PROJECTIVE CONNECTIONS AND PROJECTIVE TRANSFORMATIONS

NOBORU TANAKA

The main purpose of the present paper is to establish a theorem concerning the relation between the group of all projective transformations on an affinely connected manifold and the group of all affine transformations.

We shall say that an affine connection satisfies condition (E), if it is without torsion and affinely complete and if the Ricci tensor field $S(X, Y)$ is parallel. Our theorem states that if an affine connection satisfies condition (E) and if the quadratic form $S(X, X)$ is zero or not negative semi-definite, then the two groups coincide. This is just a generalization of the case of ordinary affine space which is well known in analytic geometry.

The proof is based on the theory of normal projective connection introduced by Elie Cartan; in particular, we make use of the "developing" process of this connection. After some preliminaries, in which we follow the book of K. Nomizu [5] for affine connections, we first formulate the normal projective connection from a global point of view, as we shall see in Proposition 1. In § 6, we prove an important lemma (Lemma 8) by using the results in the previous sections (Proposition 1, 2 and 3), from which the main theorem follows immediately. In Appendix, we shall prove a known fact on the geometric charactrization of the projective equivalence of two affine connections.

Finally we add that we have obtained some results about the relation between the group of all conformal transformations on a Riemannian manifold and the group of all isometries by using the method analogous to the projective case. We hope to deal with this problem in another paper.

The author expresses his sincere thanks to Professor K. Nomizu for his constant encouragement and interest in this work.

1. Definition of a projective transformation on an affinely connected manifold

Let M be a connected manifold of class C^∞. We asume that the dimension

Received May 24, 1957.

n of M is $\geqq 2$. Under differentiability we shall always understand that of class C^{∞}. We shall denote by M_p the tangent vector space to M at $p \in$ M. The set $\mathfrak{X}(M)$ of all vector fields on M is a module over the ring $\mathfrak{F}(M)$ of all differentiable functions on M.

An affine connection ∇ on M is given by a mapping of $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into $\mathfrak{X}(M)$ which satisfies the following conditions [5]:

a) For each $X \in \mathfrak{X}(M)$, the mapping $Y \to \nabla_Y X$ is an endomorphism of $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$;

b) For any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathfrak{F}(M)$,

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \qquad \nabla_X f \cdot Y = f \cdot \nabla_X Y + Xf \cdot Y.$$

Let $\mathfrak{A}$ be the set of all affine connections on M whose torsion are zero. We introduce a relation $\sim$ in $\mathfrak{A}$ as follows: $\bar{\nabla} \sim \nabla$, if and only if there exists a 1-form ρ on M such that

$$\bar{\nabla}_X Y = \nabla_X Y + \rho(Y)X + \rho(X)Y \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M). \tag{1.1}$$

$\sim$ is clearly an equivalent relation, by which $\mathfrak{A}$ is divided into equivalent classes. The class $\mathfrak{P}(\nabla)$ containing $\nabla \in \mathfrak{A}$ consists of all affine connections $\bar{\nabla}$ on M which can be written as (1.1) with an arbitrary 1-form ρ on M. When $\bar{\nabla} \sim \nabla$, we shall say that $\bar{\nabla}$ is *projective to* ∇ and call ρ *the associated* 1-*form* of $\bar{\nabla}$ with respect to ∇. It is known that $\bar{\nabla}$ is projective to ∇ if and only if the systems of geodesics for the two connections coincide (see Appendix).

Fix an affine connection ∇ belonging to $\mathfrak{A}$. Let f be a differentiable transformation of M onto itself. We now define a mapping $\bar{\nabla}$ of $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into $\mathfrak{X}(M)$ by

$$\bar{\nabla}_X Y = f^{-1} \nabla_{fX} fY \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where fX denotes the vector field obtained by applying the differential of f to X. We see that $\bar{\nabla}$ belongs to $\mathfrak{A}$. If $\bar{\nabla} \sim \nabla$, f is called a projective transformation of ∇. In this case, we shall call ρ *the associated* 1-*form* of f. The group $P(\nabla)$ of all projective transformations of ∇ is a Lie group with respect to compact open topology [4] and contains the group $A(\nabla)$ of all affine transformations of ∇.

In the following, we shall justify the word "projective" by considering the normal projective connection corresponding to an arbitrary class of mutually projective affine connections. For this purpose, we begin with an ordinary projective space.

2. Projective space

Let P_n be an n-dimensional real projective space constructed from an $(n+1)$-dimensional vector space F_{n+1} in the well known manner.

(2.1) We consider a fixed decomposition of F_{n+1}

$$F_{n+1} = F_1 + F_n,$$

where F_1 and F_n are 1- and n-dimensional subspaces of F_{n+1} respectively, and we choose once for all a base (ξ_0) in F_1. We denote by F_n^* the dual space of F_n and by $\langle \xi, E \rangle$ the product between $\xi \in F_n$ and $E \in F_n^*$.

(2.2) P_n is a quotient space of F'_{n+1}, where $F'_{n+1} = F_{n+1} - (o)$. More precisely, it may be regarded as the base space of a principal fiber bundle F'_{n+1} with the multiplicative group of non-zero real numbers as structure group. We denote by ω the projection of F'_{n+1} onto P_n and set $o = \omega(\xi_0)$.

(2.3) P_n may be regarded as a homogeneous space $P(n)/P'(n)$, where $P(n)$ is the so-called projective transformation group on P_n and $P'(n)$ the isotropy group of $P(n)$ at o. We know that $P(n)$ is expressible as a factor group $GL(F_{n+1})/H'_1$, where H'_1 denotes the 1-parameter subgroup $(\exp t\mathbf{1}_{n+1})$ of $GL(F_{n+1})$, $\mathbf{1}_{n+1}$ being the unit element of $GL(F_{n+1})$. The action of $P(n)$ on P_n is as follows: Let ω be the projection of $GL(F_{n+1})$ onto $P(n)$. Then,

$$\omega(\sigma)\,\omega(u) = \omega(\sigma u) \qquad \text{for all} \quad \sigma \in GL(F_{n+1}) \text{ and } u \in F'_{n+1}.$$

(2.4) We define a homomorphism φ of $GL(F_n)$ into $GL(F_{n+1})$ by

$$\varphi(a)\xi_0 = \xi_0\ ;\ \varphi(a)\eta = a\eta \qquad \text{for} \quad \eta \in F_n.$$

For each $E \in F_n^*$, let $\exp E$ be the element of $GL(F_{n+1})$ defined by

$$\exp E\xi_0 = \xi_0\ ;\ \exp E\eta = \langle \eta, E \rangle \xi_0 + \eta \qquad \text{for} \quad \eta \in F_n.$$

Then it is easily seen that $P'(n)$ can be identified with the subgroup of $GL(F_{n+1})$ composed of all the elements $\varphi(a) \cdot \exp E$ with $a \in GL(F_n)$ and $E \in F_n^*$ (the isomorphism is clear). Thus $P'(n)$ can be expressed as

$$P'(n) = \varphi(GL(F_n)) \cdot \exp F_n^*.$$

The expression $\varphi(a) \cdot \exp E$ is unique, and, in particular, φ is an isomorphism of $GL(F_n)$ into $P'(n)$.

(2.5) We define a homomorphism l of $P'(n)$ onto $GL(F_n)$ by

$$l(\varphi(a)\cdot\exp E)=a.$$

(2.6) We see from (2.3) that the Lie algebra of $P(n)$ is given by $\mathfrak{gl}(F_{n+1})/H_1$, where H_1 denotes the 1-dimensional subalgebra of $\mathfrak{gl}(F_{n+1})$ spanned by $\mathbf{1}_{n+1}$. Consider the formal direct sum $\mathfrak{p}(n)$ of three vector spaces F_n, $\mathfrak{gl}(F_n)$ and F_n^*:

$$\mathfrak{p}(n)=F_n+\mathfrak{gl}(F_n)+F_n^*.$$

For each $\xi\in F_n$, $A\in\mathfrak{gl}(F_n)$ and $E\in F_n^*$, let $\tilde{\xi}$, $\tilde{A}$ and $\tilde{E}$ be the elements of $\mathfrak{gl}(F_{n+1})$ defined respectively as follows:

$$\tilde{\xi}\xi_0=\xi,\ \tilde{\xi}\eta=0;\quad \tilde{A}\xi_0=0,\ \tilde{A}\eta=A\eta;\quad \tilde{E}\xi_0=0,\ \tilde{E}\eta=\langle\eta, E\rangle\xi_0,$$

where $\eta\in F_n$. For $A\in\mathfrak{p}(n)$, define $\tilde{A}\in\mathfrak{gl}(F_{n+1})$ by $\tilde{A}=\tilde{\xi}+\tilde{S}+\tilde{E}$ if $A=\xi+S+E$, where $\xi\in F_n$, $S\in\mathfrak{gl}(F_n)$ and $E\in F_n^*$. We now define a linear isomorphism f of $\mathfrak{p}(n)$ with $\mathfrak{gl}(F_{n+1})/H_1$ by $f(A)=\omega(\tilde{A})$, where ω denotes the projection of $\mathfrak{gl}(F_{n+1})$ onto $\mathfrak{gl}(F_{n+1})/H_1$. f being an isomorphism, we can transfer the structure of Lie algebra of $\mathfrak{gl}(F_{n+1})/H_1$ to $\mathfrak{p}(n)$ in such a way that f becomes an isomorphism of Lie algebras of $\mathfrak{p}(n)$ with $\mathfrak{gl}(F_{n+1})/H_1$. It is easy to see that the bracket operation of $\mathfrak{p}(n)$ is defined as follows:

$[\xi, \xi']=0$; $[A, B]=AB-BA$; $[E, E']=0$; $[A, \xi]=A\xi$; $[A, E]=-{}^tAE$;
$[\xi, E]=$ the element of $\mathfrak{gl}(F_n)$ defined by $[\xi, E]\eta=\langle\xi, E\rangle\eta+\langle\eta, E\rangle\xi$,

where $\eta, \xi, \xi'\in F_n$, $A, B\in\mathfrak{gl}(F_n)$ and $E, E'\in F_n^*$. In the following, we shall always identify $\mathfrak{p}(n)$ with the Lie algebra of $P(n)$. We here remark that the notation $\exp E$, introduced in (2.4), is legitimate, because we have generally

$$\exp A\omega(u)=\omega(\exp\tilde{A}u)\qquad\text{for all}\quad A\in\mathfrak{p}(n)\ \text{and}\ u\in F'_{n+1}.$$

The Lie algebra of $P'(n)$ is given by

$$\mathfrak{p}'(n)=\mathfrak{gl}(F_n)+F_n^*.$$

(2.7) The decomposition of $\mathfrak{p}(n)$

$$\mathfrak{p}(n)=F_n+\mathfrak{p}'(n)$$

is fundamental for our argument. We shall denote by A_{F_n} and $A_{\mathfrak{p}'(n)}$ the F_n- and $\mathfrak{p}'(n)$-component of $A\in\mathfrak{p}(n)$ respectively.

(2.8) The adjoint representation of $P'(n)$ in $\mathfrak{p}(n)$ is given by

$$ad\varphi(a)\xi = a\xi; \quad ad\varphi(a)A = adaA; \quad ad\varphi(a)E = {}^t a^{-1}E;$$

$$ad(\exp E)\xi = \xi + [E, \xi] + \frac{1}{2}[E, [E, \xi]]; \quad ad(\exp E)A = A + [E, A];$$

$$ad(\exp E)E' = E',$$

where $a \in P'(n)$, $\xi \in F_n$, $A \in \mathfrak{gl}(F_n)$ and $E, E' \in F_n^*$. We have $l(a)\xi = (ad(a)\xi)_{F_n}$ for all $a \in P'(n)$ and $\xi \in F_n$, which shows that l may be considered as the homomorphism of the isotropy group of $P(n)$ at o onto the linear isotropy group.

3. Connections in principal fiber bundles

We first recall definitions about principal fiber bundles and connections in them [5].

Let $P(M, G, \pi)$ be a differentiable principal fiber bundle over a base space M with structure group G and with projection π of P onto M. Denote by G_z the subspace of P_z (the tangent space to P at $z \in P$) which is tangent to the fiber through z. Let R_a be the right translation on P induced by $a \in G$. For an element A in the Lie algebra $\mathfrak{g}$ of G, we denote by A^* the vector field on P which is induced by the 1-parameter group $R_{a(t)}$ where $a(t) = \exp tA$. For each $z \in P$, the set of all elements A_z^* with $A \in \mathfrak{g}$ is equal to the subspace G_z.

A connection Q in P is a choice of a tangent subspace Q_z at each $z \in P$ which satisfies the following conditions:

(Q. 1) $\quad Q_z + G_z = P_z$ (direct sum);

(Q. 2) $\quad R_a Q_z = Q_{z \cdot a}$;

(Q. 3) $\quad Q_z$ depends differentiably on z.

Given a connection Q in P, a curve $x(t)$ in P is said to be horizontal, if, for each t, the tangent vector $x'(t)$ is contained in $Q_{x(t)}$. Let $u(t)$ be a curve through $p \in M$. Then, for each $x \in P$ such that $\pi(x) = p$, there exists one and only one horizontal curve through x which covers $u(t)$. The curve $x(t)$ is called the lift of $u(t)$ through x.

Given two principal fiber bundles $P'(M, G')$ and $P(M, G)$ with the same base space M, a mapping f of P' into P is called a homomorphism if there is a homomorphism f of G' into G such that $f(x' \cdot a') = f(x') \cdot f(a')$, where $x' \in P'$ and $a' \in G'$, and if it induces the identity transformation of M onto itself.

Let M be an n-dimensional manifold and let P_L be the bundle of frames

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

of M. By taking a base in F_n, we may regard P_L as a principal fiber bundle over the base space M with structure group $GL(F_n)$. Each element x of P_L gives an isomorphism of F_n with M_p, where $p=\pi_L(x)$, π_L being the projection of P_L onto M.

It is well known that an affine connection ∇ on M gives rise to an affine connection in P_L (in the sense of Cartan connection). Namely, to each ∇ we can associate a linear mapping of F_n into $\mathfrak{X}(P_L)$ ($\mathfrak{X}(P_L)$ may be regarded as a vector space over the field R of all real numbers) which satisfies the following conditions:

(A. 1) $B_{L_x}+GL(F_n)_x=P_{L_x}$ (direct sum), where B_{L_x} denotes the subspace of P_{L_x} composed of all the elements $B_L(\xi)_x$ where $\xi\in F_n$;

(A. 2) $$R_a B_L(\xi)=B_L(a^{-1}\xi);$$

(A. 3) $$\pi_L B_L(\xi)_x=x\cdot\xi.$$

Conversely, starting with (A. 1), (A. 2) and (A. 3), we can define an affine connection ∇ on M. For the relation between ∇ and B_L, see K. Nomizu's book [5]. In virtue of (A. 1) and (A. 2), the assignment $x\to B_{L_x}$ defines a connection in P_L, which is often called the linear connection in P_L induced by the affine connection in P_L (or on M). By (A. 1), we can set, for all $x\in P_L$ and $\xi, \xi'\in F_n$,

$$-[B_L(\xi), B_L(\xi')]_x=B_L(T_x(\xi, \xi'))_x+R_x(\xi, \xi')^*_x,$$

where $T_x(\xi, \xi')\in F_n$ and $R_x(\xi, \xi')\in\mathfrak{gl}(F_n)$. For each $x\in P_L$, T_x and R_x are linear mappings of $F_n\times F_n$ into F_n and $\mathfrak{gl}(F_n)$ respectively and are what correspond to the torsion and curvature tensor fields respectively. For example, if we denote by $R(X, Y)$ the curvature tensor field of ∇, then we have $R_x(\xi, \xi')=x^{-1}\cdot R(x\cdot\xi, x\cdot\xi')\cdot x$ for all $x\in P_L$ and $\xi, \xi'\in F_n$. Now consider, for each $x\in P_L$, a bilinear function S_x on $F_n\times F_n$ defined by

$$S_x(\xi, \xi')=Tr(\eta\to R_x(\xi, \eta)\xi').$$

If we denote by $S(X, Y)$ the Ricci tensor field of ∇, then it can be shown that $S_x(\xi, \xi')=S(x\cdot\xi, x\cdot\xi')$ for all $x\in P_L$ and $\xi, \xi'\in F_n$. For later uses, we define a linear mapping J_x of F_n into F_n^* at each $x\in P_L$ by the following formula:

(3. 1) $$\langle\xi, J_x(\xi')\rangle=\frac{1}{n^2-1}(S_x(\xi, \xi')+nS_x(\xi', \xi)).$$

