Bull. Aust. Math. Soc. **91** (2015), 202–210
doi:10.1017/S0004972714000811

RADICAL RELATED TO SPECIAL ATOMS REVISITED

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(Received 3 August 2014; accepted 5 September 2014; first published online 14 October 2014)

Abstract

A semiprime ring R is called a $*$-ring if the factor ring R/I is in the prime radical for every nonzero ideal I of R. A long-standing open question posed by Gardner asks whether the prime radical coincides with the upper radical $U(*_k)$ generated by the essential cover of the class of all $*$-rings. This question is related to many other open questions in radical theory which makes studying properties of $U(*_k)$ worthwhile. We show that $U(*_k)$ is an N-radical and that it coincides with the prime radical if and only if it is complemented in the lattice $\mathbb{L}_N$ of all N-radicals. Along the way, we show how to establish left hereditariness and left strongness of important upper radicals and give a complete description of all the complemented elements in $\mathbb{L}_N$.

2010 *Mathematics subject classification*: primary 16N80.

Keywords and phrases: $*$-ring, extraspecial radical, prime radical, essential ideal, essential cover, special class, special and supernilpotent radicals, N-radicals complemented radicals, special atoms.

1. Introduction

In this paper, all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring zero. The fundamental definitions and properties of radicals can be found in [1, 15]. A class μ of rings is called hereditary (respectively, left hereditary) if μ is closed under ideals (respectively, left ideals). If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. As usual, for a radical ρ, the ρ radical of a ring R is denoted by $\rho(R)$ and the class of all ρ-semisimple rings is denoted by $\mathcal{S}(\rho)$. π denotes the class of all prime rings and $\beta = \mathcal{U}(\pi)$ denotes the prime radical. For a radical ρ, let $\pi(\rho) = \mathcal{S}(\rho) \cap \pi$. The notation $I \triangleleft R$ (respectively, $I < R$) means that I is a two-sided ideal (respectively, a left ideal) of a ring R. An ideal I of a ring R is called essential in R if $I \cap J \neq 0$ for any nonzero two-sided ideal J of R. A ring R is called an essential extension of a ring I if I is an essential ideal of R. A class μ of rings is called essentially closed if $\mu = \mu_k$, where $\mu_k = \{R : R$ is an essential extension of some $I \in \mu\}$. A hereditary and essentially closed class of prime rings is called a special class and the upper radical generated by a special class is called a special radical. A hereditary radical containing the prime radical β is called a supernilpotent radical. A radical ρ is called left strong if $L \in \rho$ implies $L \subseteq \rho(R)$ for all $L < R$. ρ is an N-radical [26] if it is left strong, left hereditary and contains the prime radical β. A semiprime ring R is called a $*$-ring [10] if $R/I \in \beta$ for any nonzero ideal I of R. The class of all $*$-rings will be denoted by $*$. An ideal I of a ring R is called a prime (respectively, semiprime) ideal of R if $R/I \in \pi$ (respectively, $R/I \in \mathcal{S}(\beta)$). The importance of the class $*_k$ is underlined by the following two facts:

Theorem 1.1 [8, 19]. *If R is a nonzero $*$-ring, then the smallest special (respectively, supernilpotent) radical $\widehat{l_R}$ (respectively, $\bar{l}_R$) containing R is an atom of the lattice of all special (respectively, supernilpotent) radicals.*

Theorem 1.2 [10, Proposition 2]. *If $R \in *_k$ and μ is a special class of rings, then $R \in \mathcal{S}(\mathcal{U}(\mu))$ if and only if $R \in \mu$. Thus, in particular, a ring $R \in *_k$ is Jacobson semisimple if and only if R is primitive.*

Gardner [14] introduced the notion of extraspecial radicals and gave their characterisation. He showed that a special radical α is extraspecial if and only if $\alpha = \mathcal{U}(\mathrm{Sir}(\alpha))$ where $\mathrm{Sir}(\alpha)$ is the class consisting of all rings $R \in \mathcal{S}(\alpha)$ such that $\cap\{I : 0 \neq I \lhd R$ and $R/I \in \mathcal{S}(\alpha)\} \neq 0$. He asked [14, Problem 1] whether β is extraspecial. Since $\mathrm{Sir}(\beta) = *_k$ [10], Gardner's question, in fact, asks whether $\beta = \mathcal{U}(*_k)$.

