BULL. AUSTRAL. MATH. SOC. 49A27
VOL. 43 (1991) [211–224]

EXISTENCE OF OPTIMAL CONTROLS FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

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In this paper we examine a Lagrange optimal control problem driven by a nonlinear evolution equation involving a nonmonotone, state dependent perturbation term. For this problem we establish the existence of optimal admissible pairs. For the same system we also examine a time optimal control problem involving a moving target set. Finally we work out in detail an example of a strongly nonlinear parabolic distributed parameter system.

1. INTRODUCTION

In this paper we establish the existence of optimal controls for a class of strongly nonlinear, parabolic optimal control problems, with an integral cost criterion and with state dependent control constraints. Our work extends those of Ahmed [1], Ahmed and Teo [2], Avgerinos and Papageorgiou [4], Flytzanis and Papageorgiou [9], Joshi [11], Lions [12] and Vidyasagar [18]. From these works, Ahmed [1] and Ahmed and Teo [2] assumed that the differential operator $A(t)(\cdot)$ is linear (semilinear system) and in Ahmed and Teo [2] there was a state–dependent perturbation term, which though was monotone as was $A(t)(\cdot)$. In the problem studied by Avgerionos and Papageorgiou [4], the operator $A(t, \cdot)$ was nonlinear, but there were no nonmonotone terms. In Flytzanis and Papageorgiou [9] again the dynamical equation is nonlinear, but it is assumed that the partial differential operator is of the subdifferential type and in addition the semigroup of nonlinear contractions $S(t)$ generated by it is compact for $t > 0$. In Lions [12] only time invariant monotone operators were allowed, while finally Joshi [11] and Vidyasagar [18] examined systems described by Hammerstein and nonlinear finite dimensional equations respectively, but under restrictive overall hypotheses. We should also mention the recent work of Cesari [8], who studied a different class of nonlinear control problems, using results from operator theory and the nice book of Ahmed and Teo [3], which has a comprehensive introduction into the modern approaches of the theory of optimal control of nonlinear evolution equations in Banach spaces.

Received 29 March 1990
Research supported by NSF Grant DMS–8802688.

https://doi.org/10.1017/S0004972700028975 Published online by Cambridge University Press

In our problem, the important feature is the presence of a nonlinear, nonmonotone state dependent perturbation, which in concrete examples can incorporate certain partial differential operator terms of nonmonotone type. We examine a nonlinear Lagrange optimal control problem and under mild hypotheses we prove that it has a solution. We also consider a time optimal control problem, involving a moving target set and for this we establish the existence of time optimal controls. Finally we work out in detail an example of a nonlinear, parabolic distributed parameter system.

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. By $P_{f(c)}(X)$ we will be denoting the family of nonempty, closed, (convex) subsets of X. A multifunction (set valued function) $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is said to be graph measurable if $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if for all $z \in X$, $\omega \to d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\} \in L^1_+$. Measurability implies graph measurability. The converse is true if there exists a complete, σ-finite measure $\mu(\cdot)$ on (Ω, Σ).

Let Y, Z be Hausdorff topological spaces. A multifunction $G\colon Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.), if for all $U \subseteq Z$ open, $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ is open in Y. Also we say that $G(\cdot)$ is closed, if $GrG = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed in $Y \times Z$. A closed valued, u.s.c. multifunction is closed.

Let H be a separable Hilbert space and X a subspace of H, carrying the structure of a separable, reflexive Banach space and which embeds continuously and densely into H. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$ with all embeddings being continuous and dense. Such a triple of spaces is called in the literature "Gelfand triple". To have a concrete example in mind let Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial Z = \Gamma$. Set $H = L^2(Z)$ and $X = W_0^{m,p}(Z)$ with $m \in Z_+$, $2 \leqslant p < \infty$. Then $W^{m,p}(Z) \hookrightarrow L^2(Z) \hookrightarrow W^{-m,p}(Z) = [W_0^{m,p}(Z)]^*$ $(1/p + 1/q = 1)$ with all embeddings being continuous, dense and furthermore compact ("Sobolev–Kondrachov embedding theorem"). By $\|\cdot\|$ (respectively $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (respectively of H, X^*). Also by $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for (X, X^*) and by $(\cdot, \cdot)$ the inner product in H. The two are compatible in the sense that if $x \in X \subseteq H$ and $v \in H \subseteq X^*$, then $(x, v) = \langle x, v \rangle$.

3. Existence theorems

Let $T = [0, b]$ and (X, H, X^*) a Gelfand triple of spaces with all embeddings being in addition compact. Also Y is a separable, reflexive Banach space modelling the control space.

The nonlinear optimal control problem under consideration is the following:

$$(*)\quad \left\{\begin{array}{c} J(x, u) = \int_0^b L(t, x(t), u(t))dt \to \inf = m \\ \text{such that } \dot{x}(t) + A(t, x(t)) + f(t, x(t)) = B(t)u(t) \text{ almost everywhere} \\ x(0) = x_0,\ u(t) \in U(t, x(t)) \text{ almost everywhere} \\ u(\cdot) \text{ measurable} \end{array}\right\}$$

We will need the following hypotheses on the data of (*).

