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CLASSIFICATION OF IRREDUCIBLE HARISH-CHANDRA MODULES OVER GENERALIZED VIRASORO ALGEBRAS

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Abstract Let G be an arbitrary non-zero additive subgroup of the complex number field $\mathbb{C}$, and let $\mathrm{Vir}[G]$ be the corresponding generalized Virasoro algebra over $\mathbb{C}$. In this paper we determine all irreducible weight modules with finite-dimensional weight spaces over $\mathrm{Vir}[G]$. The classification strongly depends on the index group G. If G does not have a direct summand isomorphic to $\mathbb{Z}$ (the integers), then such irreducible modules over $\mathrm{Vir}[G]$ are only modules of intermediate series whose weight spaces are all one dimensional. Otherwise, there is one further class of modules that are constructed by using intermediate series modules over a generalized Virasoro subalgebra $\mathrm{Vir}[G_0]$ of $\mathrm{Vir}[G]$ for a direct summand G_0 of G with $G = G_0 \oplus \mathbb{Z}b$, where $b \in G \setminus G_0$. This class of irreducible weight modules do not have corresponding weight modules for the classical Virasoro algebra.

Keywords: generalized Virasoro algebra; weight module; Harish-Chandra module

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1. Introduction

Virasoro algebra theory has been widely used in many areas of physics and branches of mathematics, for example, string theory [**5**], modular forms [**7**], conformal field theory [**3**], Kac–Moody algebras [**6**] and vertex algebras [**2**].

Generalized Virasoro algebras were first introduced and studied by mathematicians and mathematical physicists Patera and Zassenhaus [**20**] in 1991. Because of interest in its own right as well as its close relationship to physics, the representation theory of generalized Virasoro algebras has attracted extensive attention from mathematicians and physicists. The theory has been developed particularly rapidly in the last decade. Let us first recall the definitions of these Lie algebras.

In this paper we denote by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the set of complex numbers, integers, non-negative integers and positive integers, respectively.

The *Virasoro algebra* $\mathrm{Vir} := \mathrm{Vir}[\mathbb{Z}]$ (over $\mathbb{C}$) is the Lie algebra with the basis $\{C, d_i \mid i \in \mathbb{Z}\}$ and the Lie brackets defined by

$$[d_m, d_n] = (n-m)d_{m+n} + \tfrac{1}{12}\delta_{m,-n}(m^3 - m)C \quad \text{for all } m, n \in \mathbb{Z},$$
$$[d_m, C] = 0 \quad \text{for all } m \in \mathbb{Z}.$$

The structure theory of Harish-Chandra modules over the Virasoro algebra is fairly well developed. For details, we refer the reader to [**8**, **14**, **15**, **19**] and the references therein. In particular, the Kac conjecture, i.e. the classification of irreducible Harish-Chandra modules, was obtained in [**15**], while indecomposable modules were studied in [**14**]. The classification for irreducible weight modules having a finite-dimensional non-zero weight space was given in [**19**]. These classifications were recently used to give the classification of irreducible Harish-Chandra modules over the twisted Heisenberg–Virasoro algebra [**13**].

Patera and Zassenhaus [**20**] introduced the *generalized Virasoro algebra* $\mathrm{Vir}[G]$ for any additive subgroup G of $\mathbb{C}$ from the context of mathematics and physics. This Lie algebra can be obtained from Vir by simply replacing the index group $\mathbb{Z}$ with G (see Definition 2.1). This Lie algebra $\mathrm{Vir}[G]$ is called a *rank-n Virasoro algebra* (or a *higher-rank Virasoro algebra* when $n \geqslant 2$) if $G \simeq \mathbb{Z}^n$.

Since generalized Virasoro algebras $\mathrm{Vir}[G]$ were defined in 1991, their representation theory has been extensively studied by many authors (see, for example, [**1**, **11**, **12**, **16**–**18**, **21**–**24**]). We now give a little more detail.

Su [**21**] determined Harish-Chandra modules of the intermediate series (all weight spaces are one dimensional) over high rank Virasoro algebras. Su and Zhao [**24**] determined Harish-Chandra modules of the intermediate series over generalized Virasoro algebras. Mazorchuk [**18**] proved that all irreducible Harish-Chandra modules over $\mathrm{Vir}[\mathbb{Q}]$ are intermediate series modules (where $\mathbb{Q}$ is the field of rational numbers). In [**4**], a criterion for the irreducibility of Verma modules over the generalized Virasoro algebra $\mathrm{Vir}[G]$ was obtained. In [**22**, **23**], Su gave a rough classification for irreducible Harish-Chandra modules over higher-rank Virasoro algebras (the existence of the second class of modules was not clear in [**23**]). Billig and Zhao [**1**] constructed a new class of irreducible Harish-Chandra modules over generalized Virasoro algebras different from modules of the intermediate series. Finally, a precise classification of irreducible Harish-Chandra modules over higher-rank Virasoro algebras was given by Lu and Zhao [**12**].

We shall use the results in [**12**] to give a precise classification of irreducible Harish-Chandra modules over any generalized Virasoro algebra, including higher-rank Virasoro algebras. From our proof, one can easily see that the method we have employed here is very different from those used in [**12**, **13**, **15**, **23**].

The paper is organized as follows. In § 2, for the reader's convenience, we recall some notation and collect some related results from [**12**, **14**, **18**, **24**]. In § 3, we first establish a localization method in Lemma 3.1 to localize the problem, i.e. to consider part of the module V over a finite-rank Virasoro algebra so that the partial module is still irreducible (to some extent) over a suitably smaller higher-rank Virasoro algebra. We see that any finite-dimensional subspace of an irreducible Harish-Chandra module V over $\mathrm{Vir}[G]$ can be considered as a subsapce of an irreducible submodule of V over a suitable higher-rank Virasoro algebra which is a subalgebra of $\mathrm{Vir}[G]$. This allows us to use the result in [**12**] to show in Theorem 3.4 that any irreducible uniformly bounded module over $\mathrm{Vir}[G]$ is a module of the intermediate series. The rest of the section (i.e. from Lemma 3.5 to Theorem 3.8) is devoted to handling the non-uniformly bounded case for V. In this case we cannot directly use the result in [**12**], so we have to establish new methods. In Lemma 3.5 we obtain a sufficient condition for a certain part of V to be uniformly bounded. Next, in Lemmas 3.6 and 3.7, for $\lambda \in \operatorname{supp} V \setminus \{0\}$ and subgroups $G_1, G_2 \subset G$, if both subspace $V_{\lambda+G_1}$ and $V_{\lambda+G_2}$ are uniformly bounded (as $\mathrm{Vir}[G_1]$-modules and $\mathrm{Vir}[G_2]$-modules, respectively), then we can prove that $V_{\lambda+G_1+G_2}$ is a uniformly bounded $\mathrm{Vir}[G_1+G_2]$-module. In this way we can find the maximal subgroup G_0 of G so that $V_{\lambda+G_0}$ is a uniformly bounded $\mathrm{Vir}[G_0]$-module. Then, we show that G_0 is independent of λ, and G_0 has a direct complement summand isomorphic to $\mathbb{Z}$ if $G_0 \neq G$. With the last effort in Theorem 3.8, we complete the proof.

