.r.t. the multiplication of functions.
(3) In most cases $|\alpha| \not\equiv 1$ for characters α of K. Hence, characters of K will usually fail to be partial characters of (K, G). However, if $\alpha \in \widehat{K}$ is a character of K with $\alpha(x) \neq 0$ for all $x \in K$, then obviously $\sigma := \alpha/|\alpha|$ is a partial character of (K, G).
(4) We shall show in Remark 4.21 that under certain restrictions on K and G, each character of G can be extended to a partial character of (K, G).

The canonical surjective and open projection $p\colon K \to K/G$ induces a surjective, w_*-continuous projection $M_b(K) \to M_b(K/G)$, which is also denoted by p. We shall write $\dot{x}$ for the coset $xG, x \in K$.

Lemma 4.3 *Let σ be a partial character of (K, G). Then the mapping*

$$\omega_\sigma\colon K/G \times K/G \longrightarrow M_b(K/G),$$
$$(\dot{x}, \dot{y}) \longmapsto \delta_{\dot{x}} *_\sigma \delta_{\dot{y}} := \overline{\sigma(x)} \cdot \overline{\sigma(y)} \cdot p\big(\sigma(\delta_x * \delta_y)\big)$$

is well-defined and w_*-continuous. Moreover, the total variations satisfy

$$|\delta_{\dot{x}} *_\sigma \delta_{\dot{y}}| \leq \delta_{\dot{x}} * \delta_{\dot{y}} \quad (x, y \in K). \tag{4.1}$$

Proof First note that for $x, y \in K$ and $g \in G$ we can write

$$\delta_{xg} * \delta_y = \delta_x * \delta_y * \delta_g = \int_K (\delta_w * \delta_g)\, d(\delta_x * \delta_y)(w) = \int_K \delta_{wg}\, d(\delta_x * \delta_y)(w).$$

It follows that for any $f \in C_b(K/G)$,

$$\overline{\sigma(g)} \int_K \big((f \circ p)\sigma\big)\, d(\delta_{xg} * \delta_y) = \overline{\sigma(g)} \int_K \big((f \circ p)\sigma\big)(wg)\, d(\delta_x * \delta_y)(w)$$
$$= \int_K \big((f \circ p)\sigma\big)\, d(\delta_x * \delta_y).$$

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Hence $\overline{\sigma(g)}p\big(\sigma(\delta_{xg} * \delta_y)\big) = p\big(\sigma(\delta_x * \delta_y)\big)$ for $x, y \in K$, $g \in G$, and $*_\sigma$ is well-defined. To check that $c\colon K/G \times K/G \to M_b(K/G)$, $(x, y) \mapsto \delta_x *_\sigma \delta_y$ is w_*-continuous, use the commutative diagram

$$\begin{array}{ccc} K/G \times K/G & \xrightarrow{\;c\;} & M_b(K/G) \\ {\scriptstyle p\times p}\Big\uparrow & & \Big\uparrow{\scriptstyle p} \\ K \times K & \xrightarrow{\;\tilde{c}\;} & M_b(K) \end{array}$$

where $\tilde{c}(x, y) := \overline{\sigma(x)\sigma(y)} \cdot \sigma \cdot (\delta_x * \delta_y)$. As $\tilde{c}$ and p are continuous, and as $p \times p$ is open, it follows that c is continuous. To obtain inequality (4.1), recall that $|\sigma| \equiv 1$ and hence $|\delta_{\dot{x}} *_\sigma \delta_{\dot{y}}| = \big|p\big(\sigma(\delta_x * \delta_y)\big)\big| \leq p(\delta_x * \delta_y)$. This completes the proof.

As ω_σ is w_*-continuous, its canonical continuation

$$p(\mu) *_\sigma p(\nu) := \int_{K/G\times K/G} (\delta_{\dot{x}} *_\sigma \delta_{\dot{y}})\, d\big(p(\mu) \otimes p(\nu)\big)$$

is a well-defined bilinear mapping on $M_b(K/G) \times M_b(K/G)$. In fact, we have

Lemma 4.4 *The canonical continuation $*_\sigma$ of ω_σ is given by*

$$p(\mu) *_\sigma p(\nu) = p\big(\sigma(\bar{\sigma}\mu * \bar{\sigma}\nu)\big), \quad \mu, \nu \in M_b(K). \tag{4.2}$$

With this convolution, $M_b(K/G)$ becomes a commutative Banach-$*$-algebra with

$$\|\rho *_\sigma \tau\| \leq \|\rho\| \cdot \|\tau\| \quad \textit{for all } \rho, \tau \in M_b(K/G).$$

Its neutral element is the point measure $\delta_{\dot{e}} = \delta_G$, and its involution is given by $p(\mu)^\sim := p(\mu^*)$, where $.^*$ is the involution on $(M_b(K), *)$. In particular,

$$p_\sigma\colon \big(M_b(K), *\big) \longrightarrow \big(M_b(K/G), *_\sigma\big), \quad \mu \mapsto p(\sigma\mu)$$

establishes a homomorphism of Banach-$*$-algebras.

Proof Eq. (4.2) is proved by a straightforward calculation: if $f \in C_c(K/G)$, then

$$\begin{aligned} p(\mu) *_\sigma p(\nu)(f) &= \int_{K/G\times K/G} f(\dot{x} *_\sigma \dot{y})\, d\big(p(\mu) \otimes p(\nu)\big)(\dot{x}, \dot{y}) \\ &= \int_{K\times K} f\big(p(x) *_\sigma p(y)\big)\, d(\mu \otimes \nu)(x, y) \\ &= \int_K \big((f \circ p)\sigma\big)(z)\, d(\bar{\sigma}\mu * \bar{\sigma}\nu)(z). \end{aligned}$$

Commutativity of $*_\sigma$ is clear, and associativity follows from

$$\begin{aligned} \big(p(\mu) *_\sigma p(\nu)\big) *_\sigma p(\rho) &= p\big(\sigma(\bar{\sigma}\mu * \bar{\sigma}\nu)\big) *_\sigma p(\rho) = p\Big(\sigma\big(\sigma\bar{\sigma}(\bar{\sigma}\mu * \bar{\sigma}\nu) * \bar{\sigma}\rho\big)\Big) \\ &= p\big(\sigma(\bar{\sigma}\mu * \bar{\sigma}\nu * \bar{\sigma}\rho)\big). \end{aligned}$$

The estimation of the total variation norms is clear by (4.1); moreover, $\delta_{\dot e}$ is the neutral element, as $\delta_{\dot e} *_\sigma \delta_{\dot x} = \overline{\sigma(x)} p(\sigma\delta_x) = \delta_{\dot x}$ for $x \in K$. The mapping $\tilde{}$ on $\big(M_b(K/G), *_\sigma\big)$ is well-defined, because for $f \in C_b(K/G)$ and $\mu \in M_b(K)$,

$$\int_K (f \circ p)\, d\mu^* = \int_K (f \circ p)(\bar x)\, d\bar\mu(x) = \int_{K/G} f^-(\dot x)\, d\overline{p(\mu)}(\dot x),$$

where $f^- \in C_b(K/G)$ is given by $f^-(\dot x) = f(\dot{\bar x})$. So if $p(\mu) = p(\nu)$, then $p(\mu^*) = p(\nu^*)$. Moreover, we have for $\mu, \nu \in M_b(K)$ that

$$\begin{aligned}\big(p(\mu) *_\sigma p(\nu)\big)^{\tilde{}} &= p\big(\sigma(\bar\sigma\mu * \bar\sigma\nu)\big)^{\tilde{}} = p\big(\sigma(\bar\sigma\nu * \bar\sigma\mu)^*\big) = p\big(\sigma(\bar\sigma\nu^* * \bar\sigma\mu^*)\big)\\ &= p(\nu^*) *_\sigma p(\mu^*) = p(\nu)^{\tilde{}} *_\sigma p(\mu)^{\tilde{}}.\end{aligned}$$

This shows that $\tilde{}$ is in fact an involution on $\big(M_b(K/G), *_\sigma\big)$.