Let ∇ and $\bar{\nabla}$ be two affine connections on M which are mutually projective and let ρ be the associated 1-form of $\bar{\nabla}$ with respect to ∇. Let B_L and $\bar{B}_L$ be the corresponding affine connections in P_L respectively. If we define a mapping F of P_L into F_n^* by $\langle \xi, F(x) \rangle = \rho(x \cdot \xi)$ for all $x \in P_L$ and $\xi \in F_n$, then it can be proved that (1.1) is equivalent to

$$(3.2) \qquad \bar{B}_L(\xi)_x = B_L(\xi)_x - [\xi, F(x)]_x^*.$$

4. Normal projective connection corresponding to a class of mutually projective affine connections

The main purpose of this section is to prove

PROPOSITION 1. *To each class $\mathfrak{P}$ of mutually projective affine connections on M there is associated a collection (P', l, B) as follows; P' is a principal fiber bundle over the base space M with structure group $P'(n)$; l is a homomorphism of P' onto P_L corresponding to the homomorphism l of $P'(n)$ onto $GL(F_n)$ defined in* (2.5); *B is a linear mapping of F_n into $\mathfrak{X}(P')$; moreover, the collection satisfies the following conditions:*

(P.1) *$B_z + P'(n)_z = P'_z$ (direct sum), where B_z denotes the subspace of P'_z composed of all the elements $B(\xi)_z$ where $\xi \in F_n$;*

(P.2) $R_a B(\xi) = B((ad(a^{-1})\xi)_{F_n}) + (ad(a^{-1})\xi)^*_{\mathfrak{p}'(n)}$;

(P.3) *$\pi B(\xi)_z = l(z) \cdot \xi$, where π is the projection of P' onto M;*

(P.4) *To each $\nabla \in \mathfrak{P}$ there is associated a homomorphism h of P_L into P' corresponding to the homomorphism φ of $GL(F_n)$ into $P'(n)$ defined in* (2.4) *such that*

i) $l \circ h(x) = x$;

ii) *$hB_L(\xi)_x = B(\xi)_{h(x)} + J_x(\xi)^*_{h(x)}$, where B_L is the affine connection in P_L corresponding to ∇ and J_x is given by* (3.1).

In the above proposition, let h and $\bar{h}$ be the corresponding homomorphisms of ∇ and $\bar{\nabla}$ $(\nabla, \bar{\nabla} \in \mathfrak{P})$ respectively. From (P.4) ii), we have $l \circ h(x) = l \circ \bar{h}(x) = x$ for all $x \in P_L$. It follows from (2.4) and (2.5) that there is a mapping F of P_L into F_n^* such that

$$(4.1) \qquad \bar{h}(x) = h(x) \cdot \exp F(x).$$

Let ρ be the associated 1-form of $\bar{\nabla}$ with respect to ∇. Then, it can be shown that

$$(4.2) \qquad \rho(x \cdot \xi) = \langle \xi, F(x) \rangle.$$

(4.2) will be proved later. The conditions (P.1) and (P.2) correspond to the ones for Cartan connections formulated by Ehresmann [3]. It can be proved that, for each class, the collection satisfying the conditions indicated in Proposition 1, is uniquely determined by the class up to an isomorphism (the meaning of the isomorphism is clear). We shall call the collection (P', l, B) the normal projective connection corresponding to the class $\mathfrak{P}$ [1].

Now we shall prove Proposition 1. The proof is divided into four steps.

I. We here assume that (P', l, B) satisfies only (P.1), (P.2) and (P.3). In this case, the collection will be called a projective connection. From now on, we shall derive several fundamental formulae, which will be needed for our purpose.

For each $A \in \mathfrak{p}(n)$, we define $A^\dagger \in \mathfrak{X}(P')$ by $A^\dagger = B(A_{F_n}) + A^*_{\mathfrak{p}'(n)}$. Then the following lemma follows immediately from (P.1) and (P.2).

LEMMA 1.

i) *For each* $z \in P'$, *every element of* P'_z *can be written in one and only one way in the form* $A^\dagger_z$ *with* $A \in \mathfrak{p}(n)$;

ii) *For* $a \in P'(n)$, $A' \in \mathfrak{p}'(n)$ *and* $A \in \mathfrak{p}(n)$, *we have*

$$R_a A^\dagger = (ad(a^{-1})A)^\dagger, \qquad [A'^*, A^\dagger] = [A', A]^\dagger.$$

Let f be a mapping of P' into $\mathfrak{p}(n)$. We define $f^\dagger \in \mathfrak{X}(P')$ by $f^\dagger_z = f(z)^\dagger_z$ for $z \in P'$. By a formula on bracket operation [5], P. 8, we can easily verify

LEMMA 2. *Let f and g be two mappings of P' into $\mathfrak{p}(n)$. Then we get*

$$[f^\dagger, g^\dagger]_z = [f(z)^\dagger, g(z)^\dagger]_z + (f(z)^\dagger_z g - g(z)^\dagger_z f)^\dagger_z,$$

where $f(z)^\dagger_z g$ means the result of applying $f(z)^\dagger_z$ to g.

By Lemma 1 i), we can set, for all $z \in P'$ and $\xi, \xi' \in F_n$,

$$-[B(\xi), B(\xi')]_z = A_z(\xi, \xi')^\dagger_z;$$
$$A_z(\xi, \xi') = T_z(\xi, \xi') + W_z(\xi, \xi') + J_z(\xi, \xi'),$$

where $T_z(\xi, \xi') \in F_n$, $W_z(\xi, \xi') \in \mathfrak{gl}(F_n)$ and $J_z(\xi, \xi') \in F^*_n$. Using Lemma 1 ii) and the fact that $[\xi, \xi'] = 0$ and $l(a)\xi = (ad\, a\xi)_{F_n}$, we see that

$$A_{z\cdot a}(\xi, \xi') = ad(a^{-1})A_z(l(a)\xi, l(a)\xi').$$

It follows from (2.8) that

(4.3) i) $T_{z\cdot a}(\xi, \xi') = l(a)^{-1} T_z(l(a)\xi, l(a)\xi')$;

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

ii) If $T=0$, then

$$W_{z\cdot a}(\xi, \xi') = ad(l(a)^{-1})\, W_z(l(a)\xi, l(a)\xi').$$

(4.3) means that T and W are "affine tensors" on M. T corresponds to what is usually called the torsion tensor field of the projective connection.

It can be proved that there is at least one homomorphism, say h, of P_L into P' such that $l\circ h(x) = x$ for all $x\in P_L$. We fix such a homomorphism h. We shall show that h induces an affine connection in P_L which is closely related to the given projective connection. Since, for each $z\in P'$, z and $h\circ l(z)$ lie in the same fiber in P', we can set

$$z\cdot a(z) = h\circ l(z) \tag{4.4}$$

with a mapping a of P' into $P'(\mathrm{n})$. Since $l\circ a(z) = \mathbf{1}$ (the unit element of $GL(F_n)$) for all $z\in P'$, we see from (2.5) that $a(z)$ can be expressed as follows:

$$a(z) = \exp E(z), \tag{4.5}$$

where E is a mapping of P' into F_n^*. It is easily seen that $a(z\cdot\sigma) = \sigma^{-1}a(z)\varphi\circ l(\sigma)$ holds for all $z\in P'$ and $\sigma\in P'(n)$. By differentiating the both sides of (4.4) in the direction $B(\xi)_{h(x)}$ and using (4.5), we have

$$B(\xi)_{h(x)} + (B(\xi)_{h(x)}E)^*_{h(x)} = h\circ lB(\xi)_{h(x)}.$$

We now define a linear mapping of F_n into $\mathfrak{X}(P_L)$ by $B_L(\xi)_x = lB(\xi)_{h(x)}$. In virtue of (P.1), (P.2) and (P.3), it is easily shown that B_L satisfies (A.1), (A.2) and (A.3) and hence it is an affine connection in P_L. The affine connection, thus obtained, is said to be *induced* by a homomorphism h. For each $x\in P_L$, we define a linear mapping J_x of F_n into F_n^* by $J_x(\xi) = B(\xi)_{h(x)}E$. Then we have

$$B(\xi)_{h(x)} + J_x(\xi)^*_{h(x)} = hB_L(\xi)_x. \tag{4.6}$$

If we define $J_l(\xi)^*\in\mathfrak{X}(P')$ by $J_l(\xi)^*_z = J_{l(z)}(\xi)^*_z$ for $z\in P'$ and $\xi\in F_n$, then (4.6) means that $B(\xi)+J_l(\xi)^*$ is h-related to $B_L(\xi)$ [2].