The classification strongly depends on the index group G. If G does not have a direct summand isomorphic to $\mathbb{Z}$, then irreducible modules over $\mathrm{Vir}[G]$ are only modules of intermediate series whose weight spaces are all one dimensional. Otherwise, there is one more class of modules that are constructed by using intermediate series modules over a generalized Virasoro subalgebra $\mathrm{Vir}[G_0]$ of $\mathrm{Vir}[G]$ for a direct summand G_0 of G with $G = G_0 \oplus \mathbb{Z}b$, where $b \in G \setminus G_0$ (Theorem 3.9).

Throughout this paper, a subgroup always means an additive subgroup if not specified. For any $a \in \mathbb{C}$ and $S \subset \mathbb{C}$, we define $a+S = \{a+x \mid x \in S\}$ and $aS = Sa = \{ax \mid x \in S\}$.

2. Modules over generalized Virasoro algebras

First we recall the precise definition of the generalized Virasoro algebras.

Definition 2.1. Let G be a non-zero additive subgroup of $\mathbb{C}$. The *generalized Virasoro algebra* $\mathrm{Vir}[G]$ (over $\mathbb{C}$) is the Lie algebra with the basis $\{C, d_x \mid x \in G\}$ and the Lie brackets defined by

$$\begin{aligned} [d_x, d_y] &= (y-x)d_{x+y} + \tfrac{1}{12}\delta_{x,-y}(x^3-x)C \quad \text{for all } x, y \in G, \\ [C, d_x] &= 0 \quad \text{for all } x \in G. \end{aligned}$$

It is clear that $\mathrm{Vir}[G] \cong \mathrm{Vir}[aG]$ for any $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then, for any $x \in G^* := G \setminus \{0\}$, $\mathrm{Vir}[x\mathbb{Z}]$ is a Lie subalgebra of $\mathrm{Vir}[G]$ isomorphic to $\mathrm{Vir} = \mathrm{Vir}[\mathbb{Z}]$, the classical Virasoro algebra.

A $\mathrm{Vir}[G]$-module V is called *trivial* if $\mathrm{Vir}[G]V = 0$. For any $\mathrm{Vir}[G]$-module V and $c, \lambda \in \mathbb{C}$, $V_{\lambda,c} := \{v \in V \mid d_0 v = \lambda v,\ Cv = cv\}$ is called the *weight space* of V corresponding to the weight (λ, c). When C acts as a scalar c on the whole module V, we shall simply write V_λ instead of $V_{\lambda,c}$.

A $\mathrm{Vir}[G]$-module V is called a *weight module* if V is the sum of all its weight spaces, i.e. $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$. And a weight module V is called a *Harish-Chandra module* if all the weight spaces are finite dimensional, i.e. $\dim V_\lambda < \infty$ for all $\lambda \in \mathbb{C}$. For a weight module

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V, we define $\operatorname{supp} V := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$, which is generally called the *weight set* (or the *support*) of V. Given a weight module V and any subset $S \subset \mathbb{C}$, we define $V_S = \bigoplus_{x\in S} V_x$, where $V_x = 0$ for $x \notin \operatorname{supp} V$.

Let V be a module and let $W' \subset W$ be submodules of V. The module W/W' is called a *subquotient* of V. If $W' = 0$, we consider that $W = W/W'$.

Let V be a weight module over $\mathrm{Vir}[G]$. We say that V is *uniformly bounded* if there exists $N \in \mathbb{N}$ such that $\dim V_\lambda < N$ for all $\lambda \in \operatorname{supp} V$.

Take a fixed total order '$\preceq$' on G which is compatible with the addition, i.e. $x \succeq y$ implies $x + z \succeq y + z$ for any $x, y, z \in G$. Let

$$G^+ := \{x \in G \mid x \succ 0\}, \qquad G^- := \{x \in G \mid x \prec 0\},$$
$$\mathrm{Vir}[G]^+ := \sum_{x\in G^+} \mathbb{C}d_x, \qquad \mathrm{Vir}[G]^- := \sum_{x\in G^-} \mathbb{C}d_x.$$

Let V be a weight module over $\mathrm{Vir}[G]$. A vector $v \in V_{\lambda,c}$, $\lambda \in \operatorname{supp} V$, $c \in \mathbb{C}$, is called a *highest weight* (*respectively, lowest weight*) *vector* if $\mathrm{Vir}[G]^+ v = 0$ (respectively, $\mathrm{Vir}[G]^- v = 0$). V is called a *highest weight* (*respectively, lowest weight*) *module* with highest weight (respectively, lowest weight) (λ, c) if there exists a non-zero highest (respectively, lowest) weight vector $v \in V_{\lambda,c}$ such that V is generated by v. For the natural total order on $\mathbb{Z}$, the irreducible highest weight $\mathrm{Vir}[\mathbb{Z}]$-module with highest weight (λ, c) is generally denoted by $V(c, \lambda)$.

Now we give another class of weight modules over $\mathrm{Vir}[G]$, i.e. the *modules of intermediate series* $V(\alpha, \beta, G)$. For any $\alpha, \beta \in \mathbb{C}$, the module $V(\alpha, \beta, G)$ has a basis $\{v_x \mid x \in G\}$ with actions of $\mathrm{Vir}[G]$ given by

$$Cv_y = 0, \quad d_x v_y = (\alpha + y + x\beta)v_{x+y} \quad \text{for all } x, y \in G.$$

One knows from [**24**] that $V(\alpha, \beta, G)$ is reducible if and only if $\alpha \in G$ and $\beta \in \{0, 1\}$. By $V'(\alpha, \beta, G)$ we denote the unique non-trivial irreducible subquotient of $V(\alpha, \beta, G)$. Then $\operatorname{supp}(V'(\alpha, \beta, G)) = \alpha + G$ or $\operatorname{supp}(V'(\alpha, \beta, G)) = G \setminus \{0\}$. We also define $V'(\alpha, \beta, G)$ as *intermediate series modules*. The following result is due to Su and Zhao.

