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A SIMPLIFIED PROOF OF HESSELHOLT'S CONJECTURE ON GALOIS COHOMOLOGY OF WITT VECTORS OF ALGEBRAIC INTEGERS

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Abstract

Let K be a complete discrete valuation field of characteristic zero with residue field k_K of characteristic $p > 0$. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$ and suppose that the induced extension of residue fields k_L/k_K is separable. Let $\mathbb{W}_n(\cdot)$ denote the ring of p-typical Witt vectors of length n. Hesselholt ['Galois cohomology of Witt vectors of algebraic integers', *Math. Proc. Cambridge Philos. Soc.* **137**(3) (2004), 551–557] conjectured that the pro-abelian group $\{H^1(G, \mathbb{W}_n(O_L))\}_{n\geq 1}$ is isomorphic to zero. Hogadi and Pisolkar ['On the cohomology of Witt vectors of p-adic integers and a conjecture of Hesselholt', *J. Number Theory* **131**(10) (2011), 1797–1807] have recently provided a proof of this conjecture. In this paper, we provide a simplified version of the original proof which avoids many of the calculations present in that version.

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1. Literature review

Let K be a complete discrete valuation field of characteristic zero with residue field k_K of characteristic $p > 0$. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$ and suppose that the induced extension of residue fields k_L/k_K is separable. Let $\mathbb{W}_n(\cdot)$ denote the ring of p-typical Witt vectors of length n. In Hesselholt's paper [1] it is conjectured that the pro-abelian group $\{H^1(G, \mathbb{W}_n(O_L))\}_{n\geq 1}$ is isomorphic to zero, and the conjecture is reduced to the case where L/K is a totally ramified cyclic extension of degree p. Let σ be a generator of G and let $t := v_L(\sigma(\pi_L) - \pi_L) - 1$ denote the ramification break (see [3, Ch. V, Section 3]) in the ramification filtration of G. Recall that t does not depend on the choice of generator σ.

Hesselholt shows his conjecture holds for extensions with $t > e_K/(p-1)$. Hogadi and Pisolkar have recently provided a proof of the conjecture for all Galois extensions (see [2]). In this paper, we provide a simplified version of the original proof which avoids many of the calculations present in that version. First let us recall some lemmas from [1].

Lemma 1.1. *For all* $a \in \mathcal{O}_L$, $v_K(\mathrm{tr}(a)) \geq (v_L(a) + t(p-1))/p$.

Proof. We know that $a \in \mathfrak{p}_L^{v_L(a)}$, so from [3, Ch. V, Section 3, Lemma 4], we have $\mathrm{tr}(a) = \pi_K^{\lfloor ((t+1)(p-1)+v_L(a))/p \rfloor} b$ for some $b \in \mathcal{O}_K$. Now taking K-valuations gives the desired result. □

Lemma 1.2. *For all* $a \in \mathcal{O}_L$, $v_K(\mathrm{tr}(a^p) - \mathrm{tr}(a)^p) = v_K(p) + v_L(a)$.

Proof. This follows by expanding $\mathrm{tr}(a^p) - \mathrm{tr}(a)^p$ using the multinomial formula and grouping the resulting expression into summands with distinct valuations. See the proof of [1, Lemma 2.2] for details. □

Next, we provide an alternative elementary proof of [1, Lemma 2.4].

Lemma 1.3. *Suppose that* $a \in \mathcal{O}_L^{\mathrm{tr}=0}$ *represents a nonzero class in* $\mathcal{O}_L^{\mathrm{tr}=0}/((\sigma-1)\mathcal{O}_L)$. *Then* $v_L(a) \leq t-1$.

Proof. For each $0 \leq \mu \leq p-1$, define $x_\mu = \prod_{0 \leq i < \mu} \sigma^i(\pi_L)$. It is clear that $v_L(x_\mu) = \mu$. Suppose that

$$a_0 x_0 + a_1 x_1 + \cdots + a_{p-1} x_{p-1} = 0$$

for some $a_0, a_1, \ldots, a_{p-1} \in K$. The summands on the left have distinct L-valuations modulo p and thus distinct L-valuations, implying that each summand must be zero by the nonarchimedean property. Hence the x_μ are linearly independent over K and thus span L over K. Now recall that $\ker(\mathrm{tr})/((\sigma-1)L) = H^1(G, L) = 0$ (see [3, Ch. VIII, Section 4] and [3, Ch. X, Section 1, Proposition 1]). Hence $\mathcal{O}_L^{\mathrm{tr}=0} \subseteq (\sigma-1)L$, so we can write

$$a = b_1(\sigma-1)x_1 + b_2(\sigma-1)x_2 + \cdots + b_{p-1}(\sigma-1)x_{p-1}$$

for some $b_1, b_2, \ldots, b_{p-1} \in K$. It is clear from the definition of x_μ that $\pi_L \sigma(x_\mu) = x_\mu \sigma^\mu(\pi_L)$ for each $1 \leq \mu \leq p-1$ so that