$H(A)$. $A: T \times X \to X^*$ is an operator such that

(1) $t \to A(t, x)$ is measurable,
(2) $x \to A(t, x)$ is hemicontinuous, monotone,
(3) $\|A(t, x)\|_* \leqslant c\left(\|x\|^{p-1} + 1\right)$ almost everywhere with $c > 0$, $p \geqslant 2$,
(4) $\langle A(t, x), x\rangle \geqslant c_2 \|x\|^p$ almost everywhere with $c_2 > 0$.

$H(f)$. $f: T \times X \to H$ is a map such that

(1) $t \to f(t, x)$ is measurable,
(2) $x \to f(t, x)$ is continuous and sequentially weakly continuous,
(3) there exists $c_3 > 0$ such that $-c_3 \leqslant (f(t, x), x)$ almost everywhere for all $x \in X$,
(4) $|f(t, x)| \leqslant a(t) + b\|x\|^{p-1}$ almost everywhere with $a(\cdot) \in L^q_+$, $b > 0$ $(1/p + 1/q = 1)$.

$H(B)$. $B(\cdot) \in L^\infty(T, \mathcal{L}(Y, H))$.

$H(U)$. $U: T \times H \to P_{fc}(Y)$ is a multifunction such that

(1) $U(\cdot, \cdot)$ is graph measurable,
(2) $U(t, \cdot)$ is sequentially closed in $H \times Y_w$,
(3) $|U(t, x)| \leqslant a_1(t)$ almost everywhere with $a_1(\cdot) \in L^q_+$.

$H(L)$. $L: T \times H \times Y \to \overline{\mathbb{R}}$ is an integrand such that

(1) $L(\cdot, \cdot, \cdot)$ is measurable,
(2) $L(t, \cdot, \cdot)$ is lower semicontinuous on $H \times Y$,
(3) $L(t, x, \cdot)$ is convex,
(4) $\phi(t) - M(|x| + \|u\|) \leqslant L(t, x, u)$ almost everywhere with $\phi(\cdot) \in L^1$, $M > 0$.

Finally to avoid trivial situations, we need the following admissibility hypothesis:

H_a. there exists admissible "state–control" pair (x, u) such that $J(x, u) < \infty$.

https://doi.org/10.1017/S0004972700028975 Published online by Cambridge University Press

Let $W_{pq}(T) = \{x(\cdot) \in L^p(X) : \dot{x}(\cdot) \in L^q(X^*)\}$, with the derivative involved defined in the sense of distributions. Furnished with the norm

$$\|x\|_{W_{pq}(T)} = \{\|x\|^2_{L^p(X)} + \|\dot{x}\|^2_{L^q(X^*)}\}^{1/2}$$

$W_{pq}(T)$ becomes a separable, reflexive Banach space. Furthermore we know (see Ahmed and Teo [**3**, Theorem 1.2.15, p.27]) that $W_{pq}(T) \hookrightarrow C(T, H)$ continuously. The trajectories of $(*)$ lie in $W_{pq}(T)$ (see Barbu [**6**] and Hirano [**10**]).

Theorem 3.1. *If hypotheses $H(A)$, $H(f)$, $H(B)$, $H(U)$, $H(L)$ and H_a hold, then there exists admissible "state–control" pair (x, u) such that $J(x, u) = m$.*

Proof: Let $\{(x_n, u_n)\}_{n \geqslant 1}$ be a minimising sequence of admissible pairs for (*). Then for all $n \geqslant 1$, we have:

$$\left\{\begin{array}{c} \dot{x}_n(t) + A(t, x_n(t)) + f(t, x_n(t)) = B(t)u_n(t) \text{ almost everywhere} \\ x_n(0) = x_0,\ u_n(t) \in U(t, x_n(t)) \text{ almost everywhere} \\ u_n(\cdot) \text{ measurable} \end{array}\right\}.$$

Multiply the evolution equation with $x_n(\cdot)$. We get

$$\begin{aligned} &\langle \dot{x}_n(t), x_n(t)\rangle + \langle A(t, x_n(t)), x_n(t)\rangle + (f(t, x_n(t)), x_n(t)) \\ &\quad = (B(t)u_n(t), x_n(t)) \text{ almost everywhere} \\ &\Rightarrow \frac{d}{dt}|x_n(t)|^2 + 2\langle A(t, x_n(t)), x_n(t)\rangle + 2(f(t, x_n(t)), u_n(t)) \\ &\quad = 2(B(t)u_n(t), x_n(t)) \text{ almost everywhere} . \end{aligned}$$

Using hypotheses $H(A)$ (4) and $H(f)$ (3), we get

$$\frac{d}{dt}|x_n(t)|^2 + 2c_2\|x_n(t)\|^p - 2c_3 \leqslant 2(B(t)u_n(t), x_n(t)) \text{ almost everywhere.}$$

Integrating and using Hölder's inequality, we get

$$2c_2\|x_n\|^p_{L^p(X)} \leqslant 2c_3 b + 2\|Bu_n\|_{L^q(H)}\|x_n\|_{L^p(X)} .$$

Invoking Cauchy's inequality with $\varepsilon > 0$, we get

$$\begin{aligned} 2c_2\|x_n\|^p_{L^p(X)} &\leqslant 2c_3 b + 2\frac{\varepsilon^p}{p}\|x_n\|^p_{L^p(H)} + 2\frac{1}{q\varepsilon^q}\|Bu_n\|^q_{L^q(H)} \\ &\leqslant 2c_3 b + \frac{2\varepsilon^p}{p}\|x_n\|^p_{L^p(X)} + \frac{2}{q\varepsilon^q}\|B\|^q_{L^\infty(T, \mathcal{L}(Y, H))}\|a_1\|^q_q . \end{aligned}$$

By choosing $\varepsilon > 0$ sufficiently small so that $c_2 > \varepsilon^p/p$, we get from the above inequality that there exists $M_1 > 0$ such that $\|x_n\|_{L^p(X)} \leqslant M_1$ for all $n \geqslant 1$.