Theorem 2.2 (Su and Zhao [24, Theorem 4.6]). *Let V be a non-trivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *with all weight spaces one dimensional. Then $V \cong V'(\alpha, \beta, G)$ for some $\alpha, \beta \in \mathbb{C}$.*

This result for the classical Virasoro algebra is due to Kaplansky [**10**]. The following classification of irreducible Harish-Chandra modules over the classical Virasoro algebra was obtained by Mathieu.

Theorem 2.3 (Mathieu [15]). *Every irreducible Harish-Chandra module over* Vir *is either a highest weight module, a lowest weight module or a module of intermediate series.*

We say that a $\mathrm{Vir}[\mathbb{Z}b]$-module W is *positively truncated* (*negatively truncated*) relative to b if for any $\lambda \in \operatorname{supp} V$, there exists some $x_0 \in \mathbb{Z}$ such that $\operatorname{supp}(V_{\lambda+\mathbb{Z}b}) \subset \{\lambda + xb \mid x \leqslant x_0\}$ (respectively, $\operatorname{supp}(V_{\lambda+\mathbb{Z}b}) \subset \{\lambda + xb \mid x \geqslant x_0\}$). The following result from [**14**] will be useful for our later proofs.

Theorem 2.4 (Martin and Piard [14]). *Every Harish-Chandra* $\mathrm{Vir}[\mathbb{Z}]$*-module* V *which has neither trivial submodules nor trivial quotient modules can be decomposed as a direct sum of three submodules* $V = V^+ \oplus V^0 \oplus V^-$*, where* V^+ *is positively truncated,* V^- *is negatively truncated and* V^0 *is uniformly bounded.*

Now we return to generalized Virasoro algebras. Marzorchuck [**18**] proved the following.

Theorem 2.5 (Marzorchuck [18]). *Any non-trivial irreducible Harish-Chandra module over* $\mathrm{Vir}[\mathbb{Q}]$ *is a module of intermediate series.*

In fact, Marzorchuck proved the following theorem, as he remarked at the end of [**18**].

Theorem 2.6. *Let* G *be an subgroup of* $a\mathbb{Q}$ *for some* $a \in \mathbb{C}$*, which is not finitely generated. Then any non-trivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is a module of intermediate series.*

Now we assume that $G = G_0 \oplus \mathbb{Z}b \subset \mathbb{C}$, where $0 \neq b \in \mathbb{C}$ and G_0 is a non-zero subgroup of $\mathbb{C}$. (Note that some G does not possess this property, for example, $\mathbb{Q}$, or any other subfield of $\mathbb{C}$.) Set

$$\mathrm{Vir}[G]_+ = \bigoplus_{x \in G_0, k \in \mathbb{Z}^+} \mathbb{C}d_{x+kb} \oplus \mathbb{C}C$$

and

$$\mathrm{Vir}[G]_{++} = \bigoplus_{x \in G_0, k \in \mathbb{N}} \mathbb{C}d_{x+kb}.$$

Given any $\alpha, \beta \in \mathbb{C}$, let $V'(\alpha, \beta, G_0)$ be the module of intermediate series over $\mathrm{Vir}[G_0]$. We extend the $\mathrm{Vir}[G_0]$-module structure on $V'(\alpha, \beta, G_0)$ to a $\mathrm{Vir}[G]_+$-module structure by defining $\mathrm{Vir}[G]_{++} V'(\alpha, \beta, G_0) = 0$. Then we obtain the induced $\mathrm{Vir}[G]$-module

$$M(b, G_0, V'(\alpha, \beta, G_0)) = U(\mathrm{Vir}[G]) \otimes_{U(\mathrm{Vir}[G]_+)} V'(\alpha, \beta, G_0),$$

where $U(\mathrm{Vir}[G])$ and $U(\mathrm{Vir}[G]_+)$ are the universal enveloping algebras of $\mathrm{Vir}[G]$ and $\mathrm{Vir}[G]_+$, respectively.

We see that

$$\operatorname{supp} M(b, G_0, V'(\alpha, \beta, G_0)) = \alpha - \mathbb{Z}^+ b + G_0 \quad \text{or} \quad (-\mathbb{Z}^+ b + G_0) \setminus \{0\}.$$

The $\mathrm{Vir}[G]$-module $M(b, G_0, V'(\alpha, \beta, G_0))$ has a unique maximal proper submodule J. Then we obtain the irreducible quotient module

$$V(\alpha, \beta, b, G_0) := M(b, G_0, V'(\alpha, \beta, G_0))/J.$$

It is clear that this module is uniquely determined by α, β, b and G_0 and that [**12**, Lemma 3.8]

$$\operatorname{supp} V(\alpha, \beta, b, G_0) = \alpha - \mathbb{Z}^+ b + G_0 \quad \text{or} \quad (-\mathbb{Z}^+ b + G_0) \setminus \{0\}.$$

It was proved that $V(\alpha, \beta, b, G_0)$ is a Harish-Chandra module over $\mathrm{Vir}[G]$.

Theorem 2.7 (Billig and Zhao [1, Theorem 3.1]). *The* $\mathrm{Vir}[G]$*-modules* $V(\alpha,\beta, b,G_0)$ *are Harish-Chandra modules. More precisely,*

$$\dim V_{-ib+\alpha+x} \leqslant 1\cdot 3\cdot 5\cdots(2i+1)$$

for all $i\in\mathbb{N}$, $x\in G_0$.

From this theorem, we easily deduce the following corollary, which will be used frequently in our later proofs.

Corollary 2.8. *Let* $V=V(\alpha,\beta,b,G_0)$ *and* $i\in\mathbb{Z}^+$. *Then for any subgroup* G' *of* G, *the* $\mathrm{Vir}[G']$ *module* $V_{\alpha-ib+G'}$ *is uniformly bounded if and only if* $G'\subset G_0$.

The classification of irreducible Harish-Chandra modules over higher-rank Virasoro algebras was obtained by Lu and Zhao [**12**].

Theorem 2.9. *Let* G *be an additive subgroup of* $\mathbb{C}$ *such that* $G\cong\mathbb{Z}^n$ *for some* $n\in\mathbb{N}$ *with* $n>1$. *Then any non-trivial irreducible Harish-Chandra module over* $\mathrm{Vir}[G]$ *is either of intermediate series or isomorphic to some* $V(\alpha,\beta,G_0,b)$ *for some* $\alpha,\beta\in\mathbb{C}$, $b\in G\setminus\{0\}$ *and a subgroup* G_0 *of* G *such that* $G=G_0\oplus\mathbb{Z}b$.

