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A NOTE ON ENGEL GROUPS AND LOCAL NILPOTENCE

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Abstract

This paper is concerned with the question of whether n-Engel groups are locally nilpotent. Although this seems unlikely in general, it is shown here that it is the case for the groups in a large class $\mathscr{C}$ including all residually soluble and residually finite groups (in fact all groups considered in traditional textbooks on group theory). This follows from the main result that there exist integers $c(n)$, $e(n)$ depending only on n, such that every finitely generated n-Engel group in the class $\mathscr{C}$ is both finite-of-exponent-$e(n)$–by–nilpotent-of-class$\leq c(n)$ and nilpotent-of-class$\leq c(n)$–by–finite-of-exponent-$e(n)$. Crucial in the proof is the fact that a finitely generated Engel group has finitely generated commutator subgroup.

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1. Introduction

A group is called *Engel* if for each ordered pair (g, h) of elements of the group there is a relation of the form

$$(1) \qquad [\dots[[g, h], h], \dots, h] = 1,$$

where $[x, y] := x^{-1}y^{-1}xy$, the *commutator* of x and y. Following the usual 'left-normed' convention, we write the left-hand side of (1) as $[g, h, \dots, h]$, or even more briefly as $[g, {}_nh]$, where n denotes the number of entries of h. The Engel condition (1) represents a generalization of local nilpotence: a locally nilpotent group is (clearly) Engel. Essentially the only known examples of non-locally nilpotent Engel groups are those of Golod (see [8, p. 132]): for each $d \geq 2$ there is a d-generator non-nilpotent group G_d each of whose $(d-1)$-generator subgroups is nilpotent (and G_d is in addition residually finite and a p-group). It is interesting to contrast with this the known positive results: of Wilson and Zelmanov [12, Theorem 5] that a profinite Engel

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group is locally nilpotent, of Baer (see [10, p. 360]) that an Engel group satisfying the maximal condition is nilpotent, and of Gruenberg [3] that a soluble Engel group is locally nilpotent.

However it is unknown whether or not *n-Engel* groups, that is, those satisfying the law $[x, {}_n y] \equiv 1$ for some fixed n, must be locally nilpotent (although this seems unlikely). This has been established for $n \leq 3$ (see [6]), and, for general n, for the class of residually finite n-Engel groups [11]. (Note that there are relatively easy examples of non-nilpotent n-Engel groups; see, for example, [8, p. 132] or [10, p. 362, Ex. 1].)

In the present note we call attention to a simple general fact about Engel groups which has apparently hitherto gone unnoticed, and from it infer firstly the local nilpotence of Engel 'SB-groups' (these are defined below; they include soluble groups), and then a quite specific global description of the n-Engel groups in a large class $\mathscr{C}$ of groups (including soluble and residually finite groups), yielding in particular their local nilpotence.

The 'simple fact' in question is as follows:

PROPOSITION. *A finitely generated Engel group G has finitely generated commutator subgroup $[G, G]$. Moreover if G is d-generator and n-Engel, then the rank of $[G, G]$ is bounded in terms of d and n.*

It is immediate that a finitely generated soluble Engel group is polycyclic, and therefore, in view of Baer's result mentioned above, nilpotent. In fact this argument applies to the larger class of 'SB-groups', defined as follows: An SB-*group* G is a group with a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_k = \{1\},$$

each of whose factors G_i/G_{i+1} is either soluble or locally finite of finite exponent, that is, G lies in a product $\mathfrak{S}_{l_1}\mathfrak{B}_{e_1} \cdots \mathfrak{S}_{l_r}\mathfrak{B}_{e_r}$ of varieties, where $\mathfrak{S}_l$ denotes the variety of all soluble groups of length $\leq l$ and $\mathfrak{B}_e$ the variety consisting of all locally finite groups of exponent dividing e. (That the class $\mathfrak{B}_e$ is actually a variety is a consequence of Zelmanov's solution of the restricted Burnside problem.) Thus we have the

COROLLARY 1 (Cf. Gruenberg [3]). *An Engel* SB-*group* (*in particular a soluble Engel group*) *is locally nilpotent.*

The above-mentioned class $\mathscr{C}$, originally introduced in [1], is obtained from the class of all SB-groups by closing under the operations L and R, where for any group-theoretical class $\mathscr{X}$, $L\mathscr{X}$ denotes the class of all groups locally in $\mathscr{X}$, and $R\mathscr{X}$ the class of all groups residually in $\mathscr{X}$.