Applying Lemma 2 to the case where $f(z) = \xi + J_{l(z)}(\xi)$ and $g(z) = \xi' + J_{l(z)}(\xi')$ and using Lemma 1 ii), we have

$$\begin{aligned}
&[B(\xi)+J_l(\xi)^*, B(\xi')+J_l(\xi')^*]_{h(x)} \\
&\quad = [B(\xi)+J_x(\xi)^*, B(\xi')+J_x(\xi')^*]_{h(x)} \\
&\qquad + (B_L(\xi)_xJ(\xi') - B_L(\xi')_xJ(\xi))^*_{h(x)}
\end{aligned}$$

$$= -A_{h(x)}(\xi, \xi')^{\dagger}_{h(x)} + [\xi, J_x(\xi')]^*_{h(x)} + [J_x(\xi), \xi']^*_{h(x)}$$
$$+ (B_L(\xi)_x J(\xi') - B_L(\xi')_x J(\xi))^*_{h(x)},$$

where $B_L(\xi)_x J(\xi')$ means the result of applying $B_L(\xi)_x$ to the mapping $P_L \ni x \to J_x(\xi) \in F^*_n$. On the other hand, we have

$$-[B_L(\xi), B_L(\xi')]_x = B_L(T'_x(\xi, \xi'))_x + R_x(\xi, \xi')^*_x,$$

where T' and R are what correspond to the torsion and curvature tensor fields respectively. Since $[B(\xi) + J_l(\xi)^*, B(\xi') + J_l(\xi')^*]$ is h-related to $[B_L(\xi), B_L(\xi')]$ [2], it follows that

(4.7) i) $T_{h(x)}(\xi, \xi') = T'_x(\xi, \xi')$;

ii) $W_{h(x)}(\xi, \xi') = R_x(\xi, \xi') + [\xi, J_x(\xi')] + [J_x(\xi), \xi']$.

Let $\bar{h}$ together with h be a homomorphism of P_L into P' such that $l \circ \bar{h}(x) = x$ for all $x \in P_L$. We know that there is a mapping F of P_L into F^*_n such that (4.1) holds true. Let $\bar{B}_L$ be the affine connection in P_L induced by $\bar{h}$. We shall find the relation between B_L and $\bar{B}_L$. By differentiating the both sides of (4.1) in the direction $B_L(\xi)_x$, we obtain

$$\bar{h}B_L(\xi)_x = R_{b(x)} \circ hB_L(\xi)_x + (B_L(\xi)_x F)^*_{h(x)};$$

using (4.6), Lemma 1 ii) and (2.8),

$$\bar{h}B_L(\xi)_x = B(\xi)_{\bar{h}(x)} - [F(x), \xi]^*_{\bar{h}(x)}$$
$$+ \left(\frac{1}{2}[F(x), [F(x), \xi]] + J_x(\xi) + B_L(\xi)_x F\right)^*_{\bar{h}(x)},$$

where we have set $b(x) = \exp F(x)$. Applying l to the both sides of this formula, we see that $B_L(\xi)_x = lB(\xi)_{\bar{h}(x)} - [F(x), \xi]^*_x$. But, by definition of $\bar{B}_L$, we have $lB(\xi)_{\bar{h}(x)} = \bar{B}_L(\xi)_x$. Consequently we have

$$B_L(\xi)_x = \bar{B}_L(\xi)_x - [F(x), \xi]^*_x.$$

II. We here assume that (P', l, B) satisfies (P. 1), (P. 2), (P. 3) and

(P. 4′) i) $T = 0$;

ii) $Tr(\eta \to W_z(\xi, \eta)\xi') = 0$.

In this case, the collection is called a normal projective connection.

Let h be a homomorphism of P_L into P' such that $l \circ h(x) = x$ for all $x \in P_L$. We shall show that, in the case of normal projective connection, J_x is given by (3.1). In fact, from (4.7) ii), we get

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

$$W_{h(x)}(\xi, \xi')\xi'' = R_x(\xi, \xi')\xi'' + \langle \xi'', J_x(\xi')\rangle \xi + \langle \xi, J_x(\xi')\rangle \xi'' \\ - \langle \xi'', J_x(\xi)\rangle \xi' - \langle \xi', J_x(\xi)\rangle \xi''.$$

Passing to the contraction and using (P. 4′) ii), we obtain

$$0 = S_x(\xi, \xi') + \langle \xi, J_x(\xi')\rangle - n\langle \xi', J_x(\xi)\rangle.$$

Interchanging ξ and ξ', we obtain

$$0 = S_x(\xi', \xi) - n\langle \xi, J_x(\xi')\rangle + \langle \xi', J_x(\xi)\rangle.$$

From these two formulae, it follows immediately that

$$\langle \xi, J_x(\xi')\rangle = \frac{1}{n^2-1}(S_x(\xi, \xi') + nS_x(\xi', \xi)),$$

which proves our assertion.

III. Let $\mathfrak{P}$ be a class of mutually projective affine connections. Now consider the following condition for (P', l, B):

(P. 4″) There exist a $\nabla \in \mathfrak{P}$ and a homomorphism h of P_L into P' such that

i) $l \circ h(x) = x$;

ii) $hB_L(\xi)_x = B(\xi)_{h(x)} + J_x(\xi)^*_{h(x)}$, where B_L is the affine connection in P_L corresponding to ∇ and J_x is defined by (3.1).

In the following, we shall show that if (P', l, B) satisfies (P. 1), (P. 2), (P. 3) and (P. 4″), then it also satisfies (P. 4) and hence, in this case, it is the normal projective connection corresponding to the class $\mathfrak{P}$. We first show that (P', l, B) satisfies (P. 4′). Let us make use of the results in I. First of all, we see that B_L coincides with the affine connection induced by h. From (4.7) i), we get $T_{h(x)}(\xi, \xi') = T'_x(\xi, \xi') = 0$, because ∇ has no torsion. It follows immediately from (4.3) i) that $T = 0$. Using (4.7) ii) and (3.1), it is easily shown that $Tr(\eta \to W_{h(x)}(\xi, \eta)\xi') = 0$ (the reciprocal argument of II). It follows immediately from (4.3) ii) that $Tr(\eta \to W_z(\xi, \eta)\xi') = 0$. Thus we have seen that (P', l, B) satisfies (P. 4′). Now we shall show that it satisfies (P. 4). Let $\bar{\nabla}$ be an arbitrary but fixed element of $\mathfrak{P}$ and ρ the associated 1-form of $\bar{\nabla}$ with respect to ∇. First, we define a homomorphism $\bar{h}$ of P_L into P' by (4.1) with the mapping F defined by (4.2). We have $l \circ \bar{h}(x) = x$ for all $x \in P_L$. Next, we shall show that $\bar{h}\bar{B}_L(\xi)_x = B(\xi)_{\bar{h}(x)} + \bar{J}_x(\xi)^*_{\bar{h}(x)}$, where $\bar{B}_L$ is the affine connection in P_L corresponding to $\bar{\nabla}$ and $\bar{J}_x$ is defined by (3.1) starting with $\bar{B}_L$. Let $\bar{B}'_L$ be the affine connection induced by $\bar{h}$ and let $\bar{J}'_x$ be the mapping of F_n

into F_n^* defined as in I starting with $\bar{h}$. Since (P', l, B) is a normal connection, we see from II that $\bar{J}'_x$ is identical with the one defined by (3.1) starting with $\bar{B}'_L$. Hence we have only to prove that $\bar{B}_L$ and $\bar{B}'_L$ coincide. But, by the argument in I, we see that $\bar{B}'_L(\xi)_x = B_L(\xi)_x - [\xi, F(x)]_x^*$. On the other hand, from (3.2) we have $\bar{B}_L(\xi)_x = B_L(\xi)_x - [\xi, F(x)]_x^*$. Therefore we have $\bar{B}_L = \bar{B}'_L$.

IV. We shall finally show that for each class $\mathfrak{P}$ there exists a collection (P', l, B) which satisfies (P.1), (P.2), (P.3) and (P.4″). If this is proved, Proposition 1 is an immediate consequence of III.