3. Classification of irreducible Harish-Chandra modules over generalized Virasoro algebras

In this section we give a classification of the irreducible Harish-Chandra modules over generalized Virasoro algebras $\mathrm{Vir}[G]$. Let us proceed with the convention $U(G)=U(\mathrm{Vir}[G])$, the enveloping algebra of $\mathrm{Vir}[G]$, and

$$U(G)_a=\{y\in U(\mathrm{Vir}[G])\mid [d_0,y]=ay\},\quad a\in\mathbb{C},$$

for any additive subgroup G of $\mathbb{C}$.

We recall the concept of the rank for a subgroup A of $\mathbb{C}$ from [**9**]. The *rank* of A, denoted by $\mathrm{rank}(A)$, is the maximal number r with $g_1,\dots,g_r\in A\setminus\{0\}$ such that $\mathbb{Z}g_1+\cdots+\mathbb{Z}g_r$ is a direct sum. If such an r does not exist, we define $\mathrm{rank}(A)=\infty$. For example, $\mathbb{Q}+\mathbb{Q}\sqrt{2}$ and $\mathbb{Z}+\mathbb{Z}\sqrt{2}$ both have rank 2.

From now on G is a fixed non-zero subgroup of $\mathbb{C}$ and a non-trivial irreducible Harish-Chandra module V over $\mathrm{Vir}[G]$. Because of the classifications in Theorems 2.3 and 2.6, we may also *assume that* $\mathrm{rank}\,G\geqslant 2$.

Lemma 3.1. *For any finite subset* I *of* $\mathrm{supp}\,V$, *there is a subgroup* G_I *of* G *such that*

(a) $G_I\cong\mathbb{Z}^k$ *for some* $k\in\mathbb{N}$, *and* $\mu-\mu'\in G_I$ *for any* $\mu,\mu'\in I$,

(b) $U(G_I)V_\mu=U(G_I)V_{\mu'}$ *for any* $\mu,\mu'\in I$,

(c) V_μ *is an irreducible* $U(G_I)_0$ *module for any* $\mu\in I$.

Proof. Assume that $I = \{\mu_i \mid i = 1, 2, \dots, s\}$. Since the homogeneous subspace V_{μ_i} is finite dimensional and V is an irreducible $U(G)$-module, we obtain that, for any $\mu_i, \mu_j \in I$, there are elements $y_{i,j}^{(1)}, \dots, y_{i,j}^{(d_{ij})} \in U(G)_{\mu_i - \mu_j}$ such that

$$V_{\mu_i} = \sum_{t=1}^{d_{ij}} y_{i,j}^{(t)} V_{\mu_j}, \quad \text{where } d_{ij} \in \mathbb{N}.$$

For any $\mu_i \in \operatorname{supp} V$, since V is irreducible we see that V_{μ_i} is an irreducible $U(G)_0$-module. Let $\phi_{\mu_i} : U(G)_0 \to gl(V_{\mu_i})$ be the representation of $U(G)_0$ in V_{μ_i}, where $gl(V_{\mu_i})$ is the general linear Lie algebra associated with the vector space V_{μ_i}. Since V_{μ_i} is finite dimensional, so are $gl(V_{\mu_i})$ and $\phi_{\mu_i}(U(G)_0)$. Thus, there exist $y_{i1}, \dots, y_{im_i} \in U(G)_0$ such that

$$\operatorname{span}_{\mathbb{C}}\{\phi_{\mu_i}(y_{i1}), \dots, \phi_{\mu_i}(y_{im_i})\} = \phi_{\mu_i}(U(G)_0).$$

For the finitely many elements $y_{i1}, \dots, y_{im_i}, y_{i,j}^{(1)}, \dots, y_{i,j}^{(d_{ij})}$, $1 \leqslant i, j \leqslant s$, due to the Poincaré–Birkhoff–Witt (PBW) Theorem, there are finitely many elements $g_1, \dots, g_n \in G$ such that $y_{i,j}, y_{i,j}^{(k)} \in U(G_I)$, where G_I is the subgroup of G generated by $g_1, \dots, g_n$ and all differences $\mu - \mu'$ for all $\mu, \mu' \in I$.

Since G_I is a finitely generated torsion free abelian group, it is clear that $G_I \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$ and part (a) follows. By the construction of G_I, we know that $V_{\mu_i} \subseteq U(G_I)V_{\mu_j}$ for any $\mu_i, \mu_j \in I$. Hence $U(G_I)V_{\mu_i} \subseteq U(G_I)V_{\mu_j} \subseteq U(G_I)V_{\mu_i}$ for any $\mu_i, \mu_j \in I$. Thus, $U(G_I)V_{\mu_i} = U(G_I)V_{\mu_j}$ for any $\mu_i, \mu_j \in I$ and part (b) follows.

We now prove that V_{μ_i} is an irreducible $U(G_I)_0$-module for any $\mu_i \in I$. Suppose that N is a non-trivial $U(G_I)_0$-submodule of V_{μ_i}. For any element $y \in U(G)_0$, there are some $a_j \in \mathbb{C}$ such that

$$\phi_{\mu_i}(y) = \sum_{j=1}^{m_i} a_j \phi_{\mu_i}(y_{ij}).$$

Then

$$yN = \sum_{j=1}^{m_i} a_j \phi_{\mu_i}(y_{ij})N = \left(\sum_{j=1}^{m_i} a_j y_{ij}\right)N \subset N.$$

That is, N is a non-zero $U(G)_0$-submodule of V_{μ_i}, forcing $N = V_{\mu_i}$. Hence, V_{μ_i} is an irreducible $U(G_I)_0$-module for any $\mu_i \in I$, and part (c) follows. □

Remark 3.2. From the above proof we can always choose G_I such that $\operatorname{rank}(G_I) > 1$ which will be used in the proof of Lemma 3.5–3.7 and Theorem 3.8.

Lemma 3.3. *Let I and G_I be the same as in Lemma 3.1 and let G' be a subgroup of G that contains G_I. Then, for any $\lambda \in I$, $V_{\lambda+G'}$ has a unique irreducible $\operatorname{Vir}[G']$-subquotient V' with $\dim V'_\mu = \dim V_\mu$ for any $\mu \in I$.*

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Proof. Since $G_I \subset G'$, we see that $U(G')U(G_I) = U(G')$. From parts (b) and (c) of Lemma 3.1, we know that

(b′) $U(G')V_\mu = U(G')V_{\mu'}$ for any $\mu, \mu' \in I$, which we denote by W,

(c′) V_μ is an irreducible $U_0(G')$ module for any $\mu \in I$.