Then using hypotheses $H(A)$ (3) and $H(f)$ (4) and recalling that $(p-1)q = p$, we get:

$$\|\dot{x}_n(t)\|_*^q \leqslant 4^q c^q(\|x_n(t)\|^p + 1) + 8^q(a(t)^q + b\,\|x_n(t)\|^p) \\ + 2^q\,\|B\|^q_{L^\infty(T,\,\mathcal{L}(Y,\,H))}\, a_1(t)^q \text{ almost everywhere.}$$

Integrating over $T = [0,\, b]$ and recalling that $\|x_n\|_{L^p(X)} \leqslant M_1$, we deduce that there exists $M_2 > 0$ such that $\|\dot{x}_n\|_{L^q(X^*)} \leqslant M_2$ for all $n \geqslant 1$. Hence we have proved that $\{x_n(\cdot)\}_{n\geqslant 1}$ is bounded in $W_{pq}(T)$. Recalling that $W_{pq}(T)$ is reflexive and by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $W_{pq}(T)$. Furthermore since by hypothesis $X \hookrightarrow H$ compactly, from Lions [**13**, Theorem 5.1, p.58], we have that $W_{pq}(T) \hookrightarrow L^p(H)$ compactly. So we can say that $x_n \xrightarrow{s} x$ in $L^p(H)$. Furthermore since $L^q(Y)$ is reflexive (Y being reflexive and $q > 1$), by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^q(Y)$. Then hypothesis $H(L)$ allows us to apply Theorem 2.1 of Balder [5] and get that

$$\int_0^b L(t,\, x(t),\, u(t))dt \leqslant \underline{\lim} \int_0^b L(t,\, x_n(t),\, u_n(t))dt = m.$$

So it remains to show that $(x,\, u)$ is an admissible "state–control" pair for (*). First we claim that:

$$\lim\langle A(t,\, x_n(t)) + f(t,\, x_n(t)),\, x_n(t) - x(t)\rangle \geqslant 0 \text{ almost everywhere .}$$

Suppose that this is not the case. Then we will have

$$\lim\langle A(t,\, x_n(t)) + f(t,\, x_n(t)),\, x_n(t) - x(t)\rangle < 0 \text{ for } t \in E,\, \lambda(E) > 0. \tag{1}$$

Invoking hypotheses $H(A)$ (3) and (4) and $H(f)$ (3) and (4), we have

$$\langle A(t,\, x_n(t)) + f(t,\, x_n(t)),\, x_n(t) - x(t)\rangle \geqslant c_1 + c_2\,\|x_n(t)\|^p - c_3 \\ -c\Big(\|x_n(t)\|^{p-1} + 1\Big)\,\|x(t)\| - \Big(a(t) + b\,\|x_n(t)\|^{p-1}\Big)\,\|x(t)\| \text{ almost everywhere} \tag{2}$$

Combining (1) and (2) above, we get that $\{\|x_n(t)\|\}_{n\geqslant 1}$ is bounded for $t \in E\backslash N = E'$, $\lambda(N) = 0$. Fix $t \in E'$. By passing to a subsequence (depending on $t \in E'$) if necessary, we may assume that $x_n(t) \xrightarrow{w} \widehat{x}(t)$. Since by hypothesis $X \hookrightarrow H$ compactly, we have $x_n(t) \xrightarrow{s} \widehat{x}(t)$ (the limit will depend on $t \in E'$). On the other hand recall that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ and as we have already said $W_{pq}(T) \hookrightarrow L^p(H)$ compactly. So

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we may assume that $x_n \xrightarrow{s} x$ in $L^p(H)$ and $x_n(t) \xrightarrow{s} x(t)$ almost everywhere in H. Thus we have $\widehat{x}(t) = x(t)$ for all $t \in E'' = E \setminus N_1$, $\lambda(N_1) = 0$. Then exploiting the monotonicity of $A(t, \cdot)$, $t \in E''$, we have:

$$\begin{aligned} &\langle A(t, x_n(t)), x_n(t) - x(t)\rangle \geqslant \langle A(t, x(t)), x_n(t) - x(t)\rangle \\ &\Rightarrow \lim\langle A(t, x_n(t)), x_n(t) - x(t)\rangle \geqslant 0 \; t \in E''. \end{aligned}$$

Hence finally, using hypothesis $H(f)$ (2), we have

$$\lim\langle A(t, x_n(t)) + f(t, x_n(t)), x_n(t) - x(t)\rangle \geqslant 0 \; t \in E'', \; \lambda(E'') = \lambda(E) > 0,$$

and this contradicts (1). So we have proved our claim.