Remark 4.5 If $\sigma \equiv 1$ is trivial, then $*_\sigma$ is just the convolution on the quotient hypergroup K/G. Generally however, it cannot even be expected that all the measures $\delta_{\dot x} *_\sigma \delta_{\dot y}$ are real-valued. In order to obtain a signed hypergroup structure from $*_\sigma$ on K/G, we have to assume this as an additional requirement. Notice in this context that

(4.3) $$\delta_{\dot x} *_\sigma \delta_{\dot y} \in M_b^{\mathbb{R}}(K/G) \Longleftrightarrow (\delta_{\dot x} *_\sigma \delta_{\dot y})^- = \delta_{\dot{\bar y}} *_\sigma \delta_{\dot{\bar x}}. \quad \text{for } x, y \in K$$

This is clear from the following two identities for any real $f \in C_c(K/G)$:

$$(\delta_{\dot x} *_\sigma \delta_{\dot y})^-(f) = \overline{\sigma(x)\sigma(y)} \int_K f(\dot{\bar z})\sigma(z)\, d(\delta_x * \delta_y)(z),$$

as well as

$$(\delta_{\dot{\bar x}} *_\sigma \delta_{\dot{\bar y}})(f) = \overline{\sigma(\bar x)\sigma(\bar y)} \int_K f(\dot z)\sigma(z)\, d(\delta_{\bar x} * \delta_{\bar y})(z) = \sigma(x)\sigma(y) \int_K f(\dot{\bar z})\overline{\sigma(z)}\, d(\delta_x * \delta_y)(z).$$

The proof of the following main theorem will be divided into several parts.

Theorem 4.6 *Let G be a closed subgroup of a second countable commutative hypergroup K, and let $m_{K/G}$ denote the Haar measure of the quotient hypergroup K/G. If σ is a partial character of (K,G) such that all the convolution products $\delta_{\dot x} *_\sigma \delta_{\dot y}$ are real-valued, then $(K/G, m_{K/G}, *_\sigma)$ is a commutative signed hypergroup with the same neutral element and involution as for the usual quotient hypergroup K/G. Moreover,*

(4.4) $$G = \dot e \in \operatorname{supp}(\delta_{\dot x} *_\sigma \delta_{\dot y}) \quad\Longleftrightarrow\quad \bar x G = yG \quad \textit{for all } x, y \in K.$$

As an immediate consequence of the equivalence (4.3), we have

Corollary 4.7 *If in the situation of Theorem 4.6 the quotient hypergroup K/G is hermitian, i.e., if the identity mapping is the involution on K/G, then $(K/G, m_{K/G}, *_\sigma)$ is a hermitian signed hypergroup for each partial character σ of (K,G).*

https://doi.org/10.4153/CJM-1999-006-6 Published online by Cambridge University Press

In our Laguerre-example of Section 2, $\sigma(x,t) := e^{it}$ is a partial character of $K_n \cong [0,\infty[\times \mathbb{R}$ w.r.t. the closed subgroup $G = \{0\} \times \mathbb{R}$. The quotient hypergroup $K_n/G \cong [0,\infty[$ is the hermitian Bessel-Kingman hypergroup of order $n-1$, and the signed hypergroup on $[0,\infty[$ associated with σ is just the signed Laguerre hypergroup.

Remark 4.8 Suppose that $\sigma, \tau \in C_b(K)$ are partial characters of (K,G) with $\sigma|_G = \tau|_G$. Then the function σ/τ provides a partial character of (K,G) which is constant on G-cosets. Therefore, as the projection p is open, there exists a unique $\rho \in C_b(K/G)$ with $|\rho| = 1$ and such that $\sigma/\tau = \rho \circ p$. A short calculation shows that

$$\rho(\bar{\dot{x}}) = \overline{\rho(\dot{x})} \quad \text{and} \quad \delta_{\dot{x}} *_\sigma \delta_{\dot{y}} = \overline{\rho(\dot{x})\rho(\dot{y})} \cdot \rho(\delta_{\dot{x}} *_\tau \delta_{\dot{y}}) \quad \text{for } x, y \in K.$$

Hence, $\Phi\colon \big(M_b(K/G), *_\tau\big) \longrightarrow \big(M_b(K/G), *_\sigma\big)$, $\mu \mapsto \rho\mu$ is an isomorphism of Banach-$*$-algebras. Summing up, partial characters σ, τ with $\sigma|_G = \tau|_G$ lead to signed hypergroups which are isomorphic in a canonical way.

Proof of Theorem 4.6 As $p\colon K \to K/G$ is a continuous open surjection, K/G is second countable and σ-compact. Except axiom (A5), all properties of a commutative signed hypergroup are covered by the foregoing lemmata. In particular, (A1) follows from (4.1) and the corresponding properties of the quotient hypergroup K/G. The proof of the adjoint relation (A5) requires some preparation and is postponed to the end of this section. To check (4.4), notice that (4.4) is naturally satisfied for the quotient hypergroup convolution $*_1$ on K/G. In the general case, the implication "$\Rightarrow$" now follows from (4.1). The reverse implication follows from "$\Rightarrow$" together with Corollary 2.7 in [19].

We now turn to the proof of the adjoint relation. The central idea is to involve the adjoint relation for K via a Weil formula for subgroups in [10]. Let m_K and m_G be the Haar measures of K and G respectively. For $f \in C_c(K)$ define the orbit mean

$$T_G f(\dot{x}) := \int_G f(xr)\, dm_G(r) \quad \text{with} \quad T_G f \in C_c(K/G).$$

It is well-defined since the integral on the right is constant on G-cosets. Proposition 1 of Hermann [10] says that the functional $f \mapsto \int_{K/G} T_G f \, dm_{K/G}$ establishes a non-trivial Haar measure on K. By uniqueness of m_K, we thus have

Lemma 4.9 *The Haar measures m_K, $m_{K/G}$ and m_G can be normalized such that*

$$\int_K f(x)\, dm_K(x) = \int_{K/G} T_G f(\dot{x})\, dm_{K/G}(\dot{x}) \quad \textit{for all } f \in C_c(K).$$

The proof of the adjoint relation also needs that the canonical projection $p\colon K \to K/G$ admits a Borel cross section (see p. 102 of [15] or Ch. 1 of [16]):

4.10 There exists a Borel function $\phi\colon K/G \to K$ with $p \circ \varphi = \mathrm{id}_{K/G}$ such that $\phi\big(p(F)\big)$ is a relatively compact Borel set in K for each compactum $F \subset K$.