Note that $W \subset V_{\lambda+G'}$ for all $\lambda \in I$, and $W_\mu = V_\mu$ for all $u \in I$. Suppose that W' is a proper $\mathrm{Vir}[G']$-submodule of W. For any $\mu \in I$, it is clear that W'_μ is a proper $U(G')_0$ submodule of W_μ, forcing $W'_\mu = 0$, that is, W' trivially intersects W_μ for any $\mu \in I$. Now let $V' = U(G')X/Y$, where $X = \bigoplus_{\mu\in I} V_\mu$ and Y is the sum of all $U(G')$-submodules Y' of $U(G')X$ such that $Y'_\mu = 0$ for all $\mu \in I$. Then V' is the unique irreducible $\mathrm{Vir}[G']$-subquotient of $V_{\lambda+G'}$ with $\dim V'_\mu = \dim V_\mu$ for any $\mu \in I$. □

Theorem 3.4.

(a) *If V is uniformly bounded, then V is of intermediate series.*

(b) *A Harish-Chandra module W over* $\mathrm{Vir}[G]$ *with* $\mathrm{supp}(W) \subset \lambda + G$ *for some* $\lambda \in \mathbb{C}$ *is uniformly bounded if and only if* $\dim W_\lambda = \dim W_\mu$ *for all* $\lambda, \mu \in \mathrm{supp}\, V \setminus \{0\}$.

Proof. Part (b) follows directly from (a). To prove part (a), it suffices to show that $\dim V_\lambda = 1$ for all $\lambda \in \mathrm{supp}\, V$.

Now suppose that $\dim V_\lambda \geqslant 2$ for some $\lambda \in \mathrm{supp}\, V$. Using Lemma 3.1 for $I = \{\lambda\}$, we have the subgroup G_I of G described therein. Then V_λ is an irreducible $U(G_I)_0$-module. Consider the uniformly bounded $\mathrm{Vir}[G_I]$-module $V_{\lambda+G_I}$. It has an irreducible uniformly bounded subquotient V' with $\dim V'_\lambda = \dim V_\lambda \geqslant 2$ (Lemma 3.3). By Theorem 2.9, $\dim V'_\lambda$ should be not larger than 1: a contradiction. Thus, we have that $\dim V_\lambda = 1$ for all $\lambda \in \mathrm{supp}\, V$. The proof is completed. □

Lemma 3.5. *For any $\lambda \in \mathrm{supp}\, V$ and $g \in G \setminus \{0\}$, if the $Vir[\mathbb{Z}g]$-module $V_{\lambda+\mathbb{Z}g}$ has a non-trivial uniformly bounded* $\mathrm{Vir}[\mathbb{Z}g]$*-subquotient, then $V_{\lambda+\mathbb{Z}g}$ itself is uniformly bounded.*

Proof. On the contrary, suppose that $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded. Then there exist some $\mu_1, \mu_2 \in \mathrm{supp}(V_{\lambda+\mathbb{Z}g}) \setminus \{0\}$ with $\dim V_{\mu_1} \neq \dim V_{\mu_2}$.

Applying Lemma 3.1 to $I = \{\mu_1, \mu_2\}$, we have a subgroup G_I of G described therein, and furthermore we may assume that $\mathrm{rank}\, G_I > 1$. Let $G' = G_I + \mathbb{Z}g$. Then, by Lemma 3.3, the $\mathrm{Vir}[G']$-module $V_{\lambda+G'}$ has a unique irreducible subquotient V' with $\dim V'_{\mu_i} = \dim V_{\mu_i}$, $i = 1, 2$.

Since $\dim V'_{\mu_1} \neq \dim V'_{\mu_2}$, Theorem 3.4 ensures that V' is not uniformly bounded. By Theorem 2.9, V' must be of the form $V(\alpha, \beta, G'_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G'$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Since $\dim V'_{\mu_1} \neq \dim V'_{\mu_2}$ and $\mu_1 \neq 0 \neq \mu_2$, $V'_{\lambda+\mathbb{Z}g}$ is not uniformly bounded by Theorem 3.4. Then, by Corollary 2.8, we know that $g \notin G'_0$. Thus, the $\mathrm{Vir}[\mathbb{Z}g]$-module $V'_{\lambda+\mathbb{Z}g}$ is positively truncated relative to g.

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From the definition of V' we know that $V'_{\lambda+\mathbb{Z}g}$ is a $\mathrm{Vir}[\mathbb{Z}g]$-subquotient of $V_{\lambda+\mathbb{Z}g}$, say, $V'_{\lambda+\mathbb{Z}g} = W/W'$, where $W' \subset W$ are $\mathrm{Vir}[\mathbb{Z}g]$-submodules of $V_{\lambda+\mathbb{Z}g}$. Since

$$\dim V_{\mu_1} = \dim V'_{\mu_1} = \dim W_{\mu_1},$$

$W'_{\mu_1} = 0$ and $(V_{\lambda+\mathbb{Z}g}/W)_{\mu_1} = 0$. Note that $\mu_1 \neq 0$. We see that none of $V_{\lambda+\mathbb{Z}g}/W$, W/W' and W' has uniformly bounded non-trivial $\mathrm{Vir}[\mathbb{Z}g]$-subquotients. Thus, $V'_{\lambda+\mathbb{Z}g}$ does not have any non-trivial uniformly bounded $\mathrm{Vir}[\mathbb{Z}g]$-subquotient: a contradiction. □

Lemma 3.6. *Assume that $\lambda \in \operatorname{supp} V \setminus \{0\}$ and that G_1, G_2 are any subgroups of G. If both $V_{\lambda+G_1}$ (as a $\mathrm{Vir}[G_1]$-module) and $V_{\lambda+G_2}$ (as a $\mathrm{Vir}[G_2]$-module) are uniformly bounded, then $V_{\lambda+G_1+G_2}$ is a uniformly bounded $\mathrm{Vir}[G_1+G_2]$-module.*

Proof. By Theorem 3.4, we may assume that $\dim V_\mu = m$ for non-zero $\mu \in (\lambda+G_1) \cup (\lambda+G_2)$.

On the contrary, we suppose that there are some $g_i \in G_i$, $i = 1, 2$, such that $\dim V_{\lambda+g_1+g_2} \neq m$ with $\lambda + g_1 + g_2 \neq 0$. We may assume that $g_1 + g_2 \notin G_1 \cup G_2$.