As proved in [14, Proposition 2.7] if a radical α is extraspecial, then any special class μ with the property $\alpha = \mathcal{U}(\mu)$ contains $\mathrm{Sir}(\alpha)$. In other words, $\mathrm{Sir}(\alpha)$ is the smallest special class which generates α. Thus, if β were extraspecial, then the class $*_k$ would be the smallest special class generating β. This would give a positive answer to a question put by Leavitt [13, Problem 1].

It is well known [1, 2, 29] that the family of special radicals (respectively, supernilpotent radicals, N-radicals) forms a complete lattice. We denote the lattice by $\mathbb{S}$ (respectively, $\mathbb{K}$, $\mathbb{L}_N$). The long-standing open problem of a description of special atoms (that is, atoms in $\mathbb{S}$) and supernilpotent atoms (that is, atoms in $\mathbb{K}$) was raised in [1] and then studied in [7–10, 19, 25]. The extraspeciality of β would settle this problem. Indeed, $\beta = \mathcal{U}(*_k)$ implies that every supernilpotent (respectively, special) radical strictly containing β contains a nonzero $*$-ring R and, hence, contains the supernilpotent (respectively, special) atom $\bar{l}_R$ (respectively, $\widehat{l_R}$).

Thus there is a motivation for finding a solution to Gardner's question. One way to accomplish this task is to study properties of $\mathcal{U}(*_k)$ and compare them with those enjoyed by β. This was initiated in [11, 12]. In this paper we will enrich the list of properties of the radical $\mathcal{U}(*_k)$ by showing that, just like β, $\mathcal{U}(*_k)$ is an N-radical. This enables us to obtain an equivalent reformulation of Gardner's question. Along the way, we show how to establish left hereditariness and left strongness of important upper radicals, give a full characterisation of complemented elements of the lattice $\mathbb{L}_N$ and show their connections with the question of Gardner.

2. Main results

N-radicals form an important class of radicals and have been investigated by many prominent authors [2, 16, 17, 24, 26, 27]. It is well known [15] that the prime radical β, the locally nilpotent radical $\mathcal{L}$ and the Jacobson radical $\mathcal{J}$ are N-radicals while the Brown–McCoy radical $\mathcal{G}$ is not. Moreover, the famous Koethe problem, which asks whether the nil radical $\mathcal{N}$ is left strong, is equivalent to the question whether $\mathcal{N}$ is an N-radical. So there is a good reason for studying N-radicals and therefore the search for new N-radicals continues. We will now show how to construct them.

In what follows, for a subset X of a ring A, $r(A, X) := \{a \in A : Xa = 0\}$ is the right annihilator of X in A. The left annihilator $l(A, X)$ of X in A is defined similarly. It is well known [15, page 87] that if $I \triangleleft A \in \mathcal{S}(\beta)$, then $l(A, I) = r(A, I) = \{a \in A : aI = 0 = Ia\}$. Also, it follows from [15, Example 3.17.10] that if $0 \neq L < A$ and A is a semiprime ring, then $\beta(L) = r(L, L)$.

Lemma 2.1. *For any prime number p the upper radical $\gamma = \mathcal{U}(\pi_p)$ generated by the class $\pi_p = \{R \in \pi : pR = 0\}$ is a special N-radical.*

Proof. It is easy to check that π_p is a special class with $\pi(\gamma) = \pi_p$. So γ is a special radical and, as such, contains β.

In view of [2, Theorem 18], to show that γ is left strong, it suffices to show that the class $\pi(\gamma)$ satisfies the following condition:

$$L < R \in \pi(\gamma) \text{ implies } L/r(L; L) \in \pi(\gamma). \tag{2.1}$$

Now, since $\pi(\gamma) = \pi_p$, it follows that $L < R \in \pi(\gamma)$ implies $pL = 0$ so $p(L/r(L; L)) = 0$. Moreover, since $L < R \in \pi$, it follows from [2, Lemma 3] that $L/r(L; L) \in \pi$. Thus $L/r(L; L) \in \pi(\gamma)$ and condition (2.1) holds.