$$v_L((\sigma-1)x_\mu) = v_L\bigg(\frac{(\sigma^\mu - 1)\pi_L}{\pi_L} \cdot x_\mu\bigg) = t + \mu,$$

implying that the summands on the right have distinct L-valuations modulo p, and thus distinct L-valuations. Since $a \notin (\sigma-1)\mathcal{O}_L$ by hypothesis, we must have $b_{\mu'} \notin \mathcal{O}_K$ for some μ' so that $v_L(b_{\mu'}(\sigma-1)x_{\mu'}) \leq -p + t + \mu' \leq -p + t + (p-1)$ for this μ'. Hence by the nonarchimedean property, we conclude that $v_L(a) \leq t-1$, as required. □

Lemma 1.4. *Let* $m \geq 1$ *be an integer and suppose that the map*

$$R_*^m : H^1(G, \mathbb{W}_{m+n}(\mathcal{O}_L)) \to H^1(G, \mathbb{W}_n(\mathcal{O}_L))$$

is equal to zero, for $n = 1$. *Then the same is true for all* $n \geq 1$.

Proof. This follows from the long exact sequence of cohomology. See the proof of [1, Lemma 1.1] for details. □

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2. Proof of Hesselholt's conjecture

Recall that, for each $n \geq 0$, the Witt polynomial is

$$W_n(X_0, X_1, \ldots, X_n) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}.$$

Fix any $m \geq 0$. Let

$$\sum_{i=0}^{p-1} (X_{i,0}, X_{i,1}, \ldots, X_{i,m}) = (z_0, z_1, \ldots, z_m)$$

where on the left we have a sum of Witt vectors. Then we know that each z_n is a polynomial in $\mathbb{Z}[\{X_{i,j}\}_{0\leq i\leq p-1, 0\leq j\leq n}]$ with no constant term (see [3, Ch. II, Section 6, Theorem 6]). By construction of Witt vector addition (see [3, Ch. II, Section 6, Theorem 7]),

$$\sum_{i=0}^{p-1} W_n(X_{i,0}, X_{i,1}, \ldots, X_{i,n}) = W_n(z_0, z_1, \ldots, z_n)$$

for each $0 \leq n \leq m$. Now using the expression for the Witt polynomial W_n and dividing through by p^n yields

$$f_n + \sum_{i=0}^{p-1} X_{i,n} - z_n = 0 \tag{2.1}$$

where

$$f_n = \frac{1}{p^n}\Big(\sum_{i=0}^{p-1} X_{i,0}^{p^n} - z_0^{p^n}\Big) + \frac{1}{p^{n-1}}\Big(\sum_{i=0}^{p-1} X_{i,1}^{p^{n-1}} - z_1^{p^{n-1}}\Big) + \cdots + \frac{1}{p}\Big(\sum_{i=0}^{p-1} X_{i,n-1}^{p} - z_{n-1}^{p}\Big). \tag{2.2}$$

Now for any $1 \leq n \leq m$, we may add and subtract $(1/p)(-f_{n-1})^p$ to obtain

$$f_n = g_{n-2} + \frac{1}{p}\Big(\sum_{i=0}^{p-1} X_{i,n-1}^{p} - z_{n-1}^{p} - (-f_{n-1})^p\Big) \tag{2.3}$$

where

$$g_{n-2} = \frac{1}{p^n}\Big(\sum_{i=0}^{p-1} X_{i,0}^{p^n} - z_0^{p^n}\Big) + \cdots + \frac{1}{p^2}\Big(\sum_{i=0}^{p-1} X_{i,n-2}^{p^2} - z_{n-2}^{p^2}\Big) + \frac{1}{p}(-f_{n-1})^p. \tag{2.4}$$

Lemma 2.1. *Suppose that* $(a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(O_L)$. *Then*

$$v_L(g_{n-2}|_{X_{i,j}=\sigma^i(a_j)}) \geq p^2 \cdot \min\{v_L(a_j) : 0 \leq j \leq n-2\}$$

for each $2 \leq n \leq m$.