Set $\eta_n(t) = \langle A(t, x_n(t)) + f(t, x_n(t)), x_n(t) - x(t)\rangle$. From Fatou's lemma we have

$$(3) \qquad \begin{aligned} 0 \leqslant \int_0^b \underline{\lim}\, \eta_n(t)dt &\leqslant \underline{\lim} \int_0^b \eta_n(t)dt \\ &\leqslant \overline{\lim}\left(\left(\widehat{A}(x_n) + \widehat{f}(x_n), x_n - x\right)\right)_0 \end{aligned}$$

where $\widehat{A}$: $A^P(X) \to L^q(X^*)$ is the Nemitsky operator corresponding to $A(t, x)$, $\widehat{f}$: $L^p(X) \to L^q(X^*)$ the Nemitsky operator corresponding to $f(t, x)$ and $((\cdot, \cdot))_0$ the duality brackets for the dual pair $(L^p(X), L^q(X^*))$.

Now we claim that

$$\overline{\lim}\left(\left(\widehat{A}x_n + \widehat{f}(x_n), x_n - x\right)\right)_0 = 0.$$

From the dynamics of the system for every $n \geqslant 1$, we have:

$$\left(\left(\widehat{A}x_n + \widehat{f}(x_n), x_n - x\right)\right)_0 = \left(\left(-\dot{x}_n + \widehat{B}u_n, x_n - x\right)\right)_0$$

with $\widehat{B}$ being the Nemitsky operator corresponding to $B(t)$. Recall (see for example Tanabe [17], Lemma 5.5.1, p.151) that:

$$\begin{aligned} \langle \dot{x}(t) - \dot{x}_n(t), x(t) - x_n(t)\rangle &= \frac{1}{2}\frac{d}{dt}\langle x(t) - x_n(t), x(t) - x_n(t)\rangle \\ &= \frac{1}{2}\frac{d}{dt}(x(t) - x_n(t), x(t) - x_n(t)) \\ &= \frac{1}{2}\frac{d}{dt}\left|x(t) - x_n(t)\right|^2 \\ \Rightarrow ((\dot{x} - \dot{x}_n, x - x_n))_0 &= \frac{1}{2}\left|x(b) - x_n(b)\right|^2 \\ \Rightarrow \quad -((\dot{x}_n, x - x_n))_0 &= \frac{1}{2}\left|x(b) - x_n(b)\right|^2 - ((\dot{x}, x - x_n))_0. \end{aligned}$$

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Recalling that $W_{pq}(T) \hookrightarrow C(T, H)$, we have:

$$\left(\left(-\dot{x}_n + \widehat{B}u_n, x_n - x\right)\right)_0 = ((-\dot{x}_n, x_n - x))_0 + \left(\left(\widehat{B}u_n, x_n - x\right)\right)_0$$
$$= \frac{1}{2}|x(b) - x_n(b)|^2 - ((\dot{x}, x - x_n))_0 + \left(\widehat{B}u_n, x_n - x\right)_{L^p(H), L^q(H)} \to 0 \text{ as } n \to \infty.$$

So we deduce that

$$\overline{\lim}\left(\left(\widehat{A}x_n + \widehat{f}(x_n), x_n - x\right)\right)_0 = 0$$

which proves our claim. Putting this fact back into (3) we get

$$0 = \int_0^b \underline{\lim}\, \eta_n(t) \leqslant \underline{\lim} \int_0^b \eta_n(t)dt \leqslant \overline{\lim}\left(\left(\widehat{A}x_n + \widehat{f}(x_n), x_n - x\right)\right)_0 = 0.$$

From the above inequalities, we deduce that $\int_0^b |\eta_n(t)|\, dt \to 0$ as $n \to \infty \Rightarrow \eta_n \xrightarrow{s} 0$ in $L^1(T)$ and so we may assume that $\eta_n(t) \to 0$ almost everywhere $\Rightarrow \langle A(t, x_n(t)) + f(t, x_n(t)), x_n(t) - x(t)\rangle \to 0$ almost everywhere. From this and inequality (2) above, we see that $\{\|x_n(t)\|\}_{n\geqslant 1}$ is bounded for almost all $t \in T$. Hence as before, by passing to a subsequence (depending in general on t) if necessary, we may assume that $x_n(t) \to \widehat{x}(t)$ for almost all $t \in T$. On the other hand recall that since $x_n \xrightarrow{w} x$ in $W_{pq}(T)$, we can write that $x_n(t) \xrightarrow{s} x(t)$ almost everywhere in H. Thus $\widehat{x}(t) = x(t)$ almost everywhere and thus since for almost all $t \in T$, every subsequence of $\{x_n(t)\}_{n\geqslant 1}$ for almost all $t \in T$ has a further subsequence converging weakly in X to $x(t)$, we deduce that $x_n(t) \xrightarrow{w} x(t)$ almost everywhere in X. Then $(f(t, x_n(t)), x_n(t) - x(t)) \to 0$ and so $\langle A(t, x_n(t)), x_n(t) - x(t)\rangle \to 0$ almost everywhere. Now note that since by hypothesis $H(A)$ (2), $A(t, \cdot)$ is hemicontinuous, monotone, everywhere defined on X, it is pseudomonotone (see Browder [7]). So $A(t, x_n(t)) \xrightarrow{w} A(t, x(t))$ almost everywhere in $X^* \Rightarrow \widehat{A}x_n \xrightarrow{w} \widehat{A}x$ in $L^q(X^*)$. Also from hypotheses $H(f)$ (2) and (4), we see that $\widehat{f}(x_n) \xrightarrow{w} \widehat{f}(x)$ in $L^q(H)$ (hence in $L^q(X^*)$). Finally note that since $x_n \xrightarrow{w} x$ in $W_{pq}(T) \Rightarrow \dot{x}_n \xrightarrow{w} \dot{x}$ in $L^q(X^*)$, while $\widehat{B}u_n \xrightarrow{w} \widehat{B}u$ in $L^q(H)$ (hence in $L^q(X^*)$ too). So for any $h \in L^p(X)$, we have:

$$((\dot{x}_n, h))_0 + \left(\left(\widehat{A}x_n, h\right)\right)_0 + \left(\left(\widehat{f}(x_n), h\right)\right)_0 - \left(\left(\widehat{B}u_n, h\right)\right)_0$$
$$\to ((\dot{x}, h))_0 + \left(\left(\widehat{A}x, h\right)\right)_0 + \left(\left(\widehat{f}(x), h\right)\right)_0 - \left(\left(\widehat{B}u, h\right)\right)$$
$$\Rightarrow ((\dot{x}, h))_0 + \left(\left(\widehat{A}x, h\right)\right)_0 + \left(\left(\widehat{f}(x), h\right)\right)_0 = \left(\left(\widehat{B}u, h\right)\right)_0.$$

Since $h \in L^p(X)$ was arbitrary, we deduce that

$$\left\{\begin{array}{l} \dot{x}(t) + A(t, x(t)) + f(t, x(t)) = B(t)u(t) \text{ almost everywhere} \\ x(0) = x_0 \end{array}\right\}.$$

Also recall that we have $u_n \xrightarrow{w} u$ in $L^q(Y)$ (since $\{u_n(\cdot)\}_{n\geqslant 1}$ is bounded (hypothesis $H(U)$ (3)) in the reflexive Banach space $L^q(Y)$). Then from Theorem 3.1 of [15] we have $u(t) \in \overline{\text{conv}}\, w - \overline{\lim}\, U(t, x_n(t))$ almost everywhere. But because of $H(U)$ (2) we have that $w - \overline{\lim}\, U(t, x_n(t)) \subseteq U(t, x(t))$. So $u(t) \in U(t, x(t))$ almost everywhere, $u(\cdot)$ measurable. Hence (x, u) is an admissible "state–control" pair for (*). Therefore we conclude that (x, u) is the desired optimal pair; that is, $J(x, u) = m$. ∎

We can also solve a time optimal control problem with a moving target set. So let $G: T \to 2^H \setminus \{\emptyset\}$ be the moving target. Our goal is to reach $G(\cdot)$ in minimum time moving along trajectories of (*).

We will need the following hypothesis about the moving target:

$H(G)$. $G: T \to P_{fc}(H)$ is an upper semicontinuous multifunction from T into H_w, where H_w is the Hilbert space H endowed with the weak topology.

Also hypothesis H_a will be replaced by the following controllability type hypothesis:

H_c. $E = \{t \in T: G(t) \cap P(x_0)(t) \neq \emptyset\} \neq \emptyset$, where $P(x_0)$ is the set of trajectories of (*) and $P(x_0)(t) = \{x(t): x(\cdot) \in P(x_0)\}$.

Theorem 3.2. *If hypotheses $H(A)$, $H(f)$, $H(B)$, $H(U)$, $H(G)$ and H_c hold with $p = q = 2$ and X is a Hilbert space, then there exists time optimal control.*

Proof: Let $\tau = \inf E$. It exists because of H_c. Take $\{t_n\}_{n\geqslant 1} \subset E$ such that $t_n \downarrow \tau$. Then by definition there exist $x_n(\cdot) \in P(x_0)$ such that $x_n(t_n) \in G(t_n)$ $n \geqslant 1$. Recall (see the proof of (Theorem 3.1) that $\overline{\{x_n(\cdot)\}}^{w}_{n\geqslant 1}$ is w-compact in $W_{2,2}(T)$. Since X is a Hilbert space and $X \hookrightarrow H$ compactly, from Nagy [14], we know that $W_{2,2}(T) \hookrightarrow C(T, H)$ compactly. So $\overline{\{x_n(\cdot)\}}^{s}_{n\geqslant 1}$ is compact in $C(T, H)$ and thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{s} x$ in $C(T, H) \Rightarrow x_n(t_n) \xrightarrow{s} x(\tau)$ in $H \Rightarrow x(\tau) \in w - \overline{\lim} G(t_n) \subseteq G(\tau)$ (hypothesis $H(G)$). So $x(\cdot)$ is the desired optimal trajectory and any control generating $x(\cdot)$ is a time optimal control. ∎

4. An example

In this section we work out in detail an example illustrating the applicability of our results.