Our main theorem is then as follows:

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THEOREM. *There exist integers $c(n)$, $e(n)$ depending on n only such that all n-Engel groups contained in the class $\mathscr{C}$ are actually contained in the variety*

$$(2) \qquad \mathfrak{V}_n := \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)} \cap \mathfrak{B}_{e(n)}\mathfrak{N}_{c(n)},$$

where $\mathfrak{N}_c$ denotes the variety of all groups nilpotent of class $\leq c$.

Note however that there are *Engel* (as opposed to n-Engel) groups in the class $\mathscr{C}$ which are *not* locally nilpotent, as is shown by the above-mentioned examples of Golod.

This theorem includes, in particular, the result of Gruenberg [4] that an n-Engel soluble group of derived length d must belong to $\mathfrak{B}_{\hat{e}(n)}\mathfrak{N}_{\hat{c}(d,n)}$ for some positive integers $\hat{e}(n)$ and $\hat{c}(d,n)$. Note also that a result of Groves [2, Theorem C] implies a similar conclusion to that of our theorem for n-Engel groups lying in a product of a succession of soluble or Cross varieties. The most significant improvement, in our theorem, over these results consists in the dependence of the nilpotency class $c(n)$ and the exponent $e(n)$ exclusively on the Engel class n. (Compare the result of Heineken [6] and Gupta and Newman [5] that every 3-Engel group belongs to $\mathfrak{B}_{20}\mathfrak{N}_5$. No such precise facts appear to be known for 4-Engel groups, not even whether they are all locally nilpotent.) Our theorem also generalizes [1, Corollary 2] stating in part that a residually finite, torsion-free, n-Engel group is nilpotent of class bounded in terms of n: by the theorem any n-Engel group from the class $\mathscr{C}$, and so in particular any residually finite n-Engel group, is nilpotent of class bounded in terms of n, modulo a normal subgroup of finite exponent.

The local nilpotence of n-Engel groups in $\mathscr{C}$ follows easily from the theorem; we leave the details to the reader:

COROLLARY 2. *The n-Engel groups in the class $\mathscr{C}$ are locally nilpotent.*

The proof of the above theorem is given in Section 2, and of the proposition in Section 3.

REMARK. As noted in [1], it seems reasonable to suggest the class $\mathscr{C}$ as comprising just those groups accessible to analysis using what might be called the 'classical' methods of group theory (such as those used in the textbooks [10] and [8]), in contrast with those outside $\mathscr{C}$ which, one may conjecture, require the quite distinct 'industrial' techniques of, most notably, Adjan-Novikov and Ol'shanskiĭ, used in connection with the negative solution of the general Burnside problem and related problems, and involving the construction of 'monsters'.

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2. Proof of the Theorem

Let G be an n-Engel group in the class $\mathscr{C}$. We wish to show that G lies in the intersection (2), that is, in the variety $\mathfrak{V}_n$.

We firstly prove that G belongs to $\mathfrak{N}_{c_1(n)}\mathfrak{B}_{e_1(n)}$ for some $c_1(n)$, $e_1(n)$ depending on n only. If we can prove this for an arbitrary finitely generated n-Engel SB-group then it will follow for every n-Engel group in $\mathscr{C}$ in view of this exclusive dependence on n. Hence we may assume without loss of generality that G is a finitely generated SB-group. Then by Corollary 1 above G must be nilpotent, and therefore certainly residually finite. Now it follows from a theorem of Wilson [11, Theorem 2] that every 2-generator subgroup of a residually finite n-Engel group is nilpotent of class bounded in terms of n alone, and hence, according to Mal'cev [9], such a group satisfies a (2-variable) semigroup law of degree depending only on n. Hence our group G satisfies such a semigroup law, whence by [1, Theorem A], we have

$$G \in \mathfrak{N}_{c_1(n)}\mathfrak{B}_{e_1(n)}, \tag{3}$$

for some $c_1(n)$, $e_1(n)$ depending only on n, as required.