Since γ is a special radical, in view of [2, Theorem 16], to show that γ is left hereditary it suffices to show that γ enjoys the following property:

$$L < R \in \pi \text{ and } \gamma(R) \neq 0 \text{ imply } (\gamma(L))^2 \neq 0. \tag{2.2}$$

Suppose that it does not. Then $(\gamma(L))^2 = 0$ for some nonzero $L < R \in \pi$ with $\gamma(R) \neq 0$. Then $\gamma(L) \subseteq \beta(L)$ and, since $\beta \subseteq \gamma$, it follows that $\gamma(L) = \beta(L)$. But, since $L < R \in \pi$, it follows from [2, Lemma 3] that $L/r(L; L) \in \pi$ and $\beta(L) = r(L; L)$. Thus $L/\beta(L) = L/r(L; L) = L/\gamma(L) \in \pi(\gamma)$ which implies that $pL \subseteq r(L; L)$. Then $(pL)^2 = 0$. Since, being a prime ring, R does not contain nonzero nilpotent left ideals and $pL < R$, we must have $pL = 0$. Then $pR = 0$ which implies that $R \in \pi_p = \pi(\gamma)$. But then $\gamma(R) = 0$ which is impossible. Thus γ satisfies condition (2.2) and is therefore an N-radical. □

We will now show that the radical $\mathcal{U}(*_k)$ is also an N-radical. We start with a technical but useful fact.

Lemma 2.2. *Let R be any ring and $I \triangleleft L < R$. If $0 \neq L/I \in \mathcal{S}(\beta)$ (respectively, π), then there exist a homomorphic image $\overline{R} \in \mathcal{S}(\beta)$ (respectively, π) of R and $\overline{K} < \overline{R}$ such that $L/I \cong \overline{K}/\beta(\overline{K})$.*

https://doi.org/10.1017/S0004972714000811 Published online by Cambridge University Press

Proof. Let J be an ideal of R maximal among all the ideals X of R that satisfy the condition $X \cap L \subseteq I$. First we will show that if $0 \neq L/I \in \mathcal{S}(\beta)$ (respectively, π), then $R/J \in \mathcal{S}(\beta)$ (respectively, π) and $LI \subseteq J$. Suppose that J_1 and J_2 are ideals of R strictly containing J such that $J_1J_2 \subseteq J$. Then, from the maximality of J, it follows that $J_1 \cap L \nsubseteq I$ and $J_2 \cap L \nsubseteq I$. Then $J_1 \cap L \lhd L$, $J_2 \cap L \lhd L$ and we have $(J_1 \cap L)(J_2 \cap L) \subseteq J_1J_2 \cap L \subseteq J \cap L \subseteq I$, a contradiction. Thus $R/J \in \mathcal{S}(\beta)$ (respectively, π).

Now, if $LI \nsubseteq J$, then $LIR \nsubseteq J$, as otherwise we would have $(LI)^2 \subseteq (LI)R \subseteq JR \subseteq J$ which implies $LI \subseteq J$ since $R/J \in \mathcal{S}(\beta)$, $(LI + J)/J < R/J$ and β is left strong, which is a contradiction. Thus $LI \subseteq J$. But then $I^2 \subseteq LI \subseteq L \cap J \subseteq I$. Let $\overline{R} = R/J$, $\overline{L} = L/(L \cap J) \cong (L + J)/J = \overline{K}$ and $\overline{I} = I/(L \cap J)$. Clearly, $(\overline{I})^2 = 0$ and $\overline{L} \cong \overline{K} < \overline{R}$. Moreover, since $\overline{L}/\overline{I} \cong L/I \in \mathcal{S}(\beta)$ and $\beta(\overline{L}) = \cap\{\overline{S} \lhd \overline{L} : \overline{L}/\overline{S} \in \mathcal{S}(\beta)\}$, it follows that $\beta(\overline{L}) \subseteq \overline{I}$. On the other hand, $\overline{I} \subseteq \beta(\overline{L})$ because $(\overline{I})^2 = 0$ and β, being a supernilpotent radical, contains all nilpotent rings. Thus $\beta(\overline{L}) = \overline{I}$, which gives $L/I \cong \overline{L}/\beta(\overline{L}) \cong \overline{K}/\beta(\overline{K})$. □

Our next result shows how to determine left hereditariness of many important upper radicals.