So let $T = [0, b]$ and Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial Z = \Gamma$. On $T \times Z$ we consider the following nonlinear parabolic distributed parameter optimal control problem.

(**)

$$\left\{\begin{array}{c} J(x, u) = \int_0^b \int_Z L(t, z, x(t, z), u(t, z))dzdt \rightarrow \inf = m \\ \text{such that } \frac{\partial x(t, z)}{\partial t} + \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(t, z, \eta(x(t, z))) + f(t, z, \theta(x(t, z))) \\ = (p(t, z), u(t, z)) \text{ on } T \times Z,\ D^\beta x(t, z) = 0 \text{ for } (t, z) \in T \times \Gamma,\ |\beta| \leqslant m - 1 \\ x(0, z) = x_0(z) \text{ on } \{0\} \times Z,\ \int_Z |u(t, z)|^2 \leqslant \int_Z r(t, z, x(t, z))^2 dz \\ u(\cdot, \cdot) \text{ measurable} \end{array}\right\}.$$

Here $\eta(x(z)) = \{D^\alpha x(z) \colon |\alpha| \leqslant m\}$ and $\theta(x) = \{D^\beta x(z) \colon |\beta| \leqslant m - 1\}$.

We will need the following hypotheses on the data of (**).

$H(A)'$. $A_\alpha \colon T \times Z \times \mathbf{R}^{n_m} \rightarrow \mathbf{R}$ are maps such that

(1) $(t, z) \rightarrow A_\alpha(t, z, \eta)$ is measurable,

(2) $\eta \rightarrow A(t, z, \eta)$ is continuous,

(3) $|A_\alpha(t, x, \eta)| \leqslant c\left(|\eta|^{p-1} + 1\right)$ almost everywhere, $c > 0$, $p \geqslant 2$,

(4) $\sum_{|\alpha| \leqslant m} (A_\alpha(t, z, \eta) - A_\alpha(t, z, \eta'))(\eta_\alpha - n'_\alpha) \geqslant 0$ for every $z \in Z$ and every $\eta, \eta' \in \mathbf{R}^{n_m}$, with $n_m = ((n + m)!)/(n!m!)$,

(5) $\sum_{|\alpha| \leqslant m} (A_\alpha(t, x, \eta))\eta_\alpha \geqslant c_2 \sum_{|\alpha| \leqslant m} |\eta_\alpha|^p$ almost everywhere, $c_2 > 0$.

$H(f)'$. $f \colon T \times Z \times \mathbf{R}^{n_m} \rightarrow \mathbf{R}$ is a function such that

(1) $(t, z) \rightarrow f(t, z, \theta)$ is measurable,

(2) $\theta \rightarrow f(t, z, \theta)$ is continuous,

(3) $|f(t, z, \theta)| \leqslant a(t, z) + b|\theta|^{p-1}$ almost everywhere with $a(\cdot, \cdot) \in L^q(T, L^\infty(Z))$,

(4) $f(t, z, \theta)\theta \geqslant -c_3$, $c_3 > 0$.

$H(p)$. $p(\cdot, \cdot) \in L^\infty(T \times Z)$.

$H(r)$. $T \times Z \times \mathbf{R} \rightarrow \mathbf{R}_+$ is a function such that

(1) $(t, z, v) \rightarrow r(t, z, v)$ is measurable,

(2) $v \rightarrow r(t, z, v)$ is upper semicontinuous,

(3) $|r(t, z, v)| \leqslant a_1(t, z)$ almost everywhere with $a_1(\cdot, \cdot) \in L^q\left(T, L^2(Z)\right)$.

$H(L)'$. $L \colon T \times Z \times \mathbf{R} \times \mathbf{R}^r \rightarrow \overline{\mathbf{R}}$ is an integrand such that

(1) $L(\cdot, \cdot, \cdot, \cdot)$ is measurable,

(2) $(x, u) \rightarrow L(t, z, x, u)$ is lower semicontinuous,

(3) $L(t, z, x, \cdot)$ is convex,

(4) $\phi(t, z) - M(|x| + \|u\|) \leqslant L(t, z, x, u)$ almost everywhere with $\phi(\cdot, \cdot) \in L^1(T \times Z)$ and $M > 0$.

H_0. $x_0(\cdot) \in L^2(Z)$.

H_α. There exists admissible "state–control" pair such that $J(x, u) < \infty$.

We will reduce (**) to the abstract optimal control problem (*) and then apply Theorem 3.1.

In this case $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{-m,q}(Z)$. This is a Gelfand triple and furthermore all embeddings are compact. Also $Y = L^2_r(Z)$ (the control space). We consider the time varying Dirichlet form $\alpha\colon T \times W_0^{m,p}(Z) \times W_0^{m,p}(Z) \to \mathrm{R}$ defined by

$$\alpha(t, x, y) = \sum_{|\alpha| \leqslant m} \int_Z A_\alpha(t, z, \eta(x(z))) D^\alpha y(z) dz.$$