We now use the following lemma to reduce the general situation to the special case

$$(4.5) \qquad H_x := \{y \in G : xy = x\} = \{e\} \quad \text{for all } x \in K.$$

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Lemma 4.11 *Let σ be a partial character of (K, G) with $\sigma|_H \equiv 1$ for some closed subgroup H of G. Then for the quotient hypergroup K/H with its closed subgroup G/H, the following hold:*

(i) $\sigma_H : K/H \to \mathbb{C}$, $\sigma_H(xH) := \sigma(x)$ *for $x \in K$ establishes a well-defined partial character on $(K/H, G/H)$.*

(ii) If we identify $(K/H)/(G/H)$ and K/G in the obvious way as topological spaces, then the convolution structures $(K/G, *_\sigma)$ and $\big((K/H)/(G/H), *_{\sigma_H}\big)$ are equal.

Proof Part (i) is clear. For (ii), consider the canonical projections $p: K \to K/G$, $q: K \to K/H$ and $r: K/H \to (K/H)/(G/H) = K/G$. The quotient hypergroup convolution yields that for $x, y \in K$,

$$\begin{aligned}\delta_{xG} *_{\sigma_H} \delta_{yG} &= \overline{\sigma_H(xH)} \cdot \overline{\sigma_H(yH)} \cdot r\big(\sigma_H(\delta_{xH} * \delta_{yH})\big) \\ &= \overline{\sigma(x)} \cdot \overline{\sigma(y)} \cdot r\big(\sigma_H \cdot q(\delta_x * \delta_y)\big) = \overline{\sigma(x)} \cdot \overline{\sigma(y)} \cdot r\Big(q\big(\sigma(\delta_x * \delta_y)\big)\Big) \\ &= \overline{\sigma(x)} \cdot \overline{\sigma(y)} \cdot p\big(\sigma(\delta_x * \delta_y)\big) = \delta_{xG} *_\sigma \delta_{yG}.\end{aligned}$$

Hence, $*_\sigma = *_{\sigma_H}$ as claimed.

Reduction 4.12 For each $x \in K$ the set $H_x = \{y \in G : xy = x\}$ is a compact subgroup of G (see Proposition 6.1 of [29]). Let H be the closed subgroup of G generated by all H_x. If σ is a partial character of (K, G), then $\sigma|_{H_x} \equiv 1$ holds for all $x \in K$ and thus $\sigma|_H \equiv 1$. According to Lemma 4.11, the convolution structures $(K/G, *_\sigma)$ and $\big((K/H)/(G/H), *_{\sigma_H}\big)$ are isomorphic. As condition (4.5) holds for the subgroup G/H of K/H, and as both convolution structures have the same measures as candidates for the pseudo-invariant measure, we may assume from now on that (4.5) holds. Fixing a Borel cross section ϕ as in 4.10, we thus obtain

Lemma 4.13

(i) Each $x \in K$ can be written as $x = \varphi(\dot{x}) \cdot r_x$ with a unique $r_x \in G$.

(ii) The mapping $T: K \to G$, $T(x) := r_x$ is a well-defined Borel mapping on K.

Proof (i) follows from (4.5). For (ii) notice that by (i) together with Lemma 4.1 of [11], the statements $x = \varphi(\dot{x})r$ and $\{r\} = G \cap \big(\{x\} * \{\overline{\varphi(\dot{x})}\}\big)$ are equivalent. As the convolution of sets is a continuous mapping on the space of all compacta in K w.r.t. the Michael topology (see [11]), and as this topology agrees on the subspace of all sets with exactly one element with the topology on K, it follows that T is a Borel mapping.

Finally, we recall the following variant of Følner's condition for amenable locally compact groups (see Theorem 4.16 of Paterson [17]).

4.14 The σ-compact abelian group G admits a sequence $(G_n)_{n \geq 0}$ of compact nonvoid subsets with $G_n \subset G_{n+1}$, $G = \bigcup_{n \geq 0} \overset{\circ}{G}_n$ and $\lim_{n \to \infty} m_G(G_n \setminus xG_n)/m_G(G_n) = 0$ uniformly in x on compact subsets of G.

https://doi.org/10.4153/CJM-1999-006-6 Published online by Cambridge University Press

Fix these sets $G_n \subset G$ from now on and denote the characteristic function of a set W by 1_W. Let σ be a partial character of (K, G). By Lemma 4.13, the functions

$$\sigma_n(x) := \sigma(x) \cdot 1_{G_n}(r_x), \quad n \in \mathrm{N}$$

are well-defined Borel functions on K. For $f \in C_b(K/G)$ we define $f_\sigma \in C_b(K)$ and bounded Borel functions f_{σ_n} on K by

$$f_\sigma(x) := f(\dot{x}) \cdot \sigma(x), \quad f_{\sigma_n}(x) := f(\dot{x}) \cdot \sigma_n(x).$$

Now suppose that $f \in C_b(K/G)$ has compact support. Then unless G is compact, f_σ need not even be integrable w.r.t. m_K. On the other hand, we have:

Lemma 4.15 *If $f \in C_c(K/G)$, then $f_{\sigma_n} \in L^1(K, m_K)$.*

Proof The Weil formula 4.9 yields that

$$\begin{aligned} \int_K |f(\dot{x})\sigma_n(x)|\, dm_K(x) &= \int_{K/G} \int_G |f(\dot{x})\sigma_n\big(\varphi(\dot{x})r\big)|\, dm_G(r)\, dm_{K/G}(\dot{x}) \\ &= \int_{K/G} |f|\, dm_{K/G} \cdot \int_{G_n} |\sigma|\, dm_G < \infty. \end{aligned}$$

Lemma 4.16 *For $\mu \in M_b(K)$ and $f \in C_c(K/G)$ set*

$$I_n(f, \mu) := \frac{1}{m_G(G_n)} \int_{G_n} \mu * \delta_r(f_{\sigma_n}) \cdot \overline{\sigma(r)}\, dm_G(r), \quad n \in \mathrm{N}.$$

Then $\lim_{n\to\infty} I_n(f, \mu) = \mu(f_\sigma)$.

Proof First consider the point measures $\mu = \delta_x$, $x \in K$. For $r \in G$, we write

$$f_{\sigma_n}(xr)\overline{\sigma(r)} = f(\dot{x})\sigma(x) \cdot 1_{G_n}\big(T(xr)\big) = f_\sigma(x) \cdot 1_{G_n}(r_x r) = f_\sigma(x) \cdot 1_{r_x^{-1}G_n}(r),$$

where the identity $T(xr) = T(x)r$ has been used. It follows that

$$I_n(f, \delta_x) = \frac{m_G(r_x^{-1}G_n \cap G_n)}{m_G(G_n)} \cdot f_\sigma(x).$$

Hence, by condition 4.14,

$$\lim_{n\to\infty} I_n(f, \delta_x) = f_\sigma(x). \tag{4.6}$$

For general $\mu \in M_b(K)$ we use Fubini's theorem to obtain

$$I_n(f, \mu) = \frac{1}{m_G(G_n)} \int_{G_n} \left(\int_K \delta_x * \delta_r(f_{\sigma_n})\, d\mu(x) \right) \overline{\sigma(r)}\, dm_G(r) = \int_K I_n(f, \delta_x)\, d\mu(x).$$

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Using (4.6) and $|I_n(f,\delta_x)| \le \|f\|_\infty$ for all $x \in K$, we may now apply the dominated convergence theorem to the integral on the righthand side. This completes the proof.