Theorem 2.3. *Let μ be a hereditary class of prime rings such that, for every ring R and every $L < R$, we have that $L/\beta(L) \in \mu$ implies $L^*/\beta(L^*) \in \mu$, where L^* denotes the two-sided ideal of R generated by L. Then the radical $\mathcal{U}(\mu_k)$ is left hereditary.*

Proof. Let $L < R \in \mathcal{U}(\mu_k)$ and suppose that $0 \neq L/I \in \mu_k$ for some $I \lhd L$. Then there exists $0 \neq K/I \lhd L/I$ such that $K/I \in \mu \subseteq \pi$. If $LK \subseteq I$, then we would have two nonzero ideals, namely L/I and K/I, of a prime ring K/I with $(L/I)(K/I) = 0$, which is impossible. Thus $LK \nsubseteq I$ which implies that $0 \neq (LK + I)/I \lhd K/I \in \mu$. Then, by the hereditariness of μ, we get $LK/(I \cap LK) \cong (LK + I)/I \in \mu$. Thus $0 \neq LK/(I \cap LK) \in \mu \subseteq \pi$ and so, since $LK < R$, it follows from Lemma 2.2 that there exists a homomorphic image $\overline{R} \in \pi$ of R and $\overline{K} < \overline{R}$ such that $LK/(I \cap LK) \cong \overline{K}/\beta(\overline{K}) \in \mu$. But then our assumption ensures that $(\overline{K})^*/\beta((\overline{K})^*) \in \mu$, where $(\overline{K})^*$ is the ideal of $\overline{R}$ generated by $\overline{K}$. But, since $\overline{R} \in \pi$, it follows that $\beta((\overline{K})^*) = 0$, which implies that $(\overline{K})^* \in \mu$. Moreover, since $0 \neq LK/(I \cap LK)$, it follows that $0 \neq \overline{K}$. So $0 \neq (\overline{K})^*$ because $0 \neq \overline{K} \subseteq (\overline{K})^*$. Consequently we get $0 \neq \overline{R} \in \mu_k$, which contradicts the assumption that $R \in \mathcal{U}(\mu_k)$ and concludes the proof. □

Corollary 2.4. *$\mathcal{U}(*_k)$ is a left hereditary special radical.*

Proof. We have $* \subseteq \pi$, and it was proved in [7] that the class $*$ is closed under two-sided ideals. Moreover, it was shown in [11] that $L/\beta(L) \in *$ implies $L^*/\beta(L^*) \in *$ for every ring R and every $L < R$. Thus Theorem 2.3 implies that $\mathcal{U}(*_k)$ is a left hereditary radical. Clearly, $*_k$ is a special class, so $\mathcal{U}(*_k)$ is a special radical. □

We will now show how to establish left strongness of many important upper radicals.

https://doi.org/10.1017/S0004972714000811 Published online by Cambridge University Press

Theorem 2.5. *Let μ be a hereditary class of semiprime rings that satisfies condition $(\cdot)$: $K < A \in \mu$ implies $K/\beta(K) \in \mu$ for all rings A and K. Then the class μ_k also satisfies condition $(\cdot)$ and the radical $\mathcal{U}(\mu_k)$ is left strong.*

Proof. Let μ be a hereditary class of semiprime rings that satisfies condition $(\cdot)$. First we will show that the class μ_k also satisfies condition $(\cdot)$.

Let $0 \neq K < A \in \mu_k$. We want to show that $K/\beta(K) \in \mu_k$. Since $\mu_k \subseteq \mathcal{S}(\beta)$, we have $\beta(K) = r(K, K)$ and, since β is left strong and $A \in \mu_k$, $\beta(K) \neq K$. Thus we need to show that the nonzero ring $K/r(K, K)$ is an essential extension of some ring from μ. Now, since $A \in \mu_k$, there exists an essential ideal L of A such that $L \in \mu$. We will show that the factor ring $(LK + r(K, K))/r(K, K)$ is an essential ideal of $K/r(K, K)$ and that this factor ring belongs to μ. Since $K/r(K, K) = K/\beta(K) \in \mathcal{S}(\beta)$, for the essentiality in question, it suffices to show that $r(K/r(K, K), (LK + r(K, K))/r(K, K)) = 0$. To do so, let $k \in K$ be such that $LKk \subseteq r(K, K)$. Then, since $K < A$, we have $(LKk)^2 \subseteq K(LKk) = 0$. But since $LKk < L \in \mu \subseteq \mathcal{S}(\beta)$ and β is left strong, it implies that $LKk = 0$. Now, since L is an essential ideal of the semiprime ring A, its annihilator in A is $r(A, L) = 0$. We therefore must have $Kk = 0$ which means that $k \in r(K, K)$. This proves that, indeed, $(LK + r(K, K))/r(K, K)$ is an essential ideal of $K/r(K, K)$.