Using Minkowski's and Hölder's inequalities, we have

$$\left|\int_Z A_\alpha(t, z, \eta(x(z))) D^\alpha y(z) dz\right| \leqslant \left(\int_Z |A_\alpha(t, z, \eta(x(z)))|^q \, dz\right)^{1/q} \left(\int_Z |D^\alpha y(z)|^p \, dz\right)^{1/p}$$

$$\leqslant c\left(\sum_{|\gamma| \leqslant m} \int_Z |D^\gamma x(z)|^{q(p-1)} \, dz + 1\right)^{1/q} \left(\int_Z |D^\alpha y(z)|^p \, dz\right)^{1/p}$$

$$\Rightarrow |\alpha(t, x, y)| \leqslant \widehat{c}\left(\|x\|^{p-1}_{W_0^{m,p}} + 1\right) \|y\|_{W_0^{m,p}(Z)}, \ \widehat{c} > 0.$$

Hence $\alpha(t, x, \cdot)$ is continuous and linear on $W_0^{m,p}(Z)$. Thus there exists $A\colon T \times W_0^{m,p}(Z) \to W^{-m,q}(Z)$ defined by

$$\alpha(t, x, y) = \langle A(t, x), y\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $(W_0^{m,p}(Z), W^{-m,q}(Z))$.

Observe that for all $y \in W_0^{m,p}(Z)$,

$$t \to \langle A(t, x), y\rangle = \sum_{|\alpha| \leqslant m} \int_Z A_\alpha(t, z, \eta(x(z))) D^\alpha y(z) dz$$

is by Fubini's theorem measurable. So $t \to A(t, x)$ is weakly measurable from T into $W^{-m,q}(Z)$ and since the latter is separable, invoking Pettis' theorem, we conclude that $t \to A(t, x)$ is measurable.

Next let $x_n \xrightarrow{s} x$ in $W_0^{m,p}(Z)$. Then by Krasnoselski's theorem, we have

$$\begin{aligned}&|\langle A(t, x_n) - A(t, x), y\rangle| \\ &\leqslant \sum_{|\alpha|\leqslant m} \int_Z |A_\alpha(t, z, \eta(x_n(z))) - A_\alpha(t, z, \eta(x(z)))| \cdot |D^\alpha y(z)|\, dz \to 0\end{aligned}$$

$\Rightarrow x \to A(t, x)$ is demicontinuous, in particular then hemicontinuous.

Also from hypothesis $H(A)'$ (4), we have

$$\begin{aligned}&\langle A(t, x) - A(t, y), x - y\rangle \\ &= \int_Z \sum_{|\alpha|\leqslant m} (A_\alpha(t, z, \eta(x(z))) - A_\alpha(t, z, \eta(y(z))))(\eta_\alpha(x(z)) - \eta_\alpha(y(z)))dz \geqslant 0\end{aligned}$$

$\Rightarrow x \to A(t, x)$ is monotone.

Furthermore from the growth property of $\alpha(\cdot, \cdot, \cdot)$ we have

$$\begin{aligned}&|\langle A(t, x), y\rangle| \leqslant \left(\widehat{c}\left(\|x\|^{p-1}_{W_0^{m,p}(Z)} + 1\right)\right) \cdot \|y\|_{W_0^{m,p}(Z)} \\ &\Rightarrow \|A(t, x)\|_* \leqslant \widehat{c}\left(\|x\|^{p-1}_{W_0^{m,p}(Z)} + 1\right), \widehat{c} > 0.\end{aligned}$$

Finally from hypothesis $H(A)'$ (5), we have:

$$\langle A(t, x), x\rangle \geqslant \widehat{c}_2 \|x\|^p_{W_0^{m,p}(Z)} \quad \widehat{c}_2 > 0.$$

Thus operator $A\colon T \times W_0^{m,p}(Z) \to W^{-m,q}(Z)$ defined above satisfies hypothesis $H(A)$.

Next let $F\colon T \times W_0^{m,p}(Z) \to L^2(Z)$ be defined by

$$F(t, x)(z) = f(t, z, \theta(x(z))).$$

Because of hypotheses $H(f)'$ (1), (2), (3) and since $p \geqslant 2$ and $Z \subseteq \mathbb{R}^n$ is bounded, from Krasnoselski's theorem, we have that $F(t, x)$ is well defined. Also for every $h \in L^2(Z)$, $(f(t, \cdot, \theta(x(\cdot))), h)_{L^2(Z)} = \int_Z f(t, z, \theta(x(z)))h(z)dz$ and so Fubini's theorem tells us that $t \to f(t, \cdot, \theta(x(\cdot)))$ is weakly measurable, hence by Pettis' theorem measurable. So $t \to F(t, x)$ is measurable. Furthermore, if $x_n \xrightarrow{w} x$ in $W_0^{m,p}(Z)$, then since $W_0^{m,p}(Z) \hookrightarrow W_0^{m-1,p}(Z)$ compactly, we have $x_n \xrightarrow{s} x$ in $W_0^{m-1,p}(Z)$ and then using Krasnoselski's theorem, we conclude that $F(t, \cdot)$ is completely continuous from $W_0^{m,p}(Z)$ into $L^2(Z)$. Hence $F(t, \cdot)$ is continuous and sequentially weakly continuous. In addition, because of hypothesis $H(f)'$ (3), we have

$\|F(t, x)\|_{L^2(Z)} \leqslant a(t)+b\,\|x\|^{p-1}_{W_0^{m,p}(Z)}$, with $\widehat{a}(t) = \|a(t, \cdot)\|_{L^\infty(Z)}$, so that $\widehat{a}(\cdot) \in L^q(T)$. Finally from hypothesis $H(f)'$ (4), we have $-\widehat{c}_3 \leqslant (F(t, x), x)_{L^2(Z)}$ with $\widehat{c}_3 > 0$. Thus $F\colon T \times W_0^{m,p}(Z) \to L^2(Z)$ defined above satisfies hypothesis $H(f)$.