We now complete the proof of our theorem by deducing from (3), just established, together with the assumption that G is finitely generated and n-Engel, that

$$G \in \mathfrak{B}_{e_2(n)}\mathfrak{N}_{c_2(n)}, \tag{4}$$

for some $e_2(n)$, $c_2(n)$ depending on n only. We proceed by induction on the parameter c_1 in (3). The initial case $c_1 = 0$ (that is, $G \in \mathfrak{B}_{e_1}$) is trivial; suppose that $c_1 > 0$ and inductively that a containment of the form (4) holds for classes $< c_1$. Set

$$H := \gamma_{c_1}(G^{e_1}) := [\underbrace{G^{e_1}, \dots, G^{e_1}}_{c_1}] \trianglelefteq G.$$

By (3) H is contained in the centre of G^{e_1}, so that H is certainly abelian. Since $G/H \in \mathfrak{N}_{c_1-1}\mathfrak{B}_{e_1}$, we may assume by the inductive hypothesis that $G/H \in \mathfrak{B}_{e_3}\mathfrak{N}_{c_3}$ for some functions $e_3 = e_3(n)$, $c_3 = c_3(n)$ of n only.

The next step in the proof requires the following

LEMMA 1. *For each $x \in G$, $h \in H$, and positive integer k, we have*

$$[h, x]^{e_1^t} = [h,\, _{i_1}x]^{\pm 1} \cdots [h,\, _{i_t}x]^{\pm 1} \tag{5}$$

for some $t \geq 1$, $i_1, \dots, i_t \geq k$.

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PROOF. It suffices to show that for every $i \geq 1$,

$$[h, {}_i x]^{e_1} = [h, {}_{j_1} x]^{\pm 1} \cdots [h, {}_{j_t} x]^{\pm 1} \tag{6}$$

for some $t \geq 1$, $j_1, \dots, j_t \geq i + 1$. The equation (5) follows from this by means of an easy induction on k, using the facts that H is normal in G and abelian.

For $i = 1$, (6) has the form

$$[h, x]^{e_1} = [h, {}_{j_1} x]^{\pm 1} \cdots [h, {}_{j_t} x]^{\pm 1}, \ j_1, \dots, j_t \geq 2. \tag{7}$$

This follows by repeated application of the group identity $[a, bc] \equiv [a, c][a, b][a, b, c]$, invoking the abelian-ness and normality in G of H: thus

$$[h, x^2] = [h, x]^2 [h, x, x],$$
$$[h, x^3] = [h, x^2][h, x][h, x, x^2] = [h, x]^3 [h, x, x]^3 [h, x, x, x],$$

and so on, whence, eventually,

$$[h, x^{e_1}] = [h, x]^{e_1} [h, {}_{j_1} x] \cdots [h, {}_{j_t} x],$$

where $j_1, \dots, j_t \geq 2$. Since x^{e_1} commutes with h, an equation of the form (7) follows.

Now suppose that $i \geq 1$ and inductively that (6) holds with $i - 1$ in place of i. We have

$$[h, {}_i x]^{e_1} = [h, x, {}_{i-1} x]^{e_1} = [h_1, {}_{i-1} x]^{e_1},$$

where $h_1 := [h, x] \in H$. The inductive hypothesis then gives

$$[h_1, {}_{i-1} x]^{e_1} = [h_1, {}_{l_1} x]^{\pm 1} \cdots [h_1, {}_{l_r} x]^{\pm 1},$$

where $r \geq 1$, $l_1, \dots, l_r \geq i$. Since $h_1 = [h, x]$, the desired conclusion (6) follows for i.

Returning to the proof of the theorem, we conclude from this lemma and the assumption that G is n-Engel, that $[G, H]^{e_1^n} = \{1\}$. Since this exponent depends only on n, we may work modulo $[G, H]$, that is, we may assume without loss of generality that H is central in G. Thus to summarize, we are now in the situation of a finitely generated n-Engel group G with a central subgroup H such that

$$G/H \in \mathfrak{N}_{c_1 - 1}\mathfrak{B}_{e_1} \cap \mathfrak{B}_{e_3}\mathfrak{N}_{c_3},$$

and we seek to establish (4) for such a group G. As noted before, we also have by [11, Theorem 2] that every 2-generator subgroup of G is nilpotent of class $\leq c_4$ for some c_4 depending only on n.