Note that, in fact, we have just shown that $r(K, LK) \subseteq r(K, K)$. Moreover, since $Kk = 0$ implies $LKk = 0$, we also have $r(K, K) \subseteq r(K, LK)$ which gives $r(K, LK) = r(K, K)$. Then $LK \cap r(K, K) = LK \cap r(K, LK) = r(LK, LK) = \beta(LK)$ because $LK < A \in \mathcal{S}(\beta)$. But we also have that $LK < L \in \mu$ and μ satisfies condition $(\cdot)$. Therefore $LK/\beta(LK) \in \mu$. But then we have $(LK + r(K, K))/r(K, K) \cong (LK)/(LK \cap r(K, K)) = (LK)/\beta(LK) \in \mu$.

Thus the nonzero ring $K/\beta(K)$ is an essential extension of the ring $(LK + \beta(K))/\beta(K) \in \mu$, which means that $0 \neq K/\beta(K) \in \mu_k$. We have therefore shown that any nonzero left ideal K of any ring A from μ_k can be homomorphically mapped onto a nonzero ring $K/\beta(K)$ from μ_k. In view of [6, Theorem 9], this implies that $\mathcal{U}(\mu_k)$ is left strong and completes the proof. □

Corollary 2.6. *$\mathcal{U}(*_k)$ is a left strong radical.*

Proof. It follows from [11, Proof of Theorem 3], that the hereditary class $* \subseteq \pi \subseteq \mathcal{S}(\beta)$ satisfies condition $(\cdot)$. So, by Theorem 2.5, we have that $\mathcal{U}(\mu_k)$ is left strong. □

Corollaries 2.4 and 2.6 imply the following result:

Corollary 2.7. *$\mathcal{U}(*_k)$ is a special N-radical.*

It is well known (see [2]) that inclusion on the collection $\mathbb{L}_N$ of all N-radicals of associative rings gives rise to a complete, distributive and bounded sublattice of the lattice $\mathbb{K}$ of all supernilpotent radicals. Its smallest element is the prime radical β and its greatest element is the trivial radical 1 that consists of all associative rings. As in the lattice $\mathbb{K}$, for a family $\{\rho_i\}_{i\in I}$ of N-radicals, its union $\bigvee_{i\in I} \rho_i$ is the lower radical generated by the class $\bigcup_{i\in I} \rho_i$ while its meet $\wedge_{i\in I}\rho_i$ is $\bigcap_{i\in I} \rho_i$. A supernilpotent

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radical (respectively, an N-radical) ρ is complemented in $\mathbb{K}$ (respectively, $\mathbb{L}_N$) if there exists $\rho^c \in \mathbb{K}$ (respectively, $\rho^c \in \mathbb{L}_N$) called a complement of ρ in $\mathbb{K}$ (respectively, a complement of ρ in $\mathbb{L}_N$) such that $\rho \vee \rho^c = 1$ and $\rho \wedge \rho^c = \beta$. It is well known (see [4]), that in any distributive lattice, complements are unique if they exist. Complements give a nice decomposition of rings [29] and they have been widely studied (see [1, 3, 21, 22, 28, 29]). We will now show that complements of $\mathbb{K}$ and complements of $\mathbb{L}_N$ are connected.

In [29] Snider proved that for any hereditary radicals γ and ρ, the class

$$(\gamma : \rho) = \{R : \rho(R/I) \subseteq \gamma(R/I) \text{ for every } I \lhd R\}$$