Next let $\widehat{L}\colon T \times L^2(Z) \times L^2_r(Z) \to \overline{\mathrm{R}}$ be defined by

$$\widehat{L}(t, x, u) = \int_Z L(t, z, x(z), u(z))dz.$$

From Pappas [16] we know that we can find $L_k\colon T \times Z \times \mathrm{R} \times \mathrm{R}^k \to \mathrm{R}$ Caratheodory integrands (that is, measurable in (t, z), continuous in (x, u)) such that $\phi(t, z) - M(|x| + \|u\|) \leqslant L_k(t, z, x, u) \leqslant k$ and $L_k \uparrow L$. Then set

$$\widehat{L}_k(t, x, u) = \int_Z L_k(t, z, x(z), u(z))dz.$$

Clearly $\widehat{L}_k(\cdot, \cdot, \cdot)$ is Caratheodory (that is, measurable in t, continuous in (x, u)); thus it is jointly measurable. By the monotone convergence theorem, we have $\widehat{L}_k \uparrow \widehat{L}$, so $\widehat{L}(\cdot, \cdot, \cdot)$ is jointly measurable too. Also $\widehat{L}(t, x, u) \geqslant \widehat{\phi}(t) - M(\|x\|_2 + \|u\|_2)$ while from Balder [5] we have that $\widehat{L}(t, \cdot, \cdot)$ is sequentially "strongly $\times$ weakly" lower semicontinuous on $L^2(Z) \times L^2_r(Z)$.

Finally let $U\colon T \times L^2(Z) \to P_{fc}(L^2_r(Z))$ be defined by

$$U(t, x) = \{u \in L^2_r(Z)\colon\ \|u\|_{L^2_r(Z)} \leqslant \widehat{r}(t, x)\}$$

with $\widehat{r}(t, x) = \int_Z r(t, z, x(z))dz$. As we did for integrand $\widehat{L}(\cdot, \cdot, \cdot)$ we can show that $\widehat{r}(\cdot, \cdot)$ is measurable. This time since $r(t, z, \cdot)$ is upper semicontinuous, the Caratheodory approximations are from above. Hence $GrU = \{(t, x, u) \in T \times L^2(Z) \times L^2_r(Z)\colon \widehat{r}(t, x) - \|u\|_2 \geqslant 0\} \in B(T) \times B(L^2(Z)) \times B(L^2_r(Z)) \Rightarrow U(\cdot, \cdot)$ is graph measurable. Also $|U(t, x)| \leqslant \widehat{a}_1(t)$ almost everywhere with $\widehat{a}_1(t) = \|a(t, \cdot)\|_{L^2(Z)}$; thus $\widehat{a}_1(\cdot) \in L^q_+$. Finally if $(x_n, u_n) \in GrU(t, \cdot)$ and $(x_n, u_n) \xrightarrow{s \times w} (x, u)$ in $L^2(Z) \times L^2_r(Z)$, then $\|u\|_2 \leqslant \underline{\lim}\,\|u_n\|_2 \leqslant \overline{\lim}\,\|u_n\|_2 \leqslant \overline{\lim}\,\widehat{r}(t, x_n) \leqslant \widehat{r}(t, x)$ (the last inequality coming from Fatou's Lemma). So we have satisfied hypothesis $H(U)$.

Next let $B(t)\colon L^2_r(Z) \to L^2(Z)$ be defined by $(B(t)u)(z) = (p(t, z), u(z))$. Because of hypothesis $H(p)$, $B(\cdot) \in L^\infty(T, \mathcal{L}(L^2_r(Z), L^2(Z)))$. Finally let $\widehat{x}_0 = x_0(\cdot) \in L^2(Z)$ (hypothesis H_0).

So we can rewrite (**) in the following equivalent abstract form:

$$(**)' \qquad \left\{ \begin{array}{c} \widehat{J}(x, u) = \displaystyle\int_0^b \widehat{L}(t, x(t), u(t))dt \to \inf = m \\ \text{such that } \dot{x}(t) + A(t, x(t)) + F(t, x(t)) = B(t)u(t) \text{ almost everywhere} \\ x(0) = x_0,\ u(t) \in U(\cdot, x(t)) \text{ almost everywhere} \\ u(\cdot) \text{ measurable} \end{array} \right\}.$$

https://doi.org/10.1017/S0004972700028975 Published online by Cambridge University Press

This has the same form as (*). So we can apply Theorem 3.1 and get:

Theorem 4.1. *If hypotheses $H(A)'$, $H(f)'$, $H(p)$, $H(r)$, $H(L)'$, H_0 and H_a hold, then there exists admissible "state-control" pair $(x, u) \in L^q(T, W_0^{m,p}(Z)) \times L^q(T, L_r^2(Z))$ such that $J(x, u) = m$.*

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