Case 1 ($(\lambda + g_1)(\lambda + g_2) \neq 0$). Applying Lemma 3.1 to $I = \{\lambda + g_1, \lambda + g_2, \lambda + g_1 + g_2\}$, we have the subgroup G_I as described in Lemma 3.1, and furthermore we may assume that $\operatorname{rank} G_I > 1$. Note that $g_1, g_2 \in G_I$. Lemma 3.3 ensures that the $\mathrm{Vir}[G_I]$-module $V_{\lambda+G_I}$ has a unique irreducible subquotient V' such that $\dim V'_\mu = \dim V_\mu$ for all $\mu \in I$. In particular, $\dim V'_{\lambda+g_1} \neq \dim V_{\lambda+g_1+g_2}$. Noting that $(\lambda + g_1)(\lambda + g_1 + g_2) \neq 0$, by Theorem 3.4 we know that V' is not uniformly bounded. By Theorem 2.9, $V' \cong V(\alpha, \beta, G_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G_I$ and a subgroup G_0 of G_I with $G_I = G_0 \oplus \mathbb{Z}b$. The fact that $V'_{\lambda+\mathbb{Z}g_1}$ and $V'_{\lambda+\mathbb{Z}g_2}$ are both uniformly bounded implies $g_1, g_2 \in G_0$. Thus, $g_1, g_2, g_1 + g_2 \in G_0$. By Corollary 2.8, we deduce that $V'_{\lambda+\mathbb{Z}g_1+\mathbb{Z}g_2}$ is uniformly bounded. Then, by Theorem 3.4, we see that $\dim V_{\lambda+g_1+g_2} = \dim V_{\lambda+g_1} = m$: a contradiction.

Case 2 ($(\lambda + g_1)(\lambda + g_2) = 0$). We may assume that $\lambda + g_2 = 0$ and $\lambda + g_1 \neq 0$.

From Case 1 we know that $\dim V_{\lambda+g_1+g'_2} = m$ for any $g'_2 \in G_2 \setminus \{g_2\}$ with $\lambda + g_1 + g'_2 \neq 0$. Then $V_{\lambda+g_1+\mathbb{Z}g_2}$ is a uniformly bounded $\mathrm{Vir}[\mathbb{Z}g_2]$-module (note that $g_2 \neq 0$). By Theorem 3.4 with G replaced by $\mathbb{Z}g_2$, we see that $\dim V_{\lambda+g_1+g_2} = \dim V_{\lambda+g_1} = m$: again, a contradiction.

Hence, $\dim V_{\lambda+x} = m$ for any $x \in G_1 + G_2$ with $\lambda + x \neq 0$. Therefore, $V_{G_1+G_2}$ is uniformly bounded. □

Lemma 3.7. *For any $\mu \in \operatorname{supp} V \setminus \{0\}$, there exists a unique maximal subgroup G_μ of G such that $V_{\mu+G_\mu}$ is a uniformly bounded $\mathrm{Vir}[G_\mu]$-module. Furthermore,*

(a) *$G_{\mu_1} = G_{\mu_2}$ for any $\mu_1, \mu_2 \in \operatorname{supp} V \setminus \{0\}$, which we denote by $G^{(0)}$,*

(b) *either $G^{(0)} = G$ or $G \cong G^{(0)} \oplus \mathbb{Z}b$ for some non-zero $b \in G$.*

Proof. The existence and uniqueness of G_μ for $\mu \in \operatorname{supp} V \setminus \{0\}$ follow easily from Lemma 3.6.

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(a) On the contrary, suppose there are $\mu_1 \neq \mu_2$ in $\operatorname{supp} V \setminus \{0\}$ such that $G_{\mu_1} \neq G_{\mu_2}$. We may assume that $G_{\mu_1} \setminus G_{\mu_2} \neq \emptyset$ and fix $g_1 \in G_{\mu_1} \setminus G_{\mu_2}$. Let $I = \{\mu_1, \mu_1 + g_1, \mu_2\}$ and take G_I as the subgroup described in Lemma 3.1. Furthermore, we may assume that $\operatorname{rank} G_I > 1$. Note that $g_1 \in G_I \cap G_{\mu_1}$. By Lemma 3.3, the $\operatorname{Vir}[G_I]$-module $V_{\mu_2+G_I}$ has a unique irreducible subquotient V' with $\dim V'_{\mu_i} = \dim V_{\mu_i}$ for $i = 1, 2$. Clearly, V' is non-trivial.

If V' is uniformly bounded, then $V'_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\operatorname{Vir}[\mathbb{Z}g_1]$-subquotient of $V_{\mu_2+\mathbb{Z}g_1}$. If V' is not uniformly bounded, then, as a $\operatorname{Vir}[G_I]$-module, $V' \cong V(\alpha, \beta, G'_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G_I \setminus \{0\}$ and a subgroup G'_0 of G_I with $G_I = G'_0 \oplus \mathbb{Z}b$. Note that $\mu_1 + \mathbb{Z}g_1 \subset \mu_2 + G_I$. Since $V'_{\mu_1+\mathbb{Z}g_1}$ is uniformly bounded, $g_1 \in G'_0$, and hence $V'_{\mu_2+\mathbb{Z}g_1}$ is also uniformly bounded by Corollary 2.8.

Thus, in both cases, $V'_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\operatorname{Vir}[\mathbb{Z}g_1]$-subquotient of $V_{\mu_2+\mathbb{Z}g_1}$. By Lemma 3.5, $V_{\mu_2+\mathbb{Z}g_1}$ is a uniformly bounded $\operatorname{Vir}[\mathbb{Z}g_1]$-module, forcing $g_1 \in G_{\mu_2}$ by Lemma 3.6, contradicting the choice of g_1. Thus, $G_{\mu_1} = G_{\mu_2}$ for all $\mu_1, \mu_2 \in \operatorname{supp} V \setminus \{0\}$.

(b) Suppose $G^{(0)} \neq G$. We shall prove that $G \cong G^{(0)} \oplus \mathbb{Z}b$ for some $b \in G$ in three steps.

Step 1 ($G/G^{(0)}$ is torsion free). Otherwise we may choose some $g \in G \setminus G^{(0)}$ and $k_0 \in \mathbb{N}$ such that $k_0 g \in G^{(0)}$. Take any $\mu \in \operatorname{supp} V \setminus \{0\}$. Since $\operatorname{rank} G > 1$, we have a subgroup A of G such that $A \cong \mathbb{Z}^2$. By considering the non-trivial $\operatorname{Vir}[A]$-module $V_{\mu+A}$ and using Theorem 2.9, we know that there exists $\lambda \in \operatorname{supp}(V) \setminus \mathbb{Z}g$.