To conclude the proof we shall need the following

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LEMMA 2. *Let G be as above. Then for any $x \in \gamma_{c_3}(G)$, $g \in G$, the commutator subgroup $[\langle x, g\rangle, \langle x, g\rangle]$ of $\langle x, g\rangle$, has exponent dividing $e_3^{c_4-1}$.*

PROOF. Since by [11, Theorem 2] the 2-generator subgroup $\langle x, g\rangle$ is nilpotent of class $\leq c_4$, it suffices to show that for each $i \geq 2$ the quotient $\gamma_i(\langle x, g\rangle)/\gamma_{i+1}(\langle x, g\rangle)$ has exponent dividing e_3. Now by definition $\gamma_i(\langle x, g\rangle)$ is generated by the commutators of the form $[x_1, g_1]$, $x_1 \in \gamma_{i-1}(\langle x, g\rangle)$, $g_1 \in \langle x, g\rangle$. One has

$$[x_1, g_1]^{e_3} \equiv [x_1^{e_3}, g_1] \mod \gamma_{i+1}(\langle x, g\rangle),$$

and then since $x_1 \in \gamma_{c_3}(G)$, which has exponent dividing e_3 modulo the centre of G, we have that $[x_1^{e_3}, g_1] = 1$, whence the lemma.

Using this lemma we shall now show that

$$\gamma_{c_3+1}(G)^{c_4!e_3^{c_4}} = 1. \tag{8}$$

From this the desired conclusion (4) follows, with $c_2 = c_3 + 1$, and e_2 some more complicated function of e_1, n, e_3, c_4, and so ultimately of n alone.

Write $e_4 := c_4!e_3^{c_4}$. By definition of $\gamma_{c_3+1}(G)$ each element of that group is a product $a_1 \cdots a_t$ of commutators a_i of the form $[x, g]^{\pm 1}$, $x \in \gamma_{c_3}(G)$, $g \in G$. We prove by induction on t that every such product has order dividing e_4, that is,

$$(a_1 \cdots a_t)^{e_4} = 1. \tag{9}$$

For $t = 1$ this follows from Lemma 2, since by that lemma any element of the form $[x, g]$, $x \in \gamma_{c_3}(G)$, $g \in G$, has order dividing $e_3^{c_4}$, which in turn divides e_4. Suppose that $t > 1$ and inductively that the analogue of (9) holds for such products of length $< t$. Write $a := a_1$, $b := a_2 \cdots a_t$. By the Hall-Petrescu identity (see, for example, [7, p. 317, Satz 9.4])

$$a^{e_4}b^{e_4} = (ab)^{e_4} w_2(a, b) \cdots w_{e_4}(a, b),$$

where $w_i(a, b) \in \gamma_i(\langle a, b\rangle)^{\binom{e_4}{i}}$ for each $i = 2, \ldots, e_4$. We have $a^{e_4} = 1$ by Lemma 2, and $b^{e_4} = 1$ by the inductive hypothesis. If $i > c_4$ then $\gamma_i(\langle a, b\rangle) = \{1\}$ since, as noted earlier, every 2-generator subgroup of G has class $\leq c_4$. On the other hand if $2 \leq i \leq c_4$, then it is easy to see that $\binom{e_4}{i}$ is divisible by $e_3^{c_4}$, so that for these i we have $\gamma_i(\langle a, b\rangle)^{\binom{e_4}{i}} = \{1\}$, by Lemma 2. Thus for all $i = 2, \ldots, e_4$, we have $w_i(a, b) = 1$. Hence $(ab)^{e_4} = 1$, completing the induction, and thence the proof of the theorem.

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3. Proof of the Proposition

We firstly show that if G is any Engel group, then for any $x, y \in G$, the subgroup $\langle x\rangle^{\langle y\rangle}$ is finitely generated. (This appears as Exercise 6 on p. 362 of [10]; we include a proof for completeness, since the proposition is crucial in the above argument.)