Lemma 4.17 *Let the Haar mesures m_K, $m_{K/G}$, m_G be related as in Lemma 4.9. Then for $f,g \in C_c(K/G)$ and $x \in K$, the functions $f_{\sigma_n}, g_{\sigma_n} \in L^1(K)$ satisfy*

$$\lim_{n\to\infty} \frac{1}{m_G(G_n)} \int_K f_{\sigma_n}(x*y) g_{\bar\sigma_n}(y)\, dm_K(y) = \sigma(x) \int_{K/G} f(\dot x *_\sigma \dot y) g(\dot y)\, dm_{K/G}(\dot y).$$

Proof Lemma 4.9 implies that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\frac{1}{m_G(G_n)} \int_K f_{\sigma_n}(x*y) g_{\bar\sigma_n}(y)\, dm_K(y)\\
&\quad = \int_{K/G} \frac{1}{m_G(G_n)} \int_G f_{\sigma_n}\big(x*\phi(\dot y)r\big)\cdot g_{\bar\sigma_n}\big(\phi(\dot y)r\big)\, dm_G(r)\, dm_{K/G}(\dot y)\\
&\quad = \int_{K/G} I_n(f,\delta_x*\delta_{\phi(\dot y)}) g(\dot y)\overline{\sigma\big(\varphi(\dot y)\big)}\, dm_{K/G}(\dot y),
\end{aligned}$$

where the integrand on the righthand side is bounded by $\|f\|_\infty \|g\|_\infty$ for all n. Further,

$$\lim_{n\to\infty} I_n(f,\delta_x*\delta_{\phi(\dot y)}) = f_\sigma\big(x*\phi(\dot y)\big) = \sigma(x)\sigma\big(\phi(\dot y)\big)\cdot f(\dot x *_\sigma \dot y) \quad (y\in K)$$

by Lemma 4.16. The dominated convergence theorem now yields the assertion.

4.18 Proof of the adjoint relation for $(K/G, m_{K/G}, *_\sigma)$ Interchanging f and g, and taking $\bar x$ and $\bar\sigma$ instead of x and σ in Lemma 4.17, we obtain

$$\lim_{n\to\infty} \frac{1}{m_G(G_n)} \int_K f_{\sigma_n}(y) g_{\bar\sigma_n}(\bar x*y)\, dm_K(y) = \sigma(x) \int_{K/G} f(\dot y) g(\tilde{\bar x} *_{\bar\sigma} \dot y)\, dm_{K/G}(\dot y).$$

But as the measures $\delta_{\dot x} *_\sigma \delta_{\dot y} = \overline{\sigma(x)\sigma(y)}\, p\big(\sigma(\delta_x*\delta_y)\big)$ are real-valued by the assumption of our theorem, the convolutions $*_{\bar\sigma}$ and $*_\sigma$ coincide. The adjoint relation for the hypergroup $(K,*)$ now implies that

$$\begin{aligned}
\sigma(x)\int_{K/G} f(\dot x *_\sigma \dot y) g(\dot y)\, dm_{K/G}(\dot y) &= \lim_{n\to\infty} \frac{1}{m_G(G_n)} \int_K f_{\sigma_n}(x*y) g_{\bar\sigma_n}(y)\, dm_K(y)\\
&= \lim_{n\to\infty} \frac{1}{m_G(G_n)} \int_K f_{\sigma_n}(y) g_{\bar\sigma_n}(\bar x*y)\, dm_K(y)\\
&= \sigma(x)\int_{K/G} f(\dot y) g(\tilde{\bar x} *_\sigma \dot y)\, dm_{K/G}(\dot y).
\end{aligned}$$

This yields the adjoint relation for $(K/G, m_{K/G}, *_\sigma)$ as claimed. The proof of Theorem 4.6 is now complete.

Remark 4.19

(1) An inspection of the proof of Theorem 4.6 shows that except for commutativity of $(K/G, *_\sigma)$, its assertions remain true for all second countable hypergroups K which admit a Haar measure, and for all amenable closed normal subgroups G of K.
(2) If G is a compact subgroup, then the adjoint relation can be checked directly. Moreover, the convolution algebra $\big(M_b(K/G), *_\sigma\big)$ can be regarded as a subalgebra of $\big(M_b(K/G), *\big)$ by the following result:

Proposition 4.20 *Let G be a compact subgroup of a second countable commutative hypergroup K with normalized Haar measure m_G. Let σ be a partial character of (K, G). Then $M_b(K|_\sigma G) := \{\mu \in M_b(K) : \bar{\sigma} m_G * \mu = \mu\}$ is a Banach-$*$-subalgebra of $M_b(K)$, and*

$$p_\sigma\colon M_b(K|_\sigma G) \to \big(M_b(K/G), *_\sigma\big), \mu \mapsto p(\sigma\mu)$$

is an isometric isomorphism of Banach-$*$-algebras. Finally, $m_{K/G} := p(m_G)$ is "the" pseudo-invariant measure of $(K/G, *_\sigma)$.

Proof As $\bar{\sigma}|_G$ is a character of the compact commutative group G, it follows that $\bar{\sigma} m_G * \bar{\sigma} m_G = \bar{\sigma} m_G$ and $(\bar{\sigma} m_G)^* = \bar{\sigma} m_G$. Hence, $M_b(K|_\sigma G)$ is a Banach-$*$-subalgebra of $M_b(K)$. Moreover, by Lemma 4.4, p_σ is a homomorphism of Banach-$*$-algebras. As $|\sigma| = 1$ on K, it follows that $p_\sigma|_{M_b(K|_\sigma G)}$ is an isometric isomorphism of Banach spaces. The final statement follows from Theorem 4.6 and the fact that $p(m_G)$ is the Haar measure of the quotient hypergroup K/G; *cf.* Theorem 2.5(3) of Voit [27].

Remark 4.21

(1) Assume that in the setting of Theorem 4.6 the canonical projection $p\colon K \to K/G$ admits a continuous cross-section $\phi\colon K/G \to K$ with $\phi(\dot{e}) = e$ and $\phi(\tilde{\dot{x}}) = \overline{\phi(\dot{x})}$ for $x \in K$. An inspection of the proof of Lemma 4.13 shows that in this case the mapping T is actually continuous. If $\alpha \in \hat{G}$ is any character of G, then $\sigma(x) := \alpha\big(T(x)\big)$ clearly establishes a partial character of (K, G) with $\sigma|_G = \alpha$.
(2) Consider a more concrete setting: Suppose that K is homeomorphic to a direct product $\tilde{K} \times G$ where $\tilde{K}$ is a commutative hypergroup and G a closed subgroup of K. Let $p_2\colon K \to G$, $(x, g) \mapsto g$ be the projection onto the second component. Then for any $\alpha \in \widehat{G}$, the function $\sigma := \alpha \circ p_2$ is a partial character of (K, G). This situation is satisfied for our Laguerre example of Section 2 with $K \simeq [0, \infty[\times \mathrm{R}$, $\tilde{K}$ as Bessel-Kingman hypergroup, and G being the group $(\mathrm{R}, +)$. Our partial character $\sigma(x, t) = e^{it}$ comes from the group character $\alpha(t) = e^{it}$ just as described above.