Then $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded while $V_{\lambda+\mathbb{Z}k_0 g}$ is uniformly bounded, by the definition of $G^{(0)}$. Since $V_{\lambda+\mathbb{Z}g}$ is not uniformly bounded, it has a non-trivial highest or lowest weight irreducible subquotient, say non-trivial highest weight $\operatorname{Vir}[\mathbb{Z}g]$-subquotient W. Without loss of generality, we may assume that W has a highest weight vector v with weight $\gamma \neq 0$. Then $W_{\gamma+\mathbb{Z}k_0 g}$ has a non-trivial highest weight vector v with weight $\gamma \neq 0$. Thus, $W_{\gamma+\mathbb{Z}k_0 g}$ is not uniformly bounded (this is a well-known result for the classical Virasoro algebra), contradicting the fact that $V_{\lambda+\mathbb{Z}k_0 g}$ is uniformly bounded. Thus, $G/G^{(0)}$ is torsion free. Step 1 follows.

Step 2 ($\operatorname{rank}(G/G^{(0)}) = 1$). Otherwise we assume that $\operatorname{rank}(G/G^{(0)}) > 1$ and take $g_1, g_2 \in G \setminus G^{(0)}$ such that the subgroup $G' = \langle g_1, g_2 \rangle$ is isomorphic to $\mathbb{Z}^2$ and $G' \cap G^{(0)} = 0$. Take any $\lambda \in \operatorname{supp} V \setminus \{0\}$ and consider the $\operatorname{Vir}[G']$-module $V_{\lambda+G'}$, which has an irreducible subquotient V' with $V'_\lambda \neq 0$.

If V' is uniformly bounded, then $V'_{\lambda+\mathbb{Z}g_1}$ is a non-trivial uniformly bounded $\operatorname{Vir}[\mathbb{Z}g_1]$-subquotient of $V_{\lambda+\mathbb{Z}g_1}$. Thus, by Lemma 3.5, $V_{\lambda+\mathbb{Z}g_1}$ is a uniformly bounded $\operatorname{Vir}[\mathbb{Z}g_1]$-module, forcing $g_1 \in G^{(0)}$ by Lemma 3.6: a contradiction.

If V' is not uniformly bounded, then $V' \cong V(\alpha, \beta, G'_0, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G' \setminus \{0\}$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b$. Then $V'_{\lambda+\mathbb{Z}g}$ is uniformly bounded for any $g \in G'_0 \setminus \{0\}$, and hence $V_{\lambda+\mathbb{Z}g}$ is also uniformly bounded. Thus, $g \in G^{(0)}$, contradicting the fact that $g \in G' \cap G^{(0)} = \{0\}$. Step 2 follows.

Thus, $G \subset G^{(0)} + \mathbb{Q}g$ for any $g \in G \setminus G^{(0)}$.

Fixing $g \in G \setminus G^{(0)}$, we know that $G \subset G^{(0)} + \mathbb{Q}g$. Since $G/G^{(0)}$ is torsion free and $G \neq G^{(0)}$, then $G^{(0)} \cap \mathbb{Q}g = \{0\}$. We know that $G = G^{(0)} \oplus G^{(1)}$, where $G^{(1)} = G \cap \mathbb{Q}g$.

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Step 3 ($G^{(1)} \cong \mathbb{Z}$). Otherwise, $G^{(1)}$ would be an infinitely generated abelian group of rank 1. Then by Theorem 2.6, the $\mathrm{Vir}[G^{(1)}]$ module $V_{\lambda+G^{(1)}}$ is uniformly bounded for any $\lambda \in \operatorname{supp} V$, forcing $G^{(1)} \subset G^{(0)}$: again, a contradiction. Thus, we must have $G^{(1)} \cong \mathbb{Z}$. This completes the proof. □

Theorem 3.8. *Suppose that V is an irreducible Harish-Chandra module over the generalized Virasoro algebra $\mathrm{Vir}[G]$ and that V is not uniformly bounded. Then V is isomorphic to $V(\alpha, \beta, G^{(0)}, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G \setminus \{0\}$ and a subgroup $G^{(0)}$ of G with $G = G^{(0)} \oplus \mathbb{Z}b$.*

Proof. From Theorems 2.3 and 2.6, we know that the statement holds for any G of rank 1. So we assume $\operatorname{rank} G \geqslant 2$. Since V is not uniformly bounded, we use Lemma 3.7 to obtain $G^{(0)}$ and b as described there. Then C trivially acts on V.

Take any $\mu \in \operatorname{supp} V \setminus \{0\}$. Since $\operatorname{rank} G > 1$, we have a subgroup A of G such that $A \cong \mathbb{Z}^2$. The non-trivial Harish-Chandra $\mathrm{Vir}[A]$-module $V_{\mu+A}$ has an irreducible $\mathrm{Vir}[A]$-subquotient X with $X_\mu \neq 0$. Using Theorem 2.9, we know that there exists $\lambda \in \operatorname{supp}(X) \setminus \mathbb{Z}b \subset \operatorname{supp}(V) \setminus \mathbb{Z}b$. We see that $0 \notin \lambda + \mathbb{Z}b$. The $\mathrm{Vir}[\mathbb{Z}b]$-module $W = V_{\lambda+\mathbb{Z}b}$ cannot have any non-trivial uniformly bounded subquotient, for otherwise we would have $b \in G^{(0)}$ by Lemma 3.6, contradicting the definition of $G^{(0)}$. Note that $0 \notin \operatorname{supp} W$. Then, by Theorem 2.4, $W = W^+ \oplus W^-$, where W^+ is such that $\operatorname{supp} W^+ \subset \{\lambda + kb \mid k \leqslant t_0\}$ for some $t_0 \in \mathbb{Z}$ and W^- is such that $\operatorname{supp} W^- \subset \{\lambda + kb \mid k \geqslant s_0\}$ for some $s_0 \in \mathbb{Z}$.

Since $0 \notin \lambda + \mathbb{Z}b$, it is clear that the highest weight $\lambda + k_1 b$ of any irreducible highest weight $\mathrm{Vir}[\mathbb{Z}b]$-subquotient of W must satisfy $k_1 \leqslant t_0$ and that the lowest weight $\lambda + k_2 b$ of any irreducible lowest weight $\mathrm{Vir}[\mathbb{Z}b]$ subquotient of W must satisfy $k_2 \geqslant s_0$.

We want to show that either W^+ or W^- is zero. Otherwise, we may choose $t > t_0$ and $s < s_0$ such that $\lambda + tb \in \operatorname{supp}(W^-) \subset \operatorname{supp}(V)$ and $\lambda + sb \in \operatorname{supp}(W^+) \subset \operatorname{supp}(V)$. Certainly, $\lambda + tb, \lambda + sb \in \operatorname{supp} W$. Let $I = \{\lambda + tb, \lambda + sb\}$, and take G_I as in Lemma 3.1, and furthermore we may assume that $\operatorname{rank} G_I > 1$. Set $G' = G_I + \mathbb{Z}b$. The $\mathrm{Vir}[G']$-module $V_{\lambda+G'}$ has a unique irreducible subquotient V' with $\dim V'_\mu = \dim W_\mu = \dim V_\mu$ for all $\mu \in I$.