First, making use of the bundle of frames P_L and the homomorphism φ of $GL(F_n)$ into $P'(n)$, we define a principal fiber bundle P' over the base space M with structure group $P'(n)$ together with a homomorphism h of P_L into P'. Next, we define a homomorphism l of P' into P_L by $l(z) = x \cdot l(a)$ if $z = h(x) \cdot a$ where $x \in P_L$ and $a \in P'(n)$. $l(z)$ is independent of the expression $z = h(x) \cdot a$. l becomes clearly a homomorphism, and satisfies $l \circ h(x) = x$ for all $x \in P_L$. Finally, we shall define a linear mapping B of F_n into $\mathfrak{X}(P')$. We fix a $\nabla \in \mathfrak{P}$. Let B_L be the affine connection in P_L corresponding to ∇ and let J_x be the mapping defined by (3.1). Since $l \circ h(x) = x$ for all $x \in P_L$, we can set as (4.4) and (4.5). Now define B by

$$B(\xi)_z = R_{a(z)^{-1}} \circ h B_L(\xi)_{l(z)} - (J_{l(z)}(\xi) + (ad(a(z))\xi)_{\mathfrak{p}'(n)})_z^*.$$

(P', l, B), defined above, satisfies (P.1), (P.2), (P.3) and (P.4″) with ∇ and h. Indeed, using (A.1), (A.2), (A.3) and the fact that $J_{x \cdot a} = {}^t a J_x a$ for all $x \in P_L$ and $a \in GL(F_n)$ and $\sigma^{-1} a(z) \varphi \circ l(\sigma) = a(z \cdot \sigma)$ for all $z \in P'$ and $\sigma \in P'(n)$, we can easily verify (P.1), (P.2) and (P.3). We have $B(\xi)_{h(x)} = h B_L(\xi)_x - J_x(\xi)_{h(x)}^*$, which shows that (P', l, B) satisfies (P.4″). Therefore, we have completed the proof of Proposition 1.

Under the conditions of Proposition 1, we shall prove (4.2) as we promised. Let B_L and $\bar{B}_L$ be the affine connections in P_L corresponding to ∇ and $\bar{\nabla}$ respectively. From (P.4) ii), we have $lB(\xi)_{h(x)} = B_L(\xi)_x$, which shows that B_L coincides with the affine connection in P_L induced by h. In the same way, $\bar{B}_L$ coincides with the affine connection in P_L induced by $\bar{h}$. Therefore, by the argument in I in the proof of Proposition 1, we have $\bar{B}_L(\xi)_x = B_L(\xi)_x - [\xi, F(x)]_x^*$. On the other hand, if we define a mapping F' of P_L into F_n^* by $\langle \xi, F'(x) \rangle = \rho(x \cdot \xi)$, then we know from (3.2) that $\bar{B}_L(\xi)_x = B_L(\xi)_x - [\xi, F'(x)]_x^*$. Hence

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

$[\xi, F(x)] = [\xi, F'(x)]$ holds for all $x \in P_L$ and $\xi \in F_n$. It follows that $F = F'$, which is nothing but (4.2).

Remark. As is seen from the proof of Proposition 1, to each normal projective connection over a base space M there corresponds a class of mutually projective affine connections on M such that the given normal projective connection becomes the normal projective connection corresponding to the class. The correspondence $\mathfrak{P} \to (P', l, B)$ of the set of all classes of mutually projective affine connections into the set of all normal projective connections is one-to-one and onto.

Remark. Let ∇ be an affine connection without torsion and let (P', l, B) be the normal Projective connection corresponding to the class $\mathfrak{P}(\nabla)$. It can be proved that every projective transformation f induces a bundle isomorphism $\tilde{f}$ of P' which leaves $B(\xi)$ invariant for each $\xi \in F_n$. Conversely, such an $\tilde{f}$ induces a projective transformation f in the sense of § 1.

Let (P', l, B) be a projective connection over a base space M.

We shall denote by $P_n(p)$ the fiber at $p \in M$ of the associated fiber bundle of P' with standard fiber P_n. $P_n(p)$ is often celled the tangent projective space to M at p. Each element z of P' gives a one-to-one mapping of P_n onto $P_n(p)$, where $\pi(z) = p$. The origin p^* of the tangent projective space is a point in $P_n(p)$ defined by $p^* = z \cdot o$ if $\pi(z) = p$ with $z \in P'$. The definition is independent of the choice of $z \in P'$ such that $\pi(z) = p$.

Making use of P' and the injection of $P'(n)$ into $P(n)$, we define a principal fiber bundle P over the base space M with structure group $P(n)$. In this case, we may identify P' with a submanifold of P.

As is well known, the projective connection (P', l, B) gives rise to a connection Q in P [3]. Indeed, for $z \in P'$, let Q_z be the subspace of P_z composed of all the elements $B(\xi)_z - \xi^*_z$ where $\xi \in F_n$. For $w \in P$, we define Q_w by $Q_w = R_a Q_z$ if $w = z \cdot a$ with $z \in P'$ and $a \in P(n)$. By (P.2), we see that the definition is consistent. Using (P.1) and (P.2), it is shown that the assignment $z \to Q_z$ satisfies (Q.1) and (Q.2) and hence defines a connection in P, which we shall call the projection connection in P induced by the projective connection (P', l, B).

5. Projective development

We first recall definition of affine development [5]. Let B_L be an affine connection in the bundle of frames P_L of a manifold M. Let p be a point of M and let $u(t)$ be a curve in M beginning at p. Fix a point x of P_L such that $\pi(x)=p$ and let $x(t)$ be the lift of $u(t)$ through x with respect to the linear connection. We see from the definition of linear connection that there is a curve $\xi(t)$ in F_n such that $x'(t)=B_L(\xi(t))_{x(t)}$. Then the affine development of $u(t)$ at p is defined as the curve $v(t)=x\cdot w(t)$ in the tangent space M_p where $w(t)=\int_0^t \xi(t)\,dt$. $v(t)$ is independent of the choice of x such that $\pi_L(x)=p$.

Let (P', l, B) be a projective connection over a base space M and define P and Q as in the preceding section. We now define projective development as follows [3]: Let p be a point of M and let $u(t)$ be a curve in M beginning at p. Take an arbitrary curve $y(t)$ in P' which covers $u(t)$ (such a curve necessarily exists) and let $z(t)$ be the lift of $u(t)$ through $y(0)$ with respect to the projective connection in P. There is a curve $a(t)$ in $P(n)$ such that $z(t)\cdot a(t)=y(t)$. Then the projective development of $u(t)$ at p is defined as the curve $u^*(t)=y(0)\cdot a(t)o$ in the tangent projective space $P_n(p)$. Clearly $u^*(t)$ does not depend on the choice of a curve $y(t)$ which coves $u(t)$.

Let $\mathfrak{P}$ be a class of mutually projective affine connections and let (P', l, B) be the corresponding normal projective connection. Fix an affine connection ∇ belonging to $\mathfrak{P}$. Let h be the corresponding homomorphism of P_L into P' and define J_x by (3.1). In the following proposition, we identify F_n and P_n with the tangent affine and projective spaces at $p\in M$ by x and $h(x)$ respectively, where $\pi_L(x)=p$.

PROPOSITION 2. *Fix an arbitrary point x of P_L. Let $u(t)$ be a curve in M beginning at $p=\pi_L(x)$. Let $v(t)$ and $u^*(t)$ be the developments of $u(t)$ at p into F_n and P_n with respect to the affine and projective connections respectively. Then we have $u^*(t)=a(t)o$, where $a(t)$ is a curve in $P(n)$ determined by the following differential equation*

$$a(t)^{-1}a'(t)=v'(t)+J_{x(t)}(v'(t)) \tag{5.1}$$

with the initial condition $a(0)=e$, where $x(t)$ denotes the lift of $u(t)$ through x with respect to the linear connection.

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

Proof. Let $z(t)$ be the lift of $u(t)$ through $h(x)$ with respect to the projective connection Q. Then we can set

$$z(t) \cdot a(t) = h(x(t)) \tag{5.2}$$

with a curve $a(t)$ in $P(n)$. We have $a(0) = e$. By definition of projective development, we have $u^*(t) = a(t)o$. Therefore, it is sufficient to show that $a(t)$ is the solution of (5.1). By definition of affine development, we have $B_L(v'(t))_{x(t)} = x'(t)$. From (P.4) ii), we get

$$w'(t) = B(v'(t))_{w(t)} + J_{x(t)}(v'(t))^*_{w(t)},$$

where we have set $w(t) = h(x(t))$. On the other hand, by differentiating the both sides of (5.2), we obtain

$$R_{a(t)}z'(t) + (a(t)^{-1}a'(t))^*_{w(t)} = w'(t).$$

It follows immediately from these two formulae that

$$\begin{aligned} &R_{a(t)}z'(t) + (a(t)^{-1}a'(t))^*_{w(t)} \\ &\quad = (B(v'(t)) - v'(t)^*)_{w(t)} + (v'(t) + J_{x(t)}(v'(t)))^*_{w(t)}. \end{aligned}$$

But, by definition of Q, the first term of the right side is contained in $Q_{w(t)}$. The first term of the left side is also contained in $Q_{w(t)}$, because $z(t)$ is a horizontal curve with respect to Q. Moreover, the second term in each side is contained in $P(n)_{w(t)}$. Therefore, by virtue of (Q.1), we see that $a(t)$ is the solution of (5.1). q.e.d.