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Proof. From (2.1) and (2.2) we know that f_n is a polynomial in $\mathbb{Z}[\{X_{i,j}\}_{0\le i\le p-1, 0\le j\le n-1}]$ with no constant term, and each monomial of f_n has degree at least p. From (2.1) we know that $\sum_{i=0}^{p-1} X_{i,n-1} = z_{n-1} - f_{n-1}$, implying that $\sum_{i=0}^{p-1} X_{i,n-1}^p \equiv z_{n-1}^p + (-f_{n-1})^p \pmod p$, so in view of (2.3) we see that g_{n-2} has integer coefficients. Thus from (2.4) we know that g_{n-2} is a polynomial in $\mathbb{Z}[\{X_{i,j}\}_{0\le i\le p-1, 0\le j\le n-2}]$ with no constant term, and each monomial of g_{n-2} has degree at least p^2. Hence, recalling that $v_L(\sigma^i(a_j)) = v_L(a_j)$ (see [3, Ch. II, Section 2, Corollary 3]), and using the properties of valuations, it is clear that we have the desired inequality. □

Lemma 2.2. *Suppose that* $(a_0, a_1, \dots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{\mathrm{tr}=0}$. *Then*

$$v_L(a_{n-1}) \ge \min\left\{\frac{v_L(a_n) + t(p-1)}{p}, t(p-1)\right\}$$

for each $1 \le n \le m$.

Proof. Since $(a_0, a_1, \dots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{\mathrm{tr}=0}$, by definition of the z_n we can take $z_n = 0$ for $0 \le n \le m$ and $X_{i,j} = \sigma^i(a_j)$. Then from (2.1) we see that $-f_n = \mathrm{tr}(a_n)$ for each n, and hence (2.3) reduces to

$$\frac{\mathrm{tr}(a_{n-1}^p) - \mathrm{tr}(a_{n-1})^p}{p} = -\mathrm{tr}(a_n) - g_{n-2}|_{X_{i,j}=\sigma^i(a_j)}.$$

Taking K-valuations of both sides of this equation and then applying Lemmas 1.2 and 1.1 gives

$$v_L(a_{n-1}) \ge \min\left\{\frac{v_L(a_n) + t(p-1)}{p}, v_K(g_{n-2}|_{X_{i,j}=\sigma^i(a_j)})\right\}. \tag{2.5}$$

Since $f_0 = 0$ by (2.2), we see that $g_{-1} = 0$ by (2.4). Hence taking $n = 1$ in (2.5), we see that the claim holds for $n = 1$. Now for the inductive step let $N \ge 2$ and suppose that the claim holds for all $1 \le n \le N-1$. Then we have

$$\begin{aligned} v_L(a_{N-1}) &\ge \min\left\{\frac{v_L(a_N) + t(p-1)}{p}, \frac{1}{p}\cdot v_L(g_{N-2}|_{X_{i,j}=\sigma^i(a_j)})\right\} \\ &\ge \min\left\{\frac{v_L(a_N) + t(p-1)}{p}, \frac{1}{p}\cdot p^2 \cdot \min\{v_L(a_{n-1}) : 1 \le n \le N-1\}\right\} \\ &\ge \min\left\{\frac{v_L(a_N) + t(p-1)}{p}, \frac{1}{p}\cdot p^2 \cdot \frac{t(p-1)}{p}\right\} \end{aligned}$$

where the first inequality follows from (2.5), the second by Lemma 2.1, and the third by the induction hypothesis. This completes the inductive step and the proof of the lemma. □

By Lemma 1.4, and recalling that

$$H^1(G, \mathbb{W}_{m+1}(\mathcal{O}_L)) = \frac{\mathbb{W}_{m+1}(\mathcal{O}_L)^{\mathrm{tr}=0}}{(\sigma - 1)\mathbb{W}_{m+1}(\mathcal{O}_L)}$$

(see [3, Ch. VIII, Section 4]), the following proposition (a generalisation of [1, Proposition 2.5]) proves Hesselholt's conjecture.

Proposition 2.3. *The map*

$$R_*^m : \frac{\mathbb{W}_{m+1}(\mathcal{O}_L)^{\mathrm{tr}=0}}{(\sigma - 1)\mathbb{W}_{m+1}(\mathcal{O}_L)} \to \frac{\mathcal{O}_L^{\mathrm{tr}=0}}{(\sigma - 1)\mathcal{O}_L}, \quad (a_0, a_1, \ldots, a_m) \mapsto a_0$$

is equal to zero, provided that $p^m > t$.

Proof. Suppose that $(a_0, a_1, \ldots, a_m) \in \mathbb{W}_{m+1}(\mathcal{O}_L)^{\mathrm{tr}=0}$. Note that $v_L(a_n) > t - p^n$ implies that

$$v_L(a_{n-1}) \geq \min\left\{\frac{v_L(a_n) + t(p-1)}{p}, t(p-1)\right\} > \frac{(t - p^n) + t(p-1)}{p} = t - p^{n-1}.$$

Since $v_L(a_m) > t - p^m$ by hypothesis, we see that $v_L(a_0) > t - p^0$ by downward induction. Thus by Lemma 1.3 we see that a_0 must represent the zero class in $\mathcal{O}_L^{\mathrm{tr}=0}/((\sigma - 1)\mathcal{O}_L)$. □

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