Clearly, V' is not a uniformly bounded $\mathrm{Vir}[G']$-module. Now by Theorem 2.9, we have that $V' \cong V(\alpha, \beta, G'_0, b')$ for some $\alpha, \beta \in \mathbb{C}$, $b' \in G' \setminus \{0\}$ and a subgroup G'_0 of G' with $G' = G'_0 \oplus \mathbb{Z}b'$. From Corollary 2.8 we know that $W_{\lambda+G'_0}$ is uniformly bounded. From Lemma 3.5 we see that $V_{\lambda+G'_0}$ is also uniformly bounded. Using Lemma 3.6 we deduce that $G'_0 \subset G^{(0)}$. Since $b \in G'_0 \oplus \mathbb{Z}b'$ and $b' \in G^{(0)} \oplus \mathbb{Z}b$, we have $b = x_0 + \epsilon b'$, where $x_0 \in G'_0$ and $\epsilon = \pm 1$. So we can take $b' = \epsilon b$.

Thus, $V'_{\lambda+\mathbb{Z}b}$ is either a positively truncated or a negatively truncated $\mathrm{Vir}[\mathbb{Z}b]$-module. Note that $\dim V'_{\lambda+tb} = \dim W_{\lambda+tb}$ and $\dim V'_{\lambda+sb} = \dim W_{\lambda+sb}$ are both non-zero. We may consider that $V'_{\lambda+tb} = W_{\lambda+tb}$ and $V'_{\lambda+sb} = W_{\lambda+sb}$. By definition, we know that the submodule $U(\mathbb{Z}b)W_{\lambda+tb}$ is not positively truncated and the submodule $U(\mathbb{Z}b)W_{\lambda+sb}$ is not negatively truncated. At the same time we know that $U(\mathbb{Z}b)W_{\lambda+tb} \subset V'_{\lambda+\mathbb{Z}b}$ and $U(\mathbb{Z}b)W_{\lambda+sb} \subset V'_{\lambda+\mathbb{Z}b}$. Thus, $V'_{\lambda+\mathbb{Z}b}$ is neither a positively truncated nor a negatively truncated $\mathrm{Vir}[\mathbb{Z}b]$-module, which is a contradiction.

Hence, $W = W^+$ or $W = W^-$. Without loss of generality, we assume that $W = W^+$, that is, $\dim V_{\lambda+kb} = 0$ for any $k > t_0$. But $V_{\lambda+kb+G^{(0)}}$ is a uniformly bounded

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$\mathrm{Vir}[G^{(0)}]$-module for any $k \in \mathbb{Z}$. We must have that $\dim V_{\lambda+kb+g} = \dim V_{\lambda+kb} = 0$ for any $k > t_0, g \in G^{(0)}$, provided that $\lambda + kb + g \neq 0$.

Let t_1 be the largest integer such that $\dim V_{\lambda+t_1 b} \neq 0$.

If $0 \in \lambda + t_1 b + G^{(0)}$, then $\lambda + t_1 b + G^{(0)} = G^{(0)}$. Then $\mathrm{supp}(V) \subset -\mathbb{Z}^+ b + G^{(0)}$. Any irreducible $\mathrm{Vir}[G^{(0)}]$-submodule W of $V_{G^{(0)}}$ generates V as a $\mathrm{Vir}[G]$-module. Using the PBW Theorem we know that $W = V_{G^{(0)}}$, i.e. $V_{G^{(0)}}$ is an irreducible $\mathrm{Vir}[G^{(0)}]$-module. Since $V_{G^{(0)}} \supset V_{\lambda+t_1 b} \neq 0$, then $V_{G^{(0)}}$ is a uniformly bounded non-trivial irreducible $\mathrm{Vir}[G^{(0)}]$-module. So $V_{G^{(0)}} \cong V'(\alpha, \beta, G^{(0)})$ for some $\alpha, \beta \in \mathbb{C}$. Consequently, $V \cong V(\alpha, \beta, G^{(0)}, b)$.

If $0 \notin \lambda + t_1 b + G^{(0)}$ and $0 \notin \lambda + (t_1 + 1)b + G^{(0)}$, or $0 \in \lambda + (t_1 + 1)b + G^{(0)}$ and $V_0 = 0$, then $\mathrm{Vir}_{b+G^{(0)}} V_{G^{(0)}} = 0$, where $\mathrm{Vir}_{b+G^{(0)}} = \sum_{x \in b+G^{(0)}} \mathbb{C} d_x$. So $\mathrm{Vir}_{\mathbb{N}b+G^{(0)}} V_{G^{(0)}} = 0$. Thus, $\mathrm{supp}(V) \subset -\mathbb{Z}^+ b + G^{(0)}$. Similar discussions lead to the same conclusion, that is, $V \cong V(\alpha, \beta, G^{(0)}, b)$.

If $0 \in \lambda + (t_1 + 1)b + G^{(0)}$, and $V_0 \neq 0$, take non-zero $v \in V_0$. Then $\mathrm{Vir}_{b+G^{(0)}} v = 0$ and $\mathrm{Vir}_{G^{(0)}} v = 0$. Thus, $\mathrm{Vir}_{\mathbb{Z}^+ b+G^{(0)}} v = 0$. Since V is not trivial, using the PBW Theorem and $C = 0$, we deduce that $V_{t_1 b - \mathbb{Z}b + G^{(0)}}$ is a proper submodule: a contradiction. So this case does not occur.

This proves the theorem. □

Combining Theorems 2.2, 2.6, 3.4 and 3.8, we now obtain the following classification theorem.

Theorem 3.9. *Suppose that G is an arbitrary additive subgroup of $\mathbb{C}$.*

(a) *If $\mathrm{rank}\, G = 1$ and $G \ncong \mathbb{Z}$, then any non-trivial irreducible Harish-Chandra module over $\mathrm{Vir}[G]$ is a module of the intermediate series.*

(b) *If $G \cong \mathbb{Z}$, then any non-trivial irreducible Harish-Chandra module over $\mathrm{Vir}[G]$ is a module of the intermediate series, a highest weight module or a lowest weight module.*

(c) *If $\mathrm{rank}\, G > 1$, then any non-trivial irreducible Harish-Chandra module over $\mathrm{Vir}[G]$ is either a module of the intermediate series or isomorphic to $V(\alpha, \beta, G^{(0)}, b)$ for some $\alpha, \beta \in \mathbb{C}$, $b \in G \setminus \{0\}$ and a subgroup $G^{(0)}$ of G with $G = G^{(0)} \oplus \mathbb{Z}b$.*

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