We shall use the following proposition in the next section.

PROPOSITION 3. *Let ∇ and $\bar{\nabla}$ be two affine connections on M which are mutually projective and which are both affinely complete, and let ρ be the associated 1-form of $\bar{\nabla}$ with respect to ∇. Let $u(t)$ and $\bar{u}(\bar{t})$ be geodesics with respect to ∇ and $\bar{\nabla}$ respectively such that $u'(0) = \bar{u}'(0)$. Then the mapping g of the real line R into itself defined by*

$$g(t) = \int_0^t \exp\left(\int_0^t 2\rho(u'(t))\,dt\right)dt$$

is onto, and we have $u(t) = \bar{u}(g(t))$ for all $t \in R$.

Proof. Let B_L and $\bar{B}_L$ be the affine connections in P_L corresponding to ∇ and $\bar{\nabla}$ respectively. Let F be the mapping of P_L into F_n^* defined by (4.2). Fix a point x of P_L such that $\pi_L(x) = u(0) = \bar{u}(0)$ and identify F_n with $M_{u(0)}$

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

by x. Set $u'(0)=\xi\in F_n$. Let $x(t)$ be the lift of $u(t)$ through x with respect to ∇. $u(t)$ being a geodesic such that $u'(0)=\xi$, we see that $x(t)$ is an integral curve of $B_L(\xi)$ [5]. Now define a curve $a(t)$ in $GL(F_n)$ by the following differential equation

$$a(t)^{-1}a'(t)=[\xi,\ F(x(t))] \tag{5.3}$$

with the initial condition $a(0)=e$. If we set $\lim_{t\to-\infty} g(t)=a$ and $\lim_{t\to\infty} g(t)=b$, then we see that g is a homeomorphism of R onto (a, b). It follows that there is a curve $\bar{x}(\bar{t})$ in P_L defined in (a, b) as follows:

$$\bar{x}(g(t))\cdot a(t)=x(t). \tag{5.4}$$

We show that $\bar{x}(\bar{t})$ is an integral curve of $\bar{B}_L(\xi)$. Differentiating the both sides of (5.4), we get

$$g'(t)R_{a(t)}\bar{x}'(g(t))+(a(t)^{-1}a'(t))^*_{x(t)}=x'(t).$$

From (5.3) and (3.2) and the fact $B_L(\xi)_{x(t)}=x'(t)$, we obtain

$$g'(t)\bar{x}'(g(t))=\bar{B}_L(a(t)\xi)_{\bar{x}(g(t))}.$$

Since we have $a(t)\xi=g'(t)\xi$, it follows that $\bar{x}'(\bar{t})=\bar{B}_L(\xi)_{\bar{x}(\bar{t})}$ for all $\bar{t}\in(a, b)$, which proves our assertion. $\bar{u}(\bar{t})$ being a geodesic such that $\bar{u}'(0)=\xi$, we have $\bar{u}(\bar{t})=\pi_L(\bar{x}(\bar{t}))$ [5]. On the other hand, from (5.4), we have $\pi_L(\bar{x}(g(t)))=u(t)$. Therefore we have $\bar{u}(g(t))=u(t)$ for all $t\in R$. We show that g is a homeomorphism of R onto itself. Consider the mapping g^* of R into itself defined by

$$g^*(\bar{t})=\int_0^{\bar{t}}\exp\Big(-\int_0^{\bar{t}}2\rho(\bar{u}'(\bar{t}))\,d\bar{t}\Big)d\bar{t}.$$

Then, in the same manner as above, it can be proved that $\bar{u}(\bar{t})=u(g^*(\bar{t}))$ for all $\bar{t}\in R$. An easy calculation shows that $\dfrac{dg^*\circ g(t)}{dt}=1$ for all $t\in R$, from which we see that $g^*\circ g(t)=t$ for all $t\in R$. In the same way, we have $g\circ g^*(t)=t$ for all $t\in R$. It follows that g is a homeomorphism of R onto itself. q.e.d.

6. Relation between $P(\nabla)$ and $A(\nabla)$

Given an affine connection ∇ on M, we shall say that it satisfies condition (E), if the following conditions are satisfied:

i) without torsion;

ii) affinely complete;

iii) the Ricci tensor field is parallel.

For example, a complete affine symmetric space or a complete Einstein space clearly satisfies condition (E).

The purpose of this section is to establish

THEOREM. *Let ∇ be an affine connection on M which satisfies condition (E) and let S be the Ricci tensor field of ∇.*

i) *If $S(X, X)$ is negative semi-definite, then, for each $f \in P(\nabla)$ and $p \in M$, $S_p(X, X) = 0$ is equivalent to $S_{f(p)}(fX, fX) = 0$, and, for $X \in M_p$ such that $S_p(X, X) = 0$, we have $\rho_p(X) = 0$, where ρ is the associated 1-form of f. In particular, if $S(X, X)$ is zero, then $P(\nabla)$ coincides with $A(\nabla)$.*

ii) *In all other cases, $P(\nabla)$ coincides with $A(\nabla)$.*

Proof. The proof is divided into three steps.

I. Consider a linear mapping J of F_n into F_n^*. We shall denote by Φ_J the quadric in P_n defined by the following quadratic equation on F_{n+1}:

$$-(\eta^0)^2 + \langle \eta, J(\eta) \rangle = 0 \qquad (\eta^0 \xi_0 + \eta \in F_{n+1}). \tag{6.1}$$

LEMMA 3. *Let ξ be a non-zero element of F_n and set $u^*(t) = \exp t(\xi + J(\xi))o$. Putting $\langle \xi, J(\xi) \rangle = \alpha(\xi)$, $u^*(t)$ is computed as follows:*

a) *If $\alpha(\xi) > 0$, then* $u^*(t) = \omega\left(\cosh(\sqrt{\alpha(\xi)}\,t)\xi_0 + \dfrac{\sinh(\sqrt{\alpha(\xi)}\,t)}{\sqrt{\alpha(\xi)}}\xi\right)$;

b) *If $\alpha(\xi) = 0$, then* $u^*(t) = \omega(\xi_0 + t\xi)$;

c) *If $\alpha(\xi) < 0$, then* $u^*(t) = \omega\left(\cos(\sqrt{-\alpha(\xi)}\,t)\xi_0 + \dfrac{\sin(\sqrt{-\alpha(\xi)}\,t)}{\sqrt{-\alpha(\xi)}}\xi\right)$.

Proof. We have $u^*(t) = \omega(\exp t(\tilde{\xi} + J(\tilde{\xi}))\xi_0)$. If we set $A = \tilde{\xi} + J(\tilde{\xi})$, then we have $A^{2m}\xi_0 = \alpha(\xi)^m \xi_0$ and $A^{2m+1}\xi_0 = \alpha(\xi)^m \xi$ for each integer $m \geqq 0$. It follows that

$$\exp tA\xi_0 = \left(\sum_{m=0}^{\infty} \frac{\alpha(\xi)^m}{(2m)!} t^{2m}\right)\xi_0 + \left(\sum_{m=0}^{\infty} \frac{\alpha(\xi)^m}{(2m+1)!} t^{2m+1}\right)\xi.$$

Lemma 3 follows immediately from this formula. q.e.d.

LEMMA 4. *The notation is the same as above.*

i) *If $\lim_{t \to \infty} u^*(t)$ exists, then the limit is contained in Φ_J;*

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

ii) *For any* $q^* \in \Phi_J$, *there is a* ξ *such that* $\lim_{t\to\infty} u^*(t) = q^*$.

Proof. i) Using Lemma 3, we have easily

a) If $\alpha(\xi) > 0$, then $\lim_{t\to\infty} u^*(t) = \omega\left(\xi_0 + \frac{1}{\sqrt{\alpha(\xi)}}\,\xi\right) \quad (\in \Phi_J)$;

b) If $\alpha(\xi) = 0$, then $\lim_{t\to\infty} u^*(t) = \omega(\xi) \quad (\in \Phi_J)$;

c) If $\alpha(\xi) < 0$, then $\lim_{t\to\infty} u^*(t)$ does not exist.

ii) q^* can be expressed as follows: $q^* = \omega(\eta^0\xi_0 + \eta)$ where $-(\eta^0)^2 + \alpha(\eta) = 0$. According as $\alpha(\eta) > 0$ or $\alpha(\eta) = 0$, we take $\xi = \frac{1}{\eta^0}\eta$ or $\xi = \eta$. Using this ξ, we have $\lim_{t\to\infty} u^*(t) = q^*$. q.e.d.