Write $x_i := y^i x y^{-i}$ for each integer i. We show by induction on n that $[x, {}_n y]$ has the form

$$[x, {}_n y] = u_n x_0^{\pm 1} v_n x_{-n}^{\pm 1}, \tag{10}$$

for some words u_n, v_n in $x_{-1}, \ldots, x_{-(n-1)}$. For $n = 1$ we have $[x, y] = x^{-1}y^{-1}xy = x_0^{-1}x_{-1}$, which has the right form with $u_1 = v_1 = 1$. Assuming inductively that (10) holds, we have

$$\begin{aligned} [x, {}_{n+1} y] = [[x, {}_n y], y] &= (u_n x_0^{\pm 1} v_n x_{-n}^{\pm 1})^{-1} y^{-1} u_n x_0^{\pm 1} v_n x_{-n}^{\pm 1} y \\ &= (x_{-n}^{\mp 1} v_n^{-1}) x_0^{\mp 1} (u_n^{-1} u_n^y x_{-1}^{\pm 1} v_n^y) x_{-(n+1)}^{\pm 1}. \end{aligned}$$

Since u_n^y $(:= y^{-1}u_n y)$, and v_n^y are expressions in $x_{-2}, \ldots, x_{-n}$ only, we see that $[x, {}_{n+1} y]$ has the appropriate form, completing the induction.

From (10) it follows that if $[x, {}_n y] = 1$, then

$$x_{-n} \in \langle x_0, x_{-1}, \ldots, x_{-(n-1)}\rangle, \tag{11}$$

and

$$x_0 \in \langle x_{-1}, x_{-2}, \ldots, x_{-n}\rangle. \tag{12}$$

Successive conjugations of (11) by y yields

$$x_{-i} \in \langle x_0, x_{-1}, \ldots, x_{-n}\rangle \quad \text{for all} \quad i > n,$$

and successive conjugations of (12) by y^{-1} yields

$$x_i \in \langle x_{-1}, \ldots, x_{-n}\rangle \quad \text{for all} \quad i > 0.$$

Hence

$$\langle x\rangle^{\langle y\rangle} = \langle x_{-1}, \ldots, x_{-n}\rangle,$$

showing that $\langle x\rangle^{\langle y\rangle}$ is indeed finitely generated. It follows that if H is any finitely generated subgroup of G and $g \in G$, then

$$H^{\langle g\rangle} \text{ is finitely generated.} \tag{13}$$

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The remainder of the proof, that is, the argument deducing from this that any finitely generated subgroup of G has finitely generated commutator subgroup, is identical with *that of the proposition of [1]; we reproduce the proof here since, as already noted, the present proposition is crucial to the theorem.*

One first shows that given any two elements $a, b \in G$, the commutator subgroup $\langle a, b\rangle'$ is finitely generated. The crucial fact allowing this is that $\langle a, b\rangle'$ is generated by the elements of the form $[a, b]^{a^m b^n}$ where m and n are integers. This follows in turn from the well-known fact that $\langle a, b\rangle'$ is generated by all commutators of the form $[a^r, b^s]$, r and s integers, via repeated application of the identities

$$a^{-i}[a^r, b^s]a^i = [a^{r+i}, b^s][b^s, a^i],$$
$$b^{-i}[a^r, b^s]b^i = [b^i, a^r][a^r, b^{s+i}],$$

starting with $r = s = 1$. Now $\langle [a, b]\rangle^{\langle a\rangle}$ is finitely generated by the first part of the proof, whence by (13) $(\langle [a, b]\rangle^{\langle a\rangle})^{\langle b\rangle}$ is finitely generated, as required.

This establishes the 2-generator case. Now assume inductively that the claim is valid for subgroups of G which can be generated by $\leq k$ elements, and suppose that $H < G$ requires $k + 1 > 2$ generators, say $h_1, \ldots, h_{n+1}$. Write H_i for the subgroup generated by

$$\{h_1, \ldots, h_{k+1}\}\backslash\{h_i\}, \qquad i = 1, \ldots, k + 1.$$

Then by the inductive hypothesis $[H_i, H_i]$ is finitely generated, whence so is $[H_i, H_i]^{\langle h_i\rangle}$. The conclusion now follows from the fact that $[H, H]$ is generated by the set-theoretical union of the $[H_i, H_i]^{\langle h_i\rangle}$. For this it suffices to show that the subgroup generated by this union, that is, by $U := \bigcup_i [H_i, H_i]^{\langle h_i\rangle}$, is normal in H. For instance

$$([H_1, H_1]^{h_1})^{h_2} = [H_1, H_1]^{h_2 h_1 [h_1, h_2]} = [H_1, H_1]^{h_1 [h_1, h_2]},$$

and since $[H_1, H_1]^{h_1} \subseteq U$ and $[h_1, h_2] \in [H_3, H_3]$, we have $[H_1, H_1]^{h_1 h_2} \leq \langle U\rangle$.

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