5 The Dual Space

In this section we describe the dual spaces of the signed hypergroups $(K/G, *_\sigma)$ in terms of the partial characters σ and the dual space of K. Again, we use the canonical projection $p\colon K \to K/G$ and the notation $f_\sigma := (f \circ p) \cdot \sigma$ for $f \in C_b(K/G)$.

Definition 5.1 Let G be a closed subgroup of a commutative hypergroup K. For each character $\sigma \in \hat{G}$ we define the σ-annihilator of G in K by

$$A(\hat{K}, G, \sigma) := \{\alpha \in \hat{K} : \alpha|_G = \sigma\}.$$

For $\sigma = 1$ we obtain the usual annihilator discussed in [3], [27]. The following theorem generalizes known results for quotients of commutative hypergroups (see [27]).

Theorem 5.2 *Let G be a closed subgroup of a second countable commutative hypergroup K and let σ be a partial character of (K, G) such that $(K/G, *_\sigma)$ is a signed hypergroup. If $(K/G)^{\wedge\sigma}$ denotes the dual space of $(K/G, *_\sigma)$, then the mapping*

$$\hat{p}_\sigma : (K/G)^{\wedge\sigma} \to A(\hat{K}, G, \sigma|_G), \quad \hat{p}_\sigma(\rho)(x) := \rho(\dot{x})\sigma(x) = \rho_\sigma(x)$$

is a homeomorphism.

Proof By the definition of $*_\sigma$, any $\rho \in (K/G)^{\wedge\sigma}$ satisfies $\rho_\sigma \in \hat{K}$ and $\rho_\sigma|_G = \sigma|_G$. Moreover, $\hat{p}_\sigma$ is injective and continuous. To check that $\hat{p}_\sigma$ is surjective, take $\alpha \in \hat{K}$ with $\alpha|_G = \sigma|_G$. Then α/σ is constant on G-cosets. Hence, as p is open, there exists $\rho \in C_b(K/G)$ with $\rho \circ p = \alpha/\sigma$, and it is easily checked that $\rho \in (K/G)^{\wedge\sigma}$. As $\alpha = \hat{p}_\sigma(\rho)$, the surjectivity is clear. It remains to show that $\hat{p}_\sigma$ is open, *i.e.*, that for all $\rho_0 \in (K/G)^{\wedge\sigma}$, $\epsilon > 0$, and each compactum $M \subset K/G$ the set $R := \{\hat{p}_\sigma(\rho) : \rho \in (K/G)^{\wedge\sigma}; |\rho - \rho_0| < \epsilon$ on $M\}$ is open in $A(\hat{K}, G, \sigma|_G)$. Since $p : K \to K/G$ is open and continuous, we find a compactum $L \subset K$ such that $p(L) = M$. As $|\sigma| = 1$, it follows that

$$R = \{\gamma \in A(\hat{K}, G, \sigma|_G) : |\gamma - p_\sigma(\rho_0)| < \epsilon \text{ on } L\}$$

is open in $A(\hat{K}, G, \sigma|_G)$ as claimed.

Proposition 5.3 *Assume that G is compact in the setting of Theorem 5.2. Then $A(\hat{K}, G, \sigma|_G)$ is open in $\hat{K}$. Moreover, if π_K is the Plancherel measure on $\hat{K}$ associated with the Haar measure m_K on K, and if $m_{K/G}$ is the pseudo-invariant measure on $(K/G, *_\sigma)$, as in 4.20, then the Plancherel measure $\pi^\sigma_{K/G}$ on $(K/G)^{\wedge\sigma}$ associated with $m_{K/G}$ satisfies*

$$\hat{p}_\sigma(\pi^\sigma_{K/G}) = \pi|_{A(\hat{K}, G, \sigma|_G)}. \tag{5.1}$$

Proof We abbreviate $A := A(\hat{K}, G, \sigma|_G)$ and fix some $\delta \in]0, 1[$. Then

$$P := \{\phi \in \hat{K} : |\phi(x) - \sigma(x)| < \delta \text{ for all } x \in G\}$$

is open in $\hat{K}$ with $A \subseteq P$. Moreover, as G is a compact abelian group with $\phi|_G, \sigma|_G \in \hat{G}$, we obtain $P = A$. To check (5.1), we prove that $\hat{p}_\sigma^{-1}(\pi_K|_A)$ is the Plancherel measure on $(K/G)^{\wedge\sigma}$ associated with $m_{K/G}$. Denote the Fourier transformations on K and $(K/G, *_\sigma)$ by $^\wedge$ and $^{\wedge\sigma}$ respectively, and fix $f \in C_c(K/G)$. Then $f_\sigma \in C_c(K)$. Moreover, Theorem 5.2 says that $\phi_\sigma \in \hat{K}$ for $\phi \in (K/G)^{\wedge\sigma}$, and

$$f_\sigma^\wedge(\phi_\sigma) = \int_K \sigma(x) f(\dot{x}) \cdot \overline{\sigma(x)\phi(\dot{x})}\, dm_K(x) = \int_{K/G} f(\dot{x})\overline{\phi(\dot{y})}\, dm_{K/G}(\dot{y}) = \hat{f}^\sigma(\phi). \tag{5.2}$$

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Now let $\alpha \in \hat{K}$ with $\alpha|_G \notin A$. By the Weil formula 4.9, with a suitable normalization of $m_{K/G}$,

$$f_\sigma^\wedge(\alpha) = \int_K f_\sigma(x)\bar{\alpha}(x)\,dm_K(x) = \int_{K/G} T_G(f_\sigma\bar{\alpha})(\dot{x})\,dm_{K/G}(\dot{x}).$$

But for the orbit mean $T_G(f_\sigma\bar{\alpha})$, we obtain

$$(5.3)\qquad T_G(f_\sigma\bar{\alpha})(\dot{x}) = \int_G (f_\sigma\bar{\alpha})(xr)\,dm_G(r) = f(\dot{x})\sigma(x)\overline{\alpha(x)}\int_G \sigma(r)\overline{\alpha(r)}\,dm_G(r) = 0,$$

since characters on the compact group G are orthogonal w.r.t. m_G. Formulas (5.2), (5.3) and the Plancherel formula on K [11, Theorem 7.3] thus lead to

$$\int_{K/G} |f|^2\,dm_{K/G} = \int_K |f\circ p|^2\,dm_K = \int_K |f_\sigma|^2\,dm_K = \int_{\hat{K}} |f_\sigma^\wedge|^2\,d\pi_K = \int_A |f_\sigma^\wedge|^2\,d\pi_K$$
$$= \int_A |\hat{f}^\sigma|^2\,d\big(\hat{p}_\sigma^{-1}(\pi_K|_A)\big).$$

Since this is true for all $f \in C_c(K/G)$, and since this formula determines the Plancherel measure uniquely (see [19]), the proof of Proposition 4.3 is complete.

Remark 5.4 Proposition 5.3 can be used to compute the Plancherel measure π_K for some commutative hypergroups K from the known Plancherel measures of the signed hypergroups $(K/G, *_\sigma)$. For this, assume that G is compact and that the conditions of Remark 4.21(1) are satisfied; then for any $\alpha \in \hat{G}$, the function $\sigma_\alpha := \alpha\circ T$ is a partial character of (K, G) with $\sigma_\alpha|_G = \alpha$. Now take any $\phi \in \widehat{K}$. Then $\alpha := \phi|_G \in \widehat{G}$, and $\sigma_\alpha = \alpha \circ T$ is a partial character of (K, G) with $\sigma_\alpha|_G = \alpha = \phi|_G$. Hence, $\widehat{K} = \bigcup_{\alpha\in\hat{G}} A(\widehat{K}, G, \alpha)$, where $A(\widehat{K}, G, \alpha) \cong (K/G)^{\wedge\sigma_\alpha}$. As K is σ-compact, π_K is obtained by σ-additivity from its restrictions to the open and closed subsets $A(\widehat{K}, G, \alpha)$, *i.e.*, from the Plancherel measures on the signed hypergroup duals $(K/G)^{\wedge\sigma_\alpha}$.

An example is given by a modification of the construction in Section 2, based on the reduced Heisenberg group H_n^{red} instead of H_n. Then the orbit hypergroup $(H_n^{red})^{U(n)}$ can be identified with $[0,\infty[\, \times \mathbf{T}$ and has $G := \{0\} \times \mathbf{T}$ as a compact subgroup.

6 Signed Hypergroups Related to the Gelfand Pair $\big(U(n,1), U(n)\big)$

For $n \in$ N, consider the Gelfand pair $(G, H) := \big(U(n,1), U(n)\big)$. Following Flensted-Jensen [6], we study the following further Gelfand pair $(\tilde{G}^1, \tilde{H}^1)$: Let $\tilde{G}$ be the universal covering group of G and $\pi_1\colon \tilde{G} \to G$ the covering homomorphism; let Q be the central, discrete subgroup of $\tilde{G}$ given by $Q := \pi_1^{-1}(e) \cap \pi_1^{-1}(H)_o$; here the subscript $_o$ assigns the connected component of e. Now let $\tilde{G}^1$ be the covering group $\tilde{G}^1 := \tilde{G}/Q$ of G, and $\tilde{H}^1 := \pi^{-1}(H)_o$, where $\pi\colon \tilde{G}^1 \to G$, $xQ \mapsto \pi_1(x)$ is the covering homomorphism. Then $\tilde{H}^1$ is isomorphic to H. According to Theorem 1.3 of [6], the double coset hypergroup $K := \tilde{G}^1//\tilde{H}^1$ can be identified with $[0,\infty[\, \times$ R. To describe the hypergroup convolution on K, consider the double coset hypergroup $L := G//H$. By Theorem 1.1 in [6], L can be identified with $[0,\infty[\, \times \mathbf{T}$, and hence with $Z := \{z \in$ C $: |z| \geq 1\}$ via

$$[0,\infty[\, \times \mathbf{T} \to Z, \quad (x, e^{i\theta}) \mapsto \cosh x \cdot e^{i\theta}.$$

https://doi.org/10.4153/CJM-1999-006-6 Published online by Cambridge University Press

If $\alpha := n-1 > 0$, then by Trimèche [26] the associated convolution on Z is given by

$$\delta_{z_1} * \delta_{z_2} = \frac{\alpha}{\pi}\int_{\{|z|\le 1\}} \delta_{z_1z_2+\sqrt{|z_1|^2-1}\sqrt{|z_2|^2-1}\cdot z} \cdot (1-|z|^2)^{\alpha-1}\, dm(z) \tag{6.1}$$

with $dm(z) = dx\,dy$ for $z = x+iy \in \mathbb{C}$. For $n = 1$, (6.1) degenerates to

$$\delta_{z_1} * \delta_{z_2} = \frac{1}{2\pi}\int_0^{2\pi} \delta_{z_1z_2+\sqrt{|z_1|^2-1}\sqrt{|z_2|^2-1}\cdot z}\, dz. \tag{6.2}$$

To see the precise relation between K and L, consider the central subgroup $W := \pi^{-1}(e)$ of $\tilde{G}^1$. It is naturally isomorphic to the closed subgroup $W\tilde{H}^1//\tilde{H}^1$ of K. Hence, by an obvious extension of the isomorphism theorem for double coset hypergroups in [11],

$$K/(W\tilde{H}^1//\tilde{H}^1) \cong \tilde{G}^1//W\tilde{H}^1 \cong (\tilde{G}^1/W)//(W\tilde{H}^1/W) \cong L.$$

Transferring the hypergroup structure from K to $[0,\infty[\,\times\,\mathbb{R}$ as above, W corresponds to the discrete subgroup $\{0\}\times 2\pi\mathbb{Z}$ of $[0,\infty[\,\times\,\mathbb{R}$, *cf.* Lemma 1.2(ii) in [6].

To obtain the convolution on K from that on L, denote by ln the branch of the logarithm on $\mathbb{C}_- := \mathbb{C}\setminus\{x\in\mathbb{R} : x\le 0\}$ with $\ln(1) = 0$, and set $\arg(z) := -i\cdot\ln\frac{z}{|z|} \in\,]-\pi,\pi[$ for $z\in\mathbb{C}_-$. Now take $x_1, x_2 \ge 0$, $r\in[0,1]$, and $\Psi\in[0,2\pi]$. Then

$$\operatorname{Re}(\operatorname{ch} x_1 \operatorname{ch} x_2 + \operatorname{sh} x_1 \operatorname{sh} x_2 re^{i\Psi}) > 0.$$

For $\alpha > 0$, Eq. (6.1) leads to the convolution

$$\begin{aligned} &f\big((x_1,\theta_1)*(x_2,\theta_2)\big) \\ &= \frac{\alpha}{\pi}\int_0^1\int_0^{2\pi} f\big(\operatorname{arch}|\operatorname{ch} x_1 \operatorname{ch} x_2 e^{i(\theta_1+\theta_2)} + re^{i\Psi}\operatorname{sh} x_1 \operatorname{sh} x_2|, \\ &\qquad \theta_1+\theta_2+\arg(\operatorname{ch} x_1\operatorname{ch} x_2 + re^{i\Psi}\operatorname{sh} x_1\operatorname{sh} x_2)\big) r(1-r^2)^{\alpha-1}\,dr\,d\Psi \end{aligned} \tag{6.3}$$

on K for $f\in C_b([0,\infty[\,\times\,\mathbb{R})$. (Note that the support of $\delta_{(x_1,\theta_1)} * \delta_{(x_2,\theta_2)}$ is contained in the strip $[0,\infty[\,\times\,]\theta_1+\theta_2-\pi, \theta_1+\theta_2+\pi[$ by continuity of the convolution w.r.t. the Michael topology.) In the same way, (6.2) leads for $\alpha = 0$ to the convolution

$$\begin{aligned} f\big((x_1,\theta_1)*(x_2,\theta_2)\big) &= \frac{1}{2\pi}\int_0^{2\pi} f\big(\operatorname{arch}|\operatorname{ch} x_1 \operatorname{ch} x_2 e^{i(\theta_1+\theta_2)} + re^{i\Psi}\operatorname{sh} x_1 \operatorname{sh} x_2|, \\ &\qquad \theta_1+\theta_2+\arg(\operatorname{ch} x_1\operatorname{ch} x_2 + re^{i\Psi}\operatorname{sh} x_1\operatorname{sh} x_2)\big)\,d\Psi. \end{aligned} \tag{6.4}$$

The identity of K is the point $(0,0)$, the involution is $(x,\theta)^- = (x,-\theta)$ and the Haar measure is given by

$$dm_K(x,\theta) = 2^{2(\alpha+1)}(\operatorname{sh} x)^{2\alpha+1}\operatorname{ch} x\,dx\,d\theta.$$

We next turn to the dual space of K. For this, we introduce the functions

$$\phi_{\lambda,\mu}(x,\theta) := e^{i\lambda\theta}(\operatorname{ch} x)^{\lambda}\phi_\mu^{(\alpha,\lambda)}(x) \quad (\alpha = n-1,\ \lambda,\mu\in\mathbb{C})$$

on K where $\phi_\mu^{(\alpha,\lambda)}$ is the Jacobi function of index (α,λ) as studied, for instance, by Flensted-Jensen and Koornwinder [7], [13]. In view of the symmetry $\phi_\mu^{(\alpha,\lambda)} = \phi_{-\mu}^{(\alpha,\lambda)}$, it suffices to consider $\mu\in\mathbb{C}$ with $\operatorname{Re}\mu + \operatorname{Im}\mu \ge 0$. The following theorem describes the dual space $\hat{K}$ of K in terms of the two-variable Jacobi functions $\phi_{\lambda,\mu}$:

Theorem 6.1

(1) The space of all non-trivial multiplicative continuous functions on K is given by

$$\mathcal{X}(K) := \{f \in C(K) : f \not\equiv 0,\, f(x * y) = f(x)f(y) \text{ for all } x, y \in K\} = \{\phi_{\lambda,\mu} : \lambda, \mu \in \mathbb{C}\}.$$

(2) $\phi_{\lambda,\mu} \in \mathcal{X}(K)$ is symmetric (i.e., $\phi_{\lambda,\mu}(x, -\theta) = \overline{\phi_{\lambda,\mu}(x, \theta)}$ for all $(x, \theta) \in K$) if and only if $\lambda \in \mathbb{R}$ and $\mu \in [0, \infty[\cup i\,[0, \infty[$.

(3) $\hat{K} = \{\phi_{\lambda,\mu} : \lambda \in \mathbb{R}, \mu \in [0, \infty[\cup i[0, \alpha + 1]\} \cup \{\phi_{\lambda,\mu} : (\lambda, \mu) = \big(\pm(\alpha + 1 + 2k + \eta), i\eta\big), k \in \mathbb{N}_0, \eta \in [0, \infty[\}$.

(4) The support of the Plancherel measure π on $\hat{K}$ is given by

$$\operatorname{supp} \pi = \{\phi_{\lambda,\mu} : \lambda \in \mathbb{R}, \mu \in [0, \infty[\} \\ \cup \{\phi_{\lambda,\mu} : (\lambda, \mu) = \big(\pm(\alpha + 1 + 2k + \eta), i\eta\big), k \in \mathbb{N}_0, \eta \in [0, \infty[\}.$$

Proof (1) By Theorem 2.1 of [6], the $\phi_{\lambda,\mu}$ with $\lambda, \mu \in \mathbb{C}$ are exactly the C^∞-spherical functions of the Gelfand pair $(\tilde{G}^1, \tilde{H}^1)$, *i.e.*, these functions are multiplicative on K. For the converse, we have to check $\mathcal{X}(K) \subset C^\infty(K)$. For this, take $\phi \in \mathcal{X}(K)$ and $f \in C_c^\infty(K)$ with $\int_K \bar{\phi} f \, dm_K = 1$. Then, by the definition of the convolution on K, the function $x \mapsto f(\bar{\xi} * x)$, with $\xi \in K$ fixed, belongs to $C_c^\infty(K)$. Moreover,

$$\phi(x) = \phi * f(x) = \int_K \phi(\xi) f(\bar{\xi} * x) \, dm_K(\xi).$$

Hence $\phi \in C^\infty(K)$, and the proof of (1) is finished.

(2) is clear as $\phi_\mu^{(\alpha,\lambda)}$ is real-valued for real arguments if and only if $\mu \in \mathbb{R} \cup i\mathbb{R}$. (This follows from from the hypergeometric differential equation satisfied by $\phi_\mu^{(\alpha,\lambda)}$.)

(3) follows from the asymptotic behaviour of the functions $\phi_{\lambda,\mu}$; see Eq. (2.6) of [6].

(4) is a consequence of the Plancherel formula associated with the functions $\phi_{\lambda,\mu}$ (see Theorem 2.5 of [6] or Theorem V.2 of [26]) and the fact that the Plancherel measure is determined uniquely by the Plancherel formula.

(6.3) and (6.4) show that $G = \{0\} \times \mathbb{R}$ is a closed subgroup of the hypergroup $K \cong [0, \infty[\times \mathbb{R}$. Thus, by Remark 4.21, for every $\lambda \in \mathbb{R}$ the function $\sigma_\lambda(x, \theta) := e^{i\lambda\theta}$ defines a partial character of (K, G). The convolution on the quotient hypergroup K/G is immediately seen from (6.3) and (6.4) respectively; for $\alpha = n - 1 > 0$, it is given by

$$\delta_x * \delta_y = \frac{2\alpha}{\pi} \int_0^1 \int_0^\pi \delta_{\operatorname{arch} |\Lambda(x,y;r,\Psi)|} r(1 - r^2)^{\alpha-1} \, dr \, d\Psi$$

for $x, y \geq 0$, where

$$\Lambda(x, y; r, \Psi) := \operatorname{ch} x \operatorname{ch} y + re^{i\Psi} \operatorname{sh} x \operatorname{sh} y.$$

The identity of $K/G \cong [0, \infty[$ is 0, the involution is the identity mapping and the Haar measure is given by

$$dm_{K/G}(x) = 2^{2(\alpha+1)} (\operatorname{sh} x)^{2\alpha+1} \operatorname{ch} x \, dx.$$

Thus $(K/G,*)$ is just the Jacobi hypergroup of index $(\alpha,0)$ on $[0,\infty[$, see Section 7.2 of [13] or [7]. (Note that in several formulas of [13], arcosh$(\dots)$ is missing in the argument of the integrand.) By Corollary 4.7, each partial character $\sigma_\lambda, \lambda \in \mathbb{R}$, leads to a hermitian signed hypergroup convolution $*_{(\alpha,\lambda)}$ on K/G. The following theorem summarizes properties of these signed quotient hypergroups. As the convolutions induced by σ_λ and $\sigma_{-\lambda}$ coincide (see 4.18), we restrict attention to the case $\lambda \geq 0$.

Theorem 6.2

(1) For $\alpha = n-1 > 0$, the convolution on the signed hypergroup $X_\lambda^\alpha := (K/G, *_{(\alpha,\lambda)})$ with $\lambda \geq 0$ is given by

(6.5)
$$\delta_x *_{(\alpha,\lambda)} \delta_y = \frac{2\alpha}{\pi}\int_0^1\int_0^\pi \delta_{\operatorname{arch}|\Lambda(x,y;r,\Psi)|}\cos\bigl(\lambda\arg\Lambda(x,y;r,\Psi)\bigr)r(1-r^2)^{\alpha-1}\,dr\,d\Psi$$

for $x, y \geq 0$. In the case $\alpha = 0$, it degenerates to

$$\delta_x *_{(0,\lambda)} \delta_y = \frac{1}{\pi}\int_0^\pi \delta_{\operatorname{arch}|\Lambda(x,y;1,\Psi)|}\cos\bigl(\lambda\arg\Lambda(x,y;1,\Psi)\bigr)\,d\Psi. \tag{6.6}$$

Neutral element, involution and pseudo-invariant measure are the same as for the quotient hypergroup $(K/G,*) = X_0^\alpha$.

(2) The dual space $\widehat{X}_\lambda^\alpha$ of X_λ^α is given by

$$\widehat{X}_\lambda^\alpha = \{\Psi_\mu^{(\alpha,\lambda)}, \Psi_\mu^{(\alpha,\lambda)}(x) := (\operatorname{ch} x)^\lambda \phi_\mu^{(\alpha,\lambda)}(x) \mid \mu \in I_\lambda^\alpha\}, \quad \textit{with}$$
$$I_\lambda^\alpha = [0,\infty[\,\cup\, i[0,\alpha+1] \cup \{-i(\alpha+1+2k-\lambda) : k \in \mathbb{N}_0 \textit{ and } k \leq (-\alpha-1+\lambda)/2\}.$$

Proof (1) By the definition of signed quotient hypergroup convolutions, we have for $f \in C_b([0,\infty[)$ and $x, y \geq 0$ that $(\delta_x *_{(\alpha,\lambda)} \delta_y)(f) = (\delta_{(x,0)} * \delta_{(y,0)})(f_\lambda)$; here $*$ denotes the convolution on K and $f_\lambda \in C_b(K)$ is given by $f_\lambda(x,\theta) = f(x)e^{i\lambda\theta}$. Using (6.3) and (6.4) respectively, we obtain (6.5) and (6.6). The rest is clear by Theorem 4.6.

(2) follows from $A(\widehat{K}, G, \sigma_\lambda|_G) = \{\phi_{\nu,\mu} \in \widehat{K} : \nu = \lambda\}$ and Theorems 5.2 and 6.1(3).

Remark 6.3 Eq. (6.5) on X_λ^α with $\alpha = n-1 > 0$ can be written in "kernel form" via a substitution according to [7]; more precisely, if

$$(r,\Psi) \mapsto (u,\chi) \in [0,\infty) \times [0,\pi]; \quad e^{i\chi}\operatorname{ch} u = \operatorname{ch} x \operatorname{ch} y + re^{i\Psi}\operatorname{sh} x \operatorname{sh} y = \Lambda(x,y;r,\Psi),$$

then,

$$\delta_x *_{(\alpha,\lambda)} \delta_y = \int_0^\infty \delta_u K_\lambda^\alpha(x,y,u)\,dm_{K/G}(u), \tag{6.7}$$

where the kernel K_λ^α is given by

$$K_\lambda^\alpha(x,y,u) = \frac{\alpha}{2^{2\alpha+1}\pi}\cdot\frac{1}{(\operatorname{sh} x \operatorname{sh} y \operatorname{sh} u)^{2\alpha}}\cdot\int_0^\pi \bigl[F_{x,y,u}(\chi)\bigr]_+^{\alpha-1}\cos(\lambda\chi)\,d\chi,$$

with

$$F_{x,y,u}(\chi) = 1 - (\operatorname{ch} x)^2 - (\operatorname{ch} y)^2 - (\operatorname{ch} u)^2 + 2 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} u \cos \chi;$$

here $(x)_+ := x$ for $x > 0$ and $(x)_+ := 0$ for $x \leq 0$.

For $\alpha \geq \beta > -1/2$, the Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ on $[0, \infty[$ have a well-known positive product formula (formulas (7.11) to (7.13) in [13]), which determines the Jacobi hypergroup of order (α, β) on $[0, \infty[$; see also Section 3.5 of [3]. On the other hand, the convolution formulas on X_λ^α lead immediately to new product formulas for the Jacobi functions $\phi_\mu^{(\alpha,\lambda)}$; in case $\alpha > 0$ we obtain from (6.7) that

(6.8)

$$\phi_\mu^{(\alpha,\lambda)}(x)\phi_\mu^{(\alpha,\lambda)}(y) = \frac{2^{2(\alpha+1)}}{(\operatorname{ch} x)^\lambda (\operatorname{ch} y)^\lambda} \int_0^\infty \phi_\mu^{(\alpha,\lambda)}(u) K_\lambda^\alpha(x, y, u)(\operatorname{sh} u)^{2\alpha+1}(\operatorname{ch} u)^{\lambda+1}\, du.$$

In contrast to Koornwinder's product formula (7.11) of [13] for the $\phi_\mu^{(\alpha,\lambda)}$, which cannot be extended to (α, λ) with$\lambda > \alpha$, this formula is valid for all $\lambda \geq 0$. In the case $\alpha \geq \lambda \geq 0$, it may be interpreted in the sense that the Jacobi hypergroup of order (α, λ) is obtained from the signed hypergroup X_λ^α by a deformation similar to the hypergroup renormalizations by positive semicharacters studied in Voit [28]. In fact, if $\circ_{(\alpha,\lambda)}$ denotes the convolution on the Jacobi hypergroup of order (α, λ), then

$$\delta_x \circ_{(\alpha,\lambda)} \delta_y = \frac{1}{(\operatorname{ch} x)^\lambda (\operatorname{ch} y)^\lambda} \int_0^\infty \delta_u (\operatorname{ch} u)^\lambda \, d(\delta_x *_{(\alpha,\lambda)} \delta_y)(u).$$

This in turn reveals that for $\alpha \geq \lambda \geq 0$ the convolution on X_λ^α is positivity-preserving. However, (4.1) shows that unless $\lambda = 0$, the convolution $*_{(\alpha,\lambda)}$ is not probability-preserving and therefore is not a hypergroup convolution. This is in accordance with the fact that the function $x \mapsto (\operatorname{ch} x)^\lambda$ is not multiplicative on X_λ^α for $\lambda \neq 0$.

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Zentrum Mathematik
Technische Universität München
Arcisstr. 21
D-80290 München
Germany
email: roesler@ma.tum.de

Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
Germany
email: voit@uni-tuebingen.de