II. Let $\mathfrak{P}$ be a class of mutually projective affine connections on a manifold M and let (P', l, B) be the corresponding normal projective connection. Fix an affine connection ∇ belonging to $\mathfrak{P}$ and assume that it satisfies condition (E). From now on, we use the notation in Proposition 2. For each point p of M, we define a subset $\Phi(p)$ of $P_n(p)$ as follows:

$$\Phi(p) = h(x) \cdot \Phi_{J_x} \quad \text{if} \quad \pi_L(x) = p, \tag{6.2}$$

where Φ_{J_x} is defined by (6.1) taking $J = J_x$. The definition is independent of the choice of x such that $\pi_L(x) = p$.

LEMMA 5. *Let p be a point of M and let $u(t)$ be a geodesic with respect to ∇ such that $u(0) = p$. Let $u^*(t)$ be the development of $u(t)$ at p with respect to the projective connection.*

i) *If* $\lim_{t\to\infty} u^*(t)$ *exists, then the limit is contained in* $\Phi(p)$.

ii) *For any* $q^* \in \Phi(p)$, *there is a geodesic* $u(t)$ *such that* $\lim_{t\to\infty} u^*(t) = q^*$.

Proof. Fix a point x of P_L such that $\pi_L(x) = p$. Let us make use of the result of Proposition 2. If we set $u'(0) = \xi \ (\in F_n)$, then we have $v'(t) = \xi$, because $u(t)$ is a geodesic. Since the Ricci tensor field is parallel, we see that, for any horizontal curve $x(t)$ in P_L, $S_{x(t)}$ is constant and from (3.1) that $J_{x(t)}$ is constant. Hence the differential equation (5.1) can be written as $a(t)^{-1}a'(t) = \xi + J_x(\xi)$; namely, $a(t) = \exp t(\xi + J_x(\xi))$. We have $u^*(t) = \exp t(\xi + J_x(\xi))o$. Therefore, we have only to apply Lemma 4 to $J = J_x$. q.e.d.

III. We now assume that $\bar{\nabla}$ $(\in \mathfrak{P})$ together with ∇ satisfies condition (E). Starting with $\bar{\nabla}$, we define a subset $\bar{\Phi}(p)$ of $P_n(p)$ in the same manner as $\Phi(p)$.

LEMMA 6. *$\Phi(p) = \bar{\Phi}(p)$ at each point p of M.*

Proof. We first show that $\Phi(p) \subset \bar{\Phi}(p)$. Let q^* be an arbitrary point of $\Phi(p)$. By Lemma 5 ii), we see that there is a geodesic $u(t)$ with respect to ∇ such that $\lim_{t\to\infty} u^*(t) = q^*$. We know from Proposition 3 that there is a geodesic $\bar{u}(\bar{t})$ with respect to $\bar{\nabla}$ such that $u(t) = \bar{u}(g(t))$, where $g(t)$ is a homeomorphism of R onto itself. Denoting by $\bar{u}^*(\bar{t})$ the development of $\bar{u}(\bar{t})$ with respect to the projective connection, we have $u^*(t) = \bar{u}^*(g(t))$; $\lim_{t\to\infty} \bar{u}^*(\bar{t}) = \lim_{t\to\infty} \bar{u}^*(g(t)) = \lim_{t\to\infty} u^*(t) = q^*$. By Lemma 5 i), now applied to $\bar{\nabla}$, we see that q^* is contained in $\bar{\Phi}(p)$. Thus we have shown that $\Phi(p) \subset \bar{\Phi}(p)$. In the same way, we have $\bar{\Phi}(p) \subset \Phi(p)$. Therefore, we have $\Phi(p) = \bar{\Phi}(p)$. q.e.d.

Let S and $\bar{S}$ be the Ricci tensor fields of ∇ and $\bar{\nabla}$ respectively and let ρ be the associated 1-form of $\bar{\nabla}$ with respect to ∇.

LEMMA 7. *At each point p of M, consider the following two quadratic equations on $R \times M_p$:*

$$-(X^0)^2 + \frac{1}{n-1} S_p(X, X) = 0; \tag{1}$$

$$-(X^0 - \rho_p(X))^2 + \frac{1}{n-1} \bar{S}_p(X, X) = 0. \tag{2}$$

Then, at each point p of M, in order that (X^0, X) be a solution of (1), it is necessary and sufficient that it is also a solution of (2).

Proof. Let x be a point of P_L such that $\pi_L(x) = p$. Let $\bar{h}$ be the homomorphism of P_L into P' corresponding to $\bar{\nabla}$ and let $\bar{J}_x$ be the linear mapping of F_n into F_n^* defined by (3.1) starting with $\bar{\nabla}$. We have

$$\bar{\Phi}(p) = \bar{h}(x) \cdot \Phi_{\bar{J}_x}. \tag{6.3}$$

We know that there is a mapping F of P_L into F_n^* such that

(6.4) i) $\bar{h}(x) = h(x) \cdot \exp F(x)$;

ii) $\rho(x \cdot \xi) = \langle \xi, F(x) \rangle$.

It follows from (6.2), (6.3) and (6.4) i) that $\Phi(p) = \bar{\Phi}(p)$ is equivalent to

$$\Phi_{J_x} = \exp F(x) \Phi_{\bar{J}_x}. \tag{6.5}$$

Lemma 7 follows immediately from (6.1), (6.4) ii), (6.5) and the following formulae:

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

$$\langle \xi, J_x(\xi) \rangle = \frac{1}{n-1} S(x \cdot \xi, x \cdot \xi);$$

$$\langle \xi, \bar{J}_x(\xi) \rangle = \frac{1}{n-1} \bar{S}(x \cdot \xi, x \cdot \xi).$$

q.e.d.

LEMMA 8. *Let ∇ and $\bar{\nabla}$ be two affine connections which are mutually projective and which both satisfy condition (E). Let ρ be the associated 1-form of $\bar{\nabla}$ with respect to ∇ and let S and $\bar{S}$ be the Ricci tensor fields of ∇ and $\bar{\nabla}$ respectively.*

i) *If both $S(X, X)$ and $\bar{S}(X, X)$ are negative semi-definite, then, for each point p of M, $S_p(X, X)=0$ is equivalent to $\bar{S}_p(X, X)=0$, and, for $X \in M_p$ such that $S_p(X, X)=0$, we have $\rho_p(X)=0$.*

ii) *In all other cases, ∇ and $\bar{\nabla}$ coincide.*

Proof. Let $\mathfrak{P}$ be the class of mutually projective affine connections containing ∇ and $\bar{\nabla}$. We apply Lemma 7 to ∇ and $\bar{\nabla}$.

i) The case where both $S(X, X)$ and $\bar{S}(X, X)$ are negative semi-definite. If $S_p(X, X)=0$, then $(0, X)$ is clearly a solution of (1). Using Lemma 7, we see that $(0, X)$ is a solution of (2). It follows that $\rho_p(X)=0$ and $\bar{S}_p(X, X)=0$. In the same way, if $\bar{S}_p(X, X)=0$, then we have $\rho_p(X)=0$ and $S_p(X, X)=0$.

ii) The case where either $S(X, X)$ or $\bar{S}(X, X)$ is not negative semi-definite. Without loss of generality, we can assume that $S(X, X)$ is not negative semi-definite. If $S_p(X, X)>0$, then there is a $X^0 \in R$ such that (X^0, X) becomes a solution of (1). In general, if (X^0, X) is a solution of (1), so is $(X^0, -X)$. Therefore, by Lemma 7, both (X^0, X) and $(X^0, -X)$ are solutions of (2). It follows that $X^0\rho_p(X)=0$, from which we obtain $\rho_p(X)=0$. Thus we have shown that if $S_p(X, X)>0$, then $\rho_p(X)=0$. Since the subset of M_p composed of all the elements X such that $S_p(X, X)>0$ is open in M_p, we see that $\rho_p(X)=0$ holds for all $X \in M_p$ and $p \in M$. Consequently we have $\rho=0$ and hence ∇ and $\bar{\nabla}$ coincide. q.e.d.

We are now in a position to prove the theorem. Let f be an arbitrary element of $P(\nabla)$. Let $\bar{\nabla}$ be the affine connection defined by (1.2). Then ∇ and $\bar{\nabla}$ are mutually projective, and the associated 1-form of $\bar{\nabla}$ with respect to ∇ is nothing but the associated 1-form of f. We see that $\bar{\nabla}$ satisfies condition (E) and $\bar{S}$ is given by $\bar{S}_p(X, X)=S_{f(p)}(fX, fX)$. Therefore we have only to apply Lemma 8 to ∇ and $\bar{\nabla}$. Thus we have completed the proof of the theorem.