is the largest radical among those radicals δ that satisfy the condition $\delta(R) \cap \rho(R) \subseteq \gamma(R)$ for every ring R. Moreover, if δ and γ are both hereditary, then $\delta(R) \cap \rho(R) = (\delta \wedge \rho)(R)$ for every ring R. Now, it follows from [5, Theorem 6] that if γ is a hereditary radical and ρ is any radical containing β, then $(\gamma : \rho)$ is hereditary. Thus, in the special case where $\gamma = \beta$ and $\rho \in \mathbb{K}$ we have that $(\beta : \rho) = \{R : \rho(R/I) = \beta(R/I) \text{ for every } I \lhd R\}$ is the biggest supernilpotent radical among all radicals δ that satisfy the condition $\delta \wedge \rho = \beta$. Therefore, if a supernilpotent radical ρ has a complement $\rho^c \in \mathbb{K}$, the uniqueness of the complement in $\mathbb{K}$ guarantees that $\rho^c = (\beta : \rho)$. Moreover, it was proved in [17] that if ρ and γ are both N-radicals, then so is the radical $(\rho : \gamma)$. So, since β is an N-radical, $(\beta : \rho)$ is also an N-radical for every N-radical ρ. We have therefore proved the following lemma.

Lemma 2.8. *Let ρ be a supernilpotent radical that has the complement ρ^c in $\mathbb{K}$. Then $\rho^c = (\beta : \rho) = \{R : \rho(R/I) = \beta(R/I)$ for every $I \lhd R\}$. Moreover, if in addition, ρ is an N-radical, then so is ρ^c.*

Note that $(\beta : \mathcal{J})$ rings are known as Jacobson rings and play an important role in commutative ring theory [18]; noncommutative Jacobson rings have been studied by Procesi [23] and Watters [30, 31]. Similarly, to commemorate our collaborative research, we could call the radical $\mathcal{U}(*_k)$ an *IndaH radical* and then define *IndaH rings* as $(\beta : \mathcal{U}(*_k))$ rings. Our next remark provides a very good reason for investigating IndaH rings.

Remark 2.9. $\beta = \mathcal{U}(*_k)$ if and only if $(\beta : \mathcal{U}(*_k)) = 1$.

Complements of the lattice $\mathbb{K}$ have been extensively studied by Kracilov in [20–22]. To present Kracilov's results we need to recall his notation.

For a ring A, $[A]_m$ denotes the ring of all $n \times n$ matrices over A and $\mathrm{Var}(A)$ is the variety generated by A. For $A \in \pi$ satisfying a proper polynomial identity, let $\mu(A) = \max\{m : [F]_m \in \mathrm{Var}(A) \text{ for some division ring } F\}$. Set $\mu(A) = \infty$ if A does not satisfy a proper polynomial identity. For any prime number p, let

$$\pi_p = \{A \in \pi : pA = 0\},$$

$$\kappa_p = \Big(\bigcup_{i\in I, k\in K} [Z_p^{(k)}]_i\Big) \cup \bigcup_{l\in L}\Big(\{R \in \pi : \mathrm{Var}(R) = \mathrm{Var}[Z_p^{(\infty)}]_l\} \cup \Big(\bigcup_{m\in M} [Z_p^{(m)}]_l\Big)\Big)$$

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where I, K, L and M are finite sets of positive integers, Z_p is the p-element field, $Z_p^{(n)}$ denotes the n-dimensional field over Z_p and $Z_p^{(\infty)}$ stands for the algebraic closure of Z_p.

The following theorem briefly summarises Kracilov's description of complements of $\mathbb{K}$.

Theorem 2.10 [21, 22]. *A supernilpotent radical ρ has a complement ρ^c in $\mathbb{K}$ if and only if there exist finite sets $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ of prime numbers and a finite set Δ_0 of positive integers (some of which may be empty) such that either $\pi(\rho) = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$ or $\pi(\rho^c) = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$, where $\sigma_1 = \bigcup_{p\in\Delta_1} \pi_p$, $\sigma_2 = \bigcup_{p\in\Delta_2} (\pi_p \backslash \kappa_p)$, $\sigma_3 = \bigcup_{n\in\Delta_0} \{A \in \pi : \mu(A) = n\} \backslash \bigcup_{p\in\Delta_3} \kappa_p$, $\sigma_4 = \bigcup_{p\in\Delta_4} \kappa_p$ and the classes $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are mutually disjoint.*

We are now ready to describe complements of the lattice $\mathbb{L}_N$.