Remark. An affine connection on a manifold M is called *projectively complete* if the normal projective connection corresponding to the class which contains the given affine connection is complete [3], that is, if, for each point p of M, every curve through p^* in the tangent projective space at p admits the development into the base space. We can prove the following statements:

i) *Let ∇ be an affine connection satisfying condition (E) and let S be the Ricci tensor field of ∇. If $S(X, X)$ is not negative definite, then ∇ is not projectively complete*;

ii) *A complete Einstein space with negative definite Ricci tensor field is projectively complete.*

i) and ii) indicate geometrical properties of an affine connection which satisfies condition (E).

Appendix

In §1, we remarked that the projective equivalence of two affine connections is characterized by the coincidence of the systems of geodesics for the two connections. In the following, we shall give an exact formulation of this fact and prove it.

PROPOSITION. *Let ∇ and $\bar{\nabla}$ be two affine connections on a manifold M whose torsion are zero. A necessary and sufficient condition that ∇ and $\bar{\nabla}$ be mutually profective, is given as follows: Let p be an arbitrary point of M and let V be an arbitrary 1-dimensional subspace of M_p. Let $u(t)$ be a curve in M beginning at p. Then, for all t, the tangent vector $u'(t)$ is contained in the result of parallel displacement of V along the curve with respect to ∇, if and only if, for all t, it is contained in the result of parallel displacement of V along the curve with respect to $\bar{\nabla}$.*

Proof. Let B_L and $\bar{B}_L$ be the affine connections in P_L corresponding to ∇ and $\bar{\nabla}$ respectively. To each $x \in P_L$ there is associated a linear mapping A_x of F_n into $\mathfrak{gl}(F_n)$ such that

$$\bar{B}_L(\xi)_x = B_L(\xi)_x - A_x(\xi)^*_x. \tag{1}$$

It can be proved that $A_{x \cdot a}(\xi) = ad(a^{-1})A_x(a\xi)$. Let $u(t)$ be a curve in M beginning at a point p of M. Fix a point $x \in P_L$ such that $\pi_L(x) = p$ and identify F_n with M_p by x. Let $v(t)$ and $\bar{v}(t)$ be the developments of $u(t)$ into F_n with respect to ∇ and $\bar{\nabla}$ respectively. Let $x(t)$ and $\bar{x}(t)$ be the lifts of $u(t)$ through x with respect to ∇ and $\bar{\nabla}$ respectively. We can set $x(t)=\bar{x}(t)\cdot a(t)$ with a mapping $a(t)$ in $GL(F_n)$. By the argument analogous to the proof of Proposition 2, we can derive

(2) i) $a(t)v'(t)=\bar{v}'(t)$;

ii) $a(t)^{-1}a'(t)=A_{x(t)}(v'(t))$.

It follows from these two formulae that

$$A_{x(t)}(v'(t))v'(t)=a(t)^{-1}\bar{v}''(t)-v''(t). \tag{3}$$

$u'(t)$ is contained in the result of parallel displacement of V along the curve with respect to ∇, if and only if $v'(t)$ is contained in V.

Now assume that ∇ and $\bar{\nabla}$ are mutually projective. We know that there is a mapping F of P_L into F_n^* such that $A_x(\xi)=[\xi, F(x)]$. If $v'(t)$ is contained in V, then it follows from (2) ii) that $a(t)$ leaves V invariant, because V is stable under $A_{x(t)}(v'(t))$. Therefore we see from (2) i) that $\bar{v}'(t)$ is contained in V.

Now we shall prove the converse. It is sufficient to prove that there is a differentiable mapping F of P_L into F_n^* such that $A_x(\xi)=[\xi, F(x)]$. Indeed, using the fact that $A_{x\cdot a}(\xi)=ad(a^{-1})A_x(a\xi)$, it can be proved that $F(x\cdot a)={}^taF(x)$. It follows that there is a 1-form ρ on M such that $\rho(x\cdot\xi)=\langle\xi, F(x)\rangle$ for all $x\in P_L$ and $\xi\in F_n$. Using this ρ, we have $\bar{\nabla}_XY=\nabla_XY+\rho(Y)X+\rho(X)Y$. Let ξ be an arbitrary non-zero element of F_n and denote by $V(\xi)$ the 1-dimensional subspace of F_n spanned by ξ. If $u(t)$ is a curve such that $v'(t)=\xi$, then $\bar{v}'(t)$ is contained in $V(\xi)$. Since $\bar{v}''(t)\in V(\xi)$, we see from (3) that $A_x(\xi)\xi\in V(\xi)$. Thus we have seen that, for each $x\in P_L$, there is a function α_x such that $\frac{1}{2}A_x(\xi)\xi=\alpha_x(\xi)\xi$ for all $\xi\in F_n$ and such that $\alpha_x(0)=0$. Since ∇ and $\bar{\nabla}$ are both without torsion, it can be proved that $A_x(\xi)\xi'=A_x(\xi')\xi$ for all $x\in P_L$ and $\xi, \xi'\in F_n$.

LEMMA. *Let F_n be an n-dimensional vector space. Let A be a linear mapping of $F_n\times F_n$ into F_n which satisfies the following conditions:*

i) $A(\xi, \eta)=A(\eta, \xi)$ *for all* $\xi, \eta\in F_n$;

ii) *There is a function* α *on* F_n *such that*

https://doi.org/10.1017/S0027763000021905 Published online by Cambridge University Press

a) $\frac{1}{2}A(\xi, \xi) = \alpha(\xi)\xi$ *for all* $\xi \in F_n$;

b) $\alpha(0) = 0$.

Then α *is a linear function on* F_n. *Therefore we have* $A(\xi, \eta) = \alpha(\xi)\eta + \alpha(\eta)\xi$.

Proof. We have clearly $\alpha(\lambda\xi) = \lambda\alpha(\xi)$ for all $\lambda \in R$ and $\xi \in F_n$. We shall show thst $\alpha(\xi+\eta) = \alpha(\xi) + \alpha(\eta)$ for all $\xi, \eta \in F_n$. For this purpose, it is sufficient to deal with the case where ξ and η are linearly independent. We have

$$(1) \qquad \frac{1}{2}A(x\xi+y\eta, x\xi+y\eta) = \frac{1}{2}x^2A(\xi, \xi) + xyA(\xi, \eta) + \frac{1}{2}y^2A(\eta, \eta).$$

We can set $A(\xi, \eta) = A_1\xi + A_2\eta$. Since ξ and η are linearly independent, it follows from (1) and ii) a) that

(2) i) $\alpha(\xi)x^2 + A_1xy = \alpha(x\xi+y\eta)x$;

ii) $\alpha(\eta)y^2 + A_2xy = \alpha(x\xi+y\eta)y$.

It follows immediately that

$$\alpha(\xi)x^2y + A_1xy^2 = \alpha(\eta)xy^2 + A_2x^2y \qquad \text{for all} \quad x, y \in R.$$

Since x and y are arbitrary, we get $\alpha(\xi) = A_2$ and $A_1 = \alpha(\eta)$. Setting $x = y = 1$ in (2) i), we have $\alpha(\xi+\eta) = \alpha(\xi) + \alpha(\eta)$. q.e.d.

Applying the lemma to the case where $A(\xi, \eta) = A_x(\xi)\eta$ and $\alpha = \alpha_x$, we see that α_x is a linear function and that $A_x(\xi)\eta = \alpha_x(\xi)\eta + \alpha_x(\eta)\xi$. If we define a mapping F of P_L into F_n^* by $\alpha_x(\xi) = \langle\xi, F(x)\rangle$, then we have $A_x(\xi) = [\xi, F(x)]$. The differentiability of F follows from the formula: $\langle\xi, F(x)\rangle = \frac{1}{n+1}Tr(A_x(\xi))$. Thus we have completed the proof of the proposition.

Bibliography

[1] E. Cartan, Leçons sur la théorie des espaces à connexion projective, Paris (1937).

[2] C. Chevalley, Theory of Lie groups I, Princeton, 1946.

[3] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, Colloque de Topologie, Bruxelles (1950), 29-55.

[4] S. Kobayashi, Groupe de transformations qui laissent invariante une connexion infini-

tésimale, Comptes rendus, **238** (1954), pp. 644–645.
[5] K. Nomizu, Lie groups and Differential Geometry, No. 2, Publications Math. Soc. Japan, 1956.

Mathematical Institute
Nagoya University