Theorem 2.11. *An N-radical ρ has the complement ρ^c in the lattice $\mathbb{L}_N$ if and only if there exists a finite (possibly empty) set Π of prime numbers such that either $\pi(\rho) = \bigcup_{p\in\Pi} \pi_p$ or $\pi(\rho^c) = \bigcup_{p\in\Pi} \pi_p$. If Π is empty, then we take $\bigcup_{p\in\Pi} \pi_p = \{0\}$.*

Proof. Let ρ^c be the complement of $\rho \in \mathbb{L}_N$ in $\mathbb{L}_N$. Since $\mathbb{L}_N$ is a sublattice of the lattice $\mathbb{K}$, it follows that ρ^c is the complement of ρ in $\mathbb{K}$. So, by Theorem 2.10, we may assume that $\pi(\rho) = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$, where the σ_i are the classes described in Theorem 2.10. Now, if $\sigma_2 \cup \sigma_3 \cup \sigma_4 \neq \{0\}$, then it follows from the definition of the σ_i that there are only a finite number of positive integers n such that $[\Phi]_n \in \sigma_2 \cup \sigma_3 \cup \sigma_4$, where Φ is either the field Q of rational numbers or the field $Z_p^{(m)}$ for some prime number p and some positive integer m. This implies that $[\Phi]_n \in \mathcal{S}(\rho)$. Now, since ρ is an N-radical and since every N-radical is matrix extensible [15, Corollary 4.9.7], it follows that $\Phi \in \mathcal{S}(\rho)$. Then, using the matrix extensibility of ρ again, we obtain $[\Phi]_t \in \mathcal{S}(\rho)$ for every positive integer t which contradicts the finiteness of n. Thus $\sigma_2 \cup \sigma_3 \cup \sigma_4 = \{0\}$ and then $\pi(\rho) = \sigma_1 = \bigcup_{p\in\Delta_1} \pi_p$.

The converse follows from Theorem 2.10 and Lemma 2.1. □

Corollary 2.12. *$\beta = \mathcal{U}(*_k)$ if and only if $\mathcal{U}(*_k)$ is complemented in the lattice $\mathbb{L}_N$.*

Proof. Since $\beta \wedge 1 = \beta$ and $\beta \vee 1 = 1$ in $\mathbb{L}_N$, if $\beta = \mathcal{U}(*_k)$, then $\mathcal{U}(*_k)^c = \beta^c = 1$ in $\mathbb{L}_N$ so $\mathcal{U}(*_k)$ is complemented in $\mathbb{L}_N$.

Conversely, suppose that $\mathcal{U}(*_k)$ is complemented in $\mathbb{L}_N$. Then it is complemented in $\mathbb{K}$, and, by Lemma 2.8, $\mathcal{U}(*_k)^c = (\beta : \mathcal{U}(*_k))$. Moreover, by Theorem 2.11, there exists a finite (possibly empty) set Π of prime numbers such that either $\pi(\mathcal{U}(*_k)) = \bigcup_{p\in\Pi} \pi_p$ or $\pi((\beta : \mathcal{U}(*_k))) = \bigcup_{p\in\Pi} \pi_p$.

But $\pi(\mathcal{U}(*_k))$ contains simple prime rings of characteristic zero. For example, the field Q of rational numbers is in $\pi(\mathcal{U}(*_k)) \backslash \bigcup_{p\in\Pi} \pi_p$, so we must have $\pi(\beta : \mathcal{U}(*_k)) = \bigcup_{p\in\Pi} \pi_p$. Then, for every $p \in \Pi$, we have $Z_p \in \bigcup_{p\in\Pi} \pi_p \subseteq \mathcal{S}(\beta : \mathcal{U}(*_k))$. Since $\beta(Z_p/I) = 0 = \mathcal{U}(*_k)(Z_p/I)$ for every $I \lhd Z_p$, we have $Z_p \in \mathcal{S}(\beta : \mathcal{U}(*_k)) \cap (\beta : \mathcal{U}(*_k)) = \{0\}$, a contradiction. Thus Π is empty, which means that $\pi((\beta : \mathcal{U}(*_k))) = \{0\}$. This shows that $(\beta : \mathcal{U}(*_k)) = 1$ which, in view of our Remark 2.9, means that $\beta = \mathcal{U}(*_k)$. □

https://doi.org/10.1017/S0004972714000811 Published online by Cambridge University Press

Acknowledgement

The first author wishes to express her deep gratitude and thank all the members and students of the Department of Mathematics of the Universitas Gadjah Mada for all their support and warm hospitality during the preparation of